Hamilton Cycles and Planar Graphs

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Recall: A **cycle** in a graph G is a finite sequence of vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}, $v_0, v_1, \cdots, v_{t-1}$ are distinct, and $v_0 = v_t$.

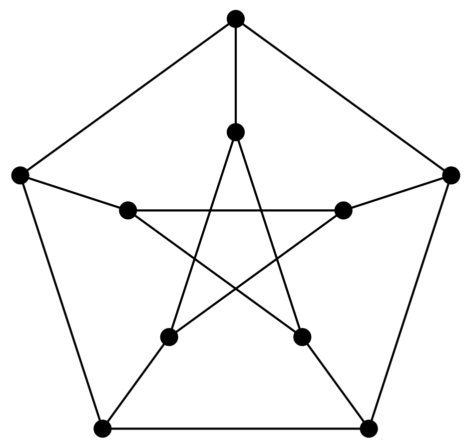
Informally, you can think of a cycle as a closed loop that start and end at the same vertex, and all intermediate vertices appear in the cycle exactly once. It does not have to use every vertex in the graph.

Brain Teaser 7 Suppose there are 20 teams in a football tournament. On the first day, each team plays exactly one game. On the second day, each team plays exactly one game as well, with a team that was not its opponent on the first day. After the first two days of the tournament, can you find 10 teams, so that no two of them have play with each other?

1 Hamilton Cycles

A cycle that uses every vertex in a graph is called a **Hamilton cycle**.

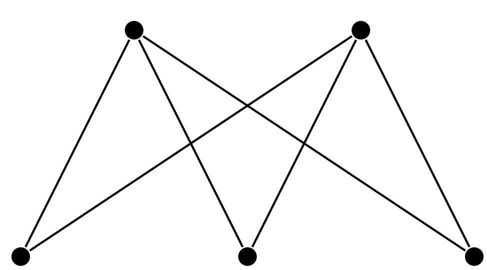
Brain Teaser 8 Does the following graph have a Hamilton cycle?



Brain Teaser 9 Given and 8×8 chessboard, find all pairs of squares on the board, such that the remaining 62 squares can be tiled using 2×1 tiles.

A **complete bipartite graph** $K_{m,n}$ is a graph with a total of $m + n$ vertices, such that its vertex set can be split into a set A of m vertices and a set B of n vertices, so that each vertex in A is adjacent to every vertex in B. Moreover, there is no edge between two vertices in A or two vertices in B.

For example, the following figure gives a drawing of $K_{2,3}$.



Brain Teaser 10 The complete bipartite graph $K_{m,n}$ has a Hamilton cycle only when $m = n$ and $m > 1$. Why?

2 Planar Graphs

Brain Teaser 11 Given three houses (House A, B, and C) and three utilities (Water, Electric, Gas,) can you connect all three of the houses to all three of the utilities without every crossing a pipeline?

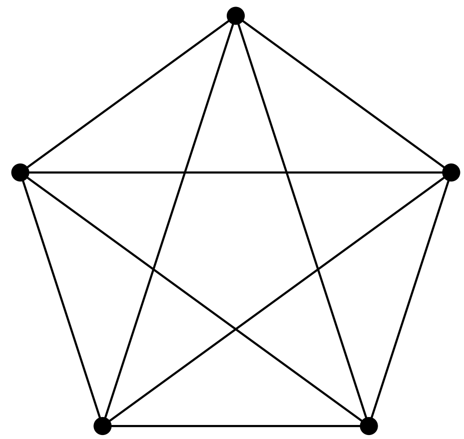
A graph is **planar** if it has a drawing in the plane so that no two vertices would coinside and no two edges would cross each other.

When a planar graph is drawn without its edges crossing each other, it divides the plane into regions. We call each region a **face**.

Theorem. *For any connected planar graph with n vertices, e edges, and f faces, we have*

$$n - e + f = 2.$$

Brain Teaser 12 Is the following graph planar?



Theorem. *For a simple connected planar graph with* $n \geq 3$ *vertices and* e *edges,* $e \leq 3n - 6$*.*

Brain Teaser 13 Is the following graph planar?