What is Graph Theory?

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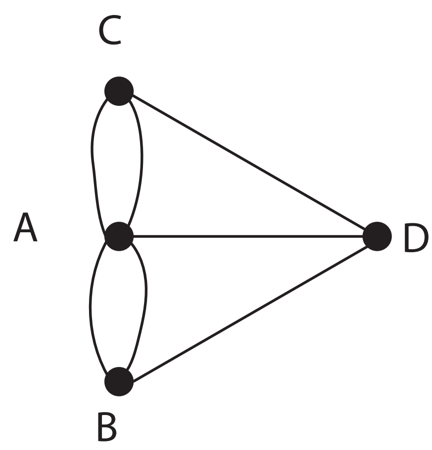
1 Let's Cross Some Bridges

Brain Teaser 1 The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands - Kneiphof and Lomse - which were connected to each other, or to the two mainland portions of the city, by a total of seven bridges. According to lore, the citizens of Königsberg used to spend Sunday afternoons walking around their beautiful city. While walking, the people of the city decided to create a game for themselves, their goal being to devise a way in which they could walk around the city, crossing each of the seven bridges only once. Can you find a way to achieve this goal?

No you wouldn't be able to find a way to do this. We will discuss more about this later. The intuition is that, we can think of Königsberg as a city with parts: the two sides of the river and the two islands.
Each part has an odd number of bridges connecting it to other parts. However, if we were to leave a part and re-enter without repeating any of the bridges, then that part must have an even number of bridges connecting it to others. So we could find a way to start from a place in a city, cross all the bridges exactly once and then come back to the staring point. This can be presented by what we call a **graph**. When A and D represent the islands and B and C represent the two sides of the river. Each line between two points represents a bridge.



As we discussed before, each point in the graph is next to an odd number of lines. So there's no way to start from one of the points in the graph, go through each line exactly once, and return to the starting point.

2 Definitions and Notations

Brain Teaser 2 Suppose there are 10 cities in a country, each of them has exactly four roads connecting it to four distinct cities in the same country. Each road connects exactly two cities. How many roads are there in total?

Each road has two ends, so if we add up all road ends at all the cities, we would get exactly twice the number of roads. In other words, $4 \times 10 = 2 \times (\# \ of \ roads)$.
Thus we can solve for the number of roads, which is 20.

Brain Teaser 3 Suppose there are 10 cities in a country, five of them have exactly four roads connecting them to four distinct cities in the same country, and the other five of them have five roads connecting them to five distinct cities in the same country. In this case, how many roads are there in total?

$$4 \times 5 + 5 \times 5 = 2 \times (\# \ of \ roads)$$

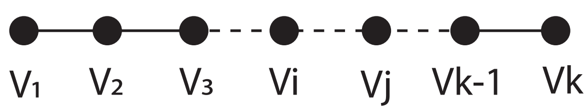
Solving this gives a fraction as the number of roads, which is impossible. Therefore such a situation wouldn't exist.

Brain Teaser 4 Can you find a graph over 7 vertices, with the degree of the vertices being 0, 1, 2, 3, 4, 5, 6?

No. One explanation is the sum of degrees of all vertices would be odd, which is impossible.
Another explanation would be, since we are only considering simple graphs, for a vertex to have degree 6, it must be adjacent to every other vertex in the graph. This implies every vertex has degree at least 1, which would be a contradiction. So such a graph wouldn't exist.

Brain Teaser 5 For some positive integer $k > 2$, suppose we have a graph G where every vertex has degree at least k. Show that there is a cycle in G with even length.

Consider a longest path in this graph, which we denote as $v_1, v_2, ..., v_k$.



Since this is one of the longest paths in the graph, we can't find a longer path. If v_1 has a neighbour u that is outside of this path, then $u, v_1, v_2, ..., v_k$ forms a longer path, which contradicts our assumption. Therefore, all the neighbours of v_1 are in this path. The degree of v_1 is greater than 2, so other than v_2, it has at least two other neighbours. Suppose two of its neighbours are v_i and v_j with $i < j$, $i \neq 2$, $j \neq 2$. Then $v_1, v_2, ..., v_i, v_1$ forms a cycle, and $v_1, v_2, ..., v_j, v_1$ form a cycle.

If one of these two cycles are of even length then we have already found an even cycle in this graph.
If both of these cycles are of odd length, then both i and j are odd numbers. Consider the cycle $v_1, v_i, v_{i+1}..., v_j, v_1$. Since both i and j are odd, this is a cycle of even length.

Brain Teaser 6 Suppose we have a graph G with n vertices, such that for any two vertices u and v that are not adjacent to each other, we have $deg(u) + deg(v) \geq n$. Can you show that this graph is connected?

To show the graph is connected, we need to show that, if we pick any two vertices u and v in the graph, then we can find a path that connects u to v.
For any two vertices u and v, if they are neighbours, then there is a path from u to v.
If u and v are not neighbours, then $deg(u) + deg(v) \geq n$. If u and v do not have a common neighbour, then the graph has $2 + deg(u) + deg(v) \geq n + 2$ vertices. This contradicts with the condition that this graph has n vertices, so u and v must have a common neighbour. Therefore, we can find a path from u to v through this common neighbour. Thus this graph is connected.