What is Graph Theory?

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1 Let's Cross Some Bridges

Brain Teaser 1 The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands - Kneiphof and Lomse - which were connected to each other, or to the two mainland portions of the city, by a total of seven bridges.

According to lore, the citizens of Königsberg used to spend Sunday afternoons walking around their beautiful city. While walking, the people of the city decided to create a game for themselves, their goal being to devise a way in which they could walk around the city, crossing each of the seven bridges only once. Can you find a way to achieve this goal?

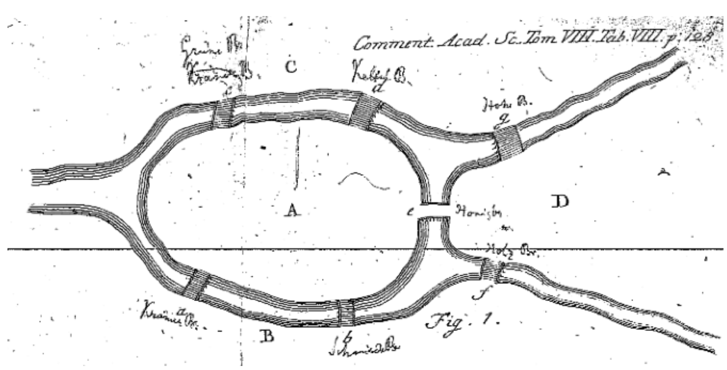

Figure 1: Euler's Figure of the seven bridges of Königsberg

"This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it. - Leonard Euler"

2 Definitions and Notations

1. A **graph** is a pair of sets $G = (V, E)$ where V is a set of vertices and E is a set of edges whose endpoints are in V.

2. Two vertices are **adjacent** if there is an edge with them as its endpoints.

3. A vertex and an edge are **incident** to each other if the vertex is an endpoint of the edge.

4. An edge whose ends points are the same is a **loop**. Multiple edges connecting a pair of vertices are called **parallel edges**. A graph without loops and parallel edges is a **simple graph**. Unless stated otherwise, we assume all graphs are simple.

5. A graph G is **bipartite** if the vertices of G can be partitioned into two sets A and B, such that every edge of G connects a vertex in A with a vertex in B.

6. The **degree** $deg(v)$ of a vertex v is the number of edges it is incident to.

7. A **path** in a graph G is a finite sequence of distinct vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}. The **length** of a path is the number of edges in it.

8. A **cycle** in a graph G is a finite sequence of vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}, $v_0, v_1, \cdots, v_{t-1}$ are distinct, and $v_0 = v_t$.

9. A **walk** in a graph G is a finite sequence of vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}.

10. A graph is **connected** if there is a path between every pair of distinct vertices.

11. A **tree** is a connected graph with no cycles.

12. The **complete graph** over n vertices, or K_n is the graph with n vertices such that there is an edge between every pair of vertices.

13. The **complete bipartite graph** is a bipartite graph with all vertices in A adjacent to all vertices in B.

Brain Teaser 2 Suppose there are 10 cities in a country, each of them has exactly four roads connecting it to four distinct cities in the same country. Each road connects exactly two cities. How many roads are there in total?

Brain Teaser 3 Suppose there are 10 cities in a country, five of them have exactly four roads connecting them to four distinct cities in the same country, and the other five of them have five roads connecting them to five distinct cities in the same country. In this case, how many roads are there in total?

Brain Teaser 4 Can you find a graph over 7 vertices, with the degree of the vertices being $0, 1, 2, 3, 4, 5, 6$?

Brain Teaser 5 For some positive integer $k > 2$, suppose we have a graph G where every vertex has degree at least k. Show that there is a cycle in G with even length.

Brain Teaser 6 Suppose we have a graph G with n vertices, such that for any two vertices u and v that are not adjacent to each other, we have $deg(u) + deg(v) \geq n$. Can you show that this graph is connected?

3 Eularian Circuits and Eulerian Walks

An **Eulerian circuit** of a graph G is a walk that starts and ends at the same vertex and goes through each edge of G exactly once.

An **Eulerian walk** of a graph G is a walk that goes through each edge of G exactly once.

Theorem. *Let G be a connected graph, then*

(a) G contains an Eularian circuit if and only if the degree of every vertex is even; and

(b) G contains an Eulerian walk if and only if it has at most 2 vertices with odd degree.