Ramsey Theory and Matchings

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1 Ramsey Theory

Brain Teaser 14 Is the following statement true?

"Given any six people, then there are three of them all know each other, or three of them such that no two of the three know each other. (Assuming that "knowing" is symmetric, meaning that if A knows B, then B knows A as well.)"

Recall: The **complete graph** over n vertices, or K_n is the graph with n vertices such that there is an edge between every pair of vertices.

Let s and t be positive integers. The **Ramsey number** $R(s,t)$ is the minimum integer for which every red-blue coloring of K_n contains either a red K_s or a blue K_t.

Theorem. *For any positive integers s and t, the Ramsey number $R(s,t)$ is finite.*

Brain Teaser 15 For any positive integer $n \geq 2$, what is $R(2, n)$?

Brain Teaser 16 Can you show $R(3, 3) > 5$? (And therefore $R(3, 3) = 6$.)

Brain Teaser 17 Show that $R(s, t) \leq R(s - 1, t) + R(s, t - 1)$.

2 Matchings

Brain Teaser 18 The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

A **matching** in a graph is a collection of edges which have no endpoints in common. A matching in a graph G is a **perfect matching** if every vertex is incident to an edge in the matching.

Brain Teaser 19 How many perfect matchings are there in K_9?

Brain Teaser 20 How many perfect matchings are there in K_6?

Consider a bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$. We say that A has a **perfect matching** to B if there is a matching which hits every vertex in A.

Theorem (Hall's Marriage Theorem)**.** *For any set $S \subseteq A$, let $N(S)$ denote the set of vertices which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for every $S \subseteq A$.*

Brain Teaser 21 Prove that if we have a bipartite graph where each vertex has degree k, where $k \geq 2$ is a positive integer, then this graph has a perfect matching.