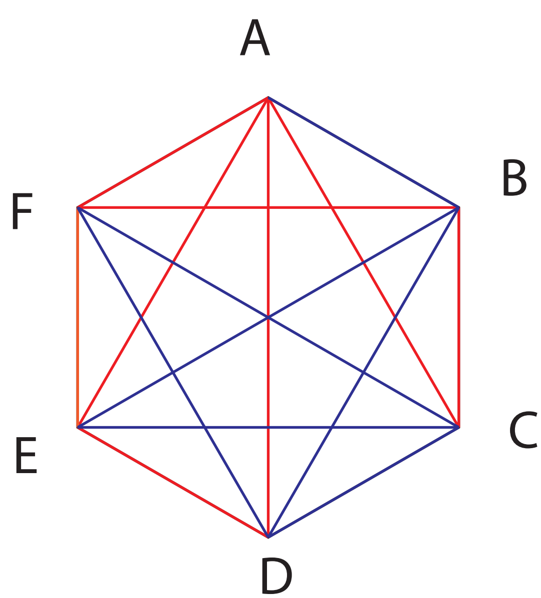
Ramsey Theory and Matchings

Caelan Wang

October 23, 2019

Brain Tease 14 Is the following statement true?
"Given any six people, then there are three of them that all know each other, or three of them such that no two of the three know each other. (Assuming that "knowing" is symmetric, meaning that if A knows B, then B knows A as well.)"

The answer is yes. To model this as a graph, we can use a vertex to represent a person, connect two people using a blue edge if they know each other, and connect two people using a red edge if they don't know each other. The graph below is an example of how the relationships between these people could look like.



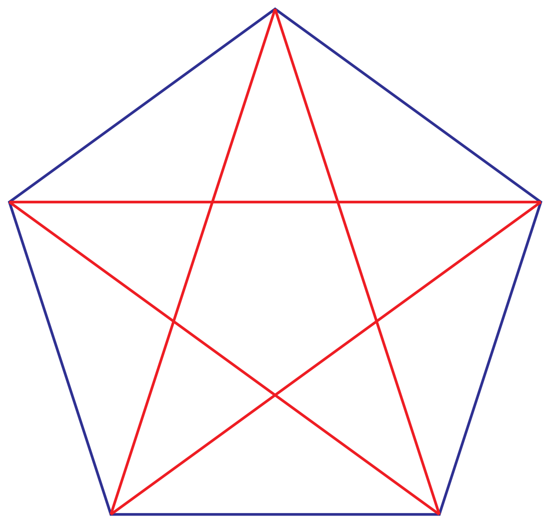
Every vertex of this graph has degree 5. Then at any vertex, there are at least three edges of the same colour.
Let's consider an arbitrary vertex A, and we consider the other ends of three edges incident to A that have the same colour (say red). Without loss of generality, we assume the other end of these three edges are the vertices B, C, D. If there is a red edge between any two of B, C, D, then along with the edges connecting them to A, we have found a red triangle, i.e. three people who don't know each other.

Brain Teaser 15 For any positive integer $n \geq 2$, what is $R(2, n)$?

$R(2, n) = n$.
If we were to colour the edges of K_n, then it either has a red edge (red K_2) or not (blue K_n) so $R(2, n) \geq n$.
We can colour all edges of K_{n-1} blue, and it wouldn't contain a red K_2 or a blue K_n. So $R(2, n) > n - 1$. Therefore $R(2, n) = n$.

Brain Teaser 16 Can you show $R(3, 3) > 5$? (And therefore $R(3, 3) = 6$.)

This edge colouring of K_5 has no red K_3 or blue K_3 so $R(3, 3) > 5$.



Brain Teaser 17 Show that $R(s, t) \leq R(s - 1, t) + R(s, t - 1)$.

Consider the complete graph on $R(s-1, t) + R(s, t-1)$ vertices whose edges are coloured red and blue. Pick any vertex v from this graph, and separate the rest of vertices into two sets: a vertex is in the set M if a red edge connects it to v, and a vertex is in the set N if a blue edge connects it to v.
Use $|M|$ to represent the number of vertices M and use $|N|$ to represent the number of vertices in N. Then:

$$|M| + |N| = R(s - 1, t) + R(s, t - 1)$$

This implies either $|M| \geq R(s - 1, t)$ or $|N| \geq R(s, t - 1)$.
Suppose $|M| \geq R(s - 1, t)$. If the vertices in M form a blue K_t in it, then the original graph also has a blue K_t. Otherwise, the vertices in M form a red K_{s-1}. Along with the vertex v, we get a red K_5 in the graph.
If $|M| \geq R(s - 1, t)$ is not true, then$|N| \geq R(s, t - 1)$ and a similar argument applies.

Brain Teaser 18 The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

We use a graph to represent this tournament, where the vertices are the players, and there's an edge between two vertices if the corresponding two players played a game. Suppose we can find at most $t \leq 5$ games without common players, then there are $20 - 2t$ players not in these t games. Moreover, since t is the most we can find, there cannot be any edges between the vertices representing these $20 - 2t$ players. But these players must each play at least one game, so each of them would have played at least one game with a player in the t games we have picked. Then we must have:

$$\# \text{ of edges} \geq t + (20 - 2t)$$

$$\# \text{ of edges} = 20 - t$$

Where t is the games we picked and $20 - 2t$ is the smallest number of games we didn't pick.
Since $t \leq 5$, # of edges $\geq 20 - t \geq 15$.
However, only 14 games were played, which is a contradiction. Therefore, our assumption of $t \leq 5$ is false, and we can pick 6 games with 12 distinct players.

Brain Teaser 19 How many perfect matchings are there in K_9?

0. Only graphs with an even number of vertices can have a perfect matching.

Brain Teaser 20 How many perfect matchings are there in K_6?

Starting from a vertex, there are 5 choices of a vertex it can be matched to. Once we pick one of these 5 choices, we proceed by trying to match one of the unmatched vertex to another, and there would be 3 choices. Once we pick this second edge, the remaining two vertices must be matches. So the total number of choices is:

$$3 \times 5 = 15$$

Brain Teaser 21 Prove that if we have a bipartite graph where each vertex has degree k, where $k \geq 2$ is a positive integer, then this graph has a perfect matching.

We use Hall's Marriage Theorem to prove this. For any $S \subseteq A$, there are $k|S|$ edges coming out of it, and each one of these edges has the other end in $N(S)$.
For every vertex in $N(S)$, it must have at most k neighbours in S. So, to be able to take all $k|S|$ edges from S, $|N(S)|$ must be at least $|S|$. Thus Hall's Marriage Theorem applies and A has a perfect matching to B.
Moreover, there are $k|A|$ edges coming out of A and $k|B|$ edges coming out of B. Since this is a bipartite graph, each edge has exactly one end in A and one in B, so $k|A| = k|B|$, or $|A| = |B|$. Since A has a perfect matching to B, we can say this graph has a perfect matching, where every vertex in A and every vertex in B is matched.