Hamilton Cycles and Planar Graphs

Caelan Wang

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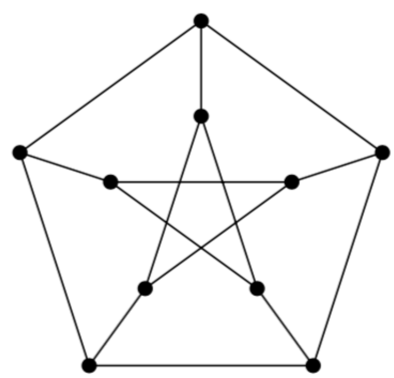
Brain Teaser 7 Suppose there are 20 teams in a football tournament. On the first day, each team plays exactly one game. On the second day, each team plays exactly one game as well, with a team that was not its opponent on the first day. After the first two days of the tournament, can you find 10 teams, so that no two of them have played with each other?

We use a vertex to represent each team and draw an edge between two teams if they've played each other on the first two days.
Note each vertex in this graph has degree 2, so this graph is a cycle or a collection of cycles. Moreover, each cycle in this graph would be of even length. Picking every other vertex in these cycles would give us 10 teams such that none of them have played each other.

Brain Teaser 8 Does the following graph have a Hamilton Cycle?



No. We consider the outer five vertices and the inner five vertices separately. There are three types of edges: the edges between two outer vertices, the edges between two inner vertices, and the edges between an outer vertex and an inner vertex. If this graph has a Hamilton cycle, then the Hamilton cycle must contain an even number (two or four) of edges between an outer vertex and an inner vertex. A case analysis (try it out!) can show that no such Hamilton cycle exists.

Brain Teaser 9 Given an 8×8 chessboard, find all pairs of squares on the board, such that the remaining 62 squares can be tiled using 2×1 tiles.

We can represent this board by a graph, where each square is a vertex and there's an edge between two vertices if the two squares they represent share a side. Note that this graph has a Hamilton cycle. *In fact, it has more than one, but all we need is one. Could you find one?*
Deleting any pair of vertices from this Hamilton cycle would leave us a path of 62 vertices or two paths with 62 vertices in total.
In the case of one path, as long as we tile along the path, all 62 vertices can be tiles by 2×1 tiles.
In the case of two paths, if we tile along the two paths and cover all 62 vertices only if both of these two paths have an even number of vertices.
In both of these cases, the squares being deleted are one black and one white, and deleting one black and one white tile would always give one path or two paths with an even number of vertices, so all choices of a black and a white square would work.

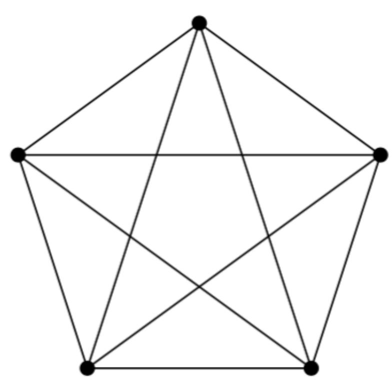
Brain Teaser 10 The complete bipartite graph $K_{m,n}$ has a Hamilton cycle only when $m = n$ and $m > 1$. Why?

When $m = n$, clearly $K_{m,n}$ doesn't have a Hamilton cycle. If a bipartite graph is to have a Hamilton cycle, then the vertices in this cycle must alternate between vertices in the two sides. This requires the two sides to have the same number of vertices.
Although not part of this question, it is true that $K_{n,n}$ has a Hamilton cucle, provided that $n \geq 2$.

Brain Teaser 11 Given three houses (House A, B, and C) and three utilities (Water, Electric, Gas), can you connect all three of the houses to all of the utilities without ever crossing a pipeline?

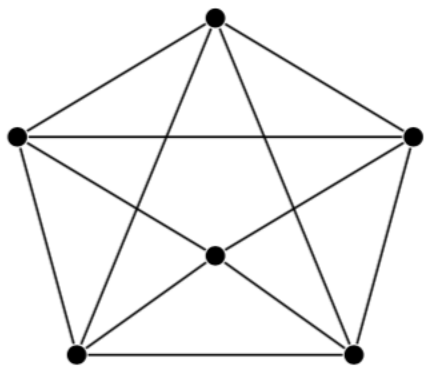
The answer is no. We'll say more about why once we discuss more about planar graphs.

Brain Teaser 12 Is the following graph planar?



This graph is not planar.
Suppose in contrary that it is, then this is a way to draw it without the edges crossing each other. We use f to denote the number of faces in this drawing. Then by the formula in the previous theorem, we have $5-10+f=2$, so $f=7$. Meanwhile, note that each face is bounded by at least 3 edges. Moreover, if we add the number of edges bounding each face, then every edge is counted exactly twice. So $3f=21 \leq 2e=20$. This is a contradiction which implies our assumption is false. Thus, the graph is not planar. The previous brain teaser can be solved in a similar way. *Just be careful: can a face be bounded by 3 edges in a bipartite graph?*

Brain Teaser 13 Is the following graph planar?



Yes, the graph is planar. Here is a planar drawing of it: