Problem Set 1: Integers Modulo n

1) Determine which elements have multiplicative inverses in the following sets of integers modulo n:

(a) $\mathbb{Z}_{12}$

(b) $\mathbb{Z}_3$

(c) $\mathbb{Z}_4$

(d) $\mathbb{Z}_5$

(e) $\mathbb{Z}_6$

(f) $\mathbb{Z}_7$

(g) $\mathbb{Z}_2$

(h) If every non-zero element a of $\mathbb{Z}_n$ has a multiplicative inverse, we say that $\mathbb{Z}_n$ is a **field**. Which of the above sets were fields? See any patterns?

(i) How do the elements **without** inverses relate to the modulus? Have you noticed any patterns?

2) In $\mathbb{Z}_{12}$, we already saw that $-6 \equiv 6 \pmod{12}$, which is a really strange property to see! The only number in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$ whose negative is equal to itself is 0.

(a) In which of the sets listed in Question **1)** can you find a non-zero number x such that $-x \equiv x \pmod{n}$?

(b) Can you determine the necessary condition for $-x \equiv x \pmod{n}$ to be possible? Prove your condition works!

3) For our standard calendar, there are 7 days in a week, 365 days in a year, and 366 days in a leap year.

(a) Determine the values of 365 and 366 modulo 7.

(b) Today is Wednesday, October 30$^{\text{th}}$, 2019. What day of the week will October 30$^{\text{th}}$ be in

(i) 2020?

(ii) 2021?

(iii) 2024?

(iv) 2030?

(c) What is the first year in the future that will have the exact same calendar as 2019? By this, I mean that every day of the year falls on the same day of the week as it does in 2019, and has the same number of days.

4) We have yet to consider square roots in $\mathbb{Z}_n$ (and will do so in great detail later!) For now, for the following sets, determine for which values a the equation $x^2 \equiv a \pmod{n}$ has a solution, and give all possible solutions when one exists!

(a) $\mathbb{Z}_5$

(b) $\mathbb{Z}_6$

(c) $\mathbb{Z}_8$

(d) $\mathbb{Z}_{10}$

(e) When a solution exists, how many solutions do you get? Why does this happen?