Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

February 4/5/6 2020

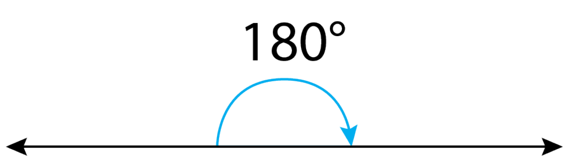
Circumference of the Earth

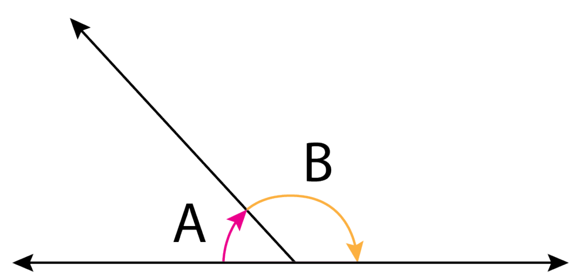
Introduction

Eratosthenes was a Greek mathematician and geographer born in 276 BC. He is most well known for being the first person to measure the circumference of the Earth. He had no planes, no satellites, and no means of travelling around the world. How did he do it?

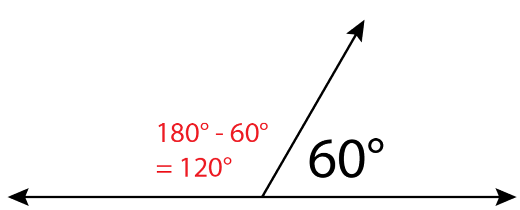
Angles

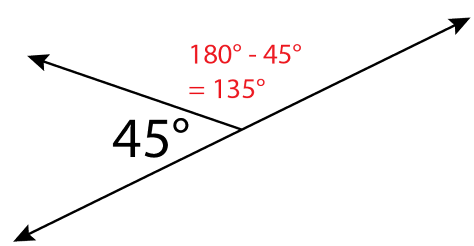
Supplementary angles are angles which add to 180°. In the diagram below, angles A and B are supplementary, since A + B = 180°.



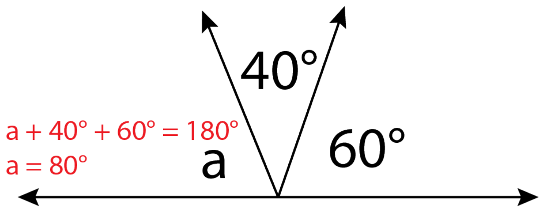


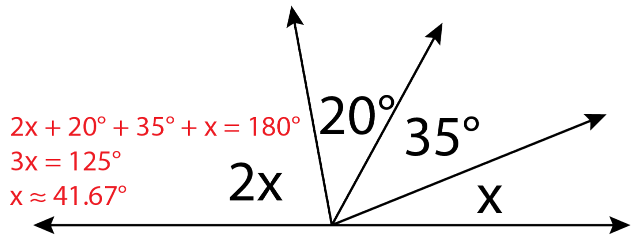
Example 1. Fill in the missing angles in the diagrams below:



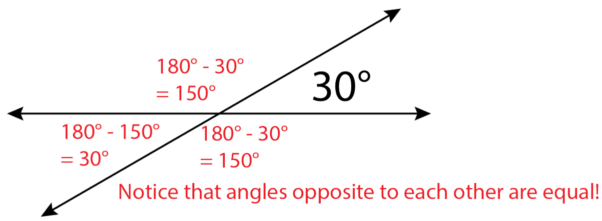


Example 2. Find the value of angles a and x in the diagrams below:



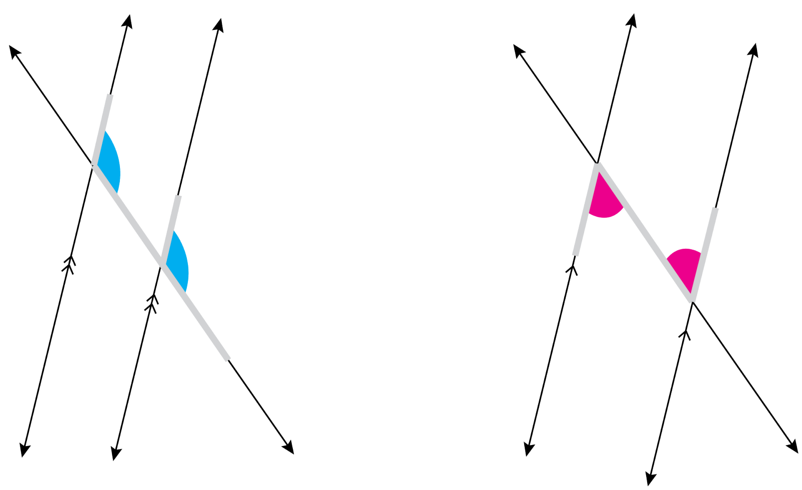


Example 3. Fill in the missing angles. What do you notice?



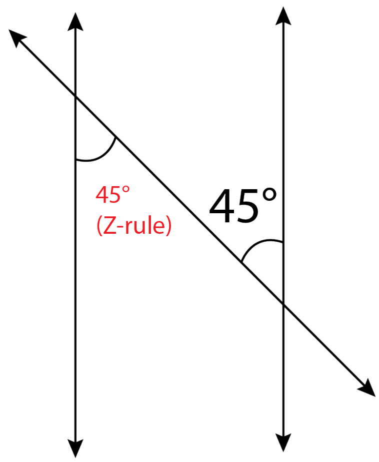
This pattern is called the **vertically opposite angles rule**.

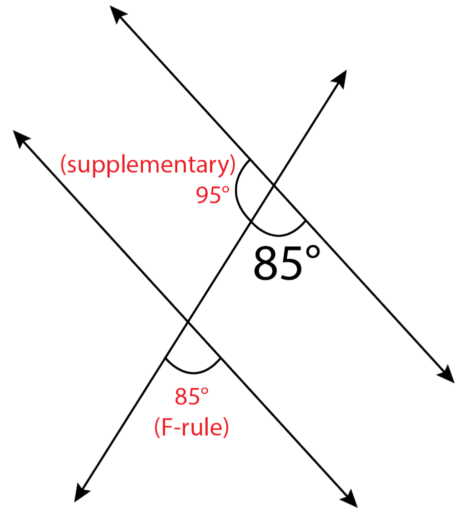
Parallel lines are lines which will never intersect. The diagram below shows two parallel lines which are intersected by a third line, called a **transversal**. The coloured angles are equal to each other.



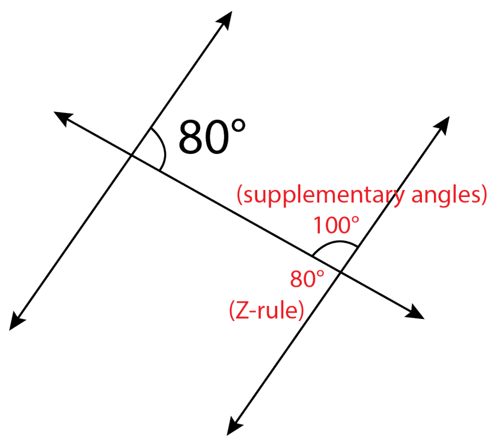
the "F" rule the "Z" rule

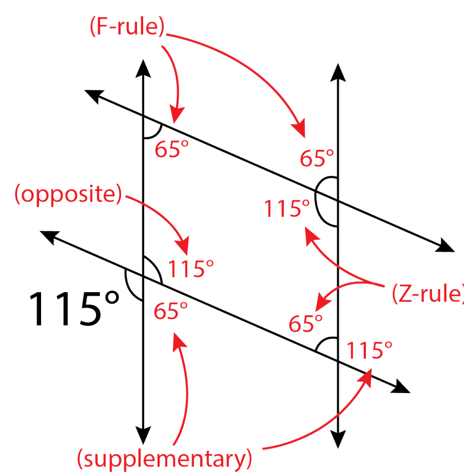
Example 4. Find the value of angles x, y, and z in the diagrams below:



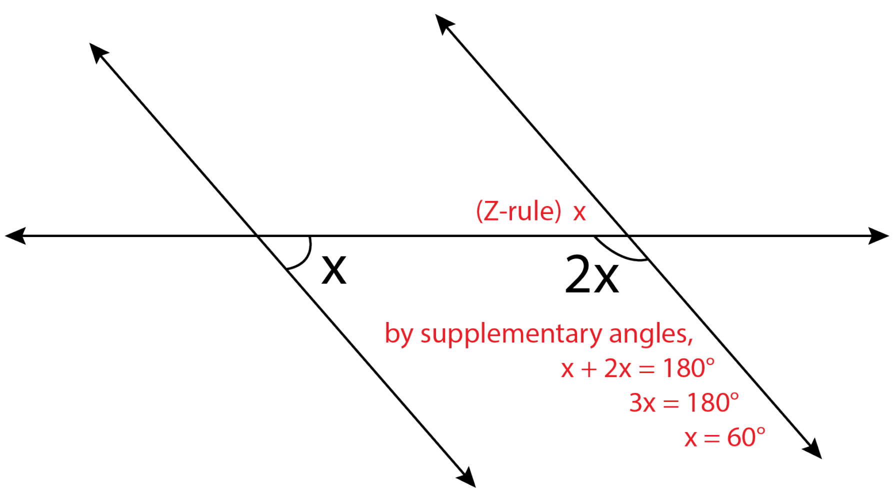


Example 5. Find the value of angles a,b,c,d,e, and w in the diagrams below:

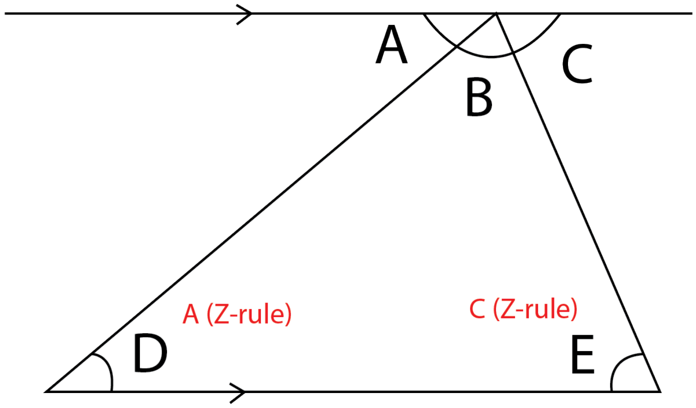




Example 6. Find the value of x in the diagram below:



Example 7. Find the value of angles D and E in the diagram below:



What is D + B + E = ?

D + B + E = A + B + C = 180°

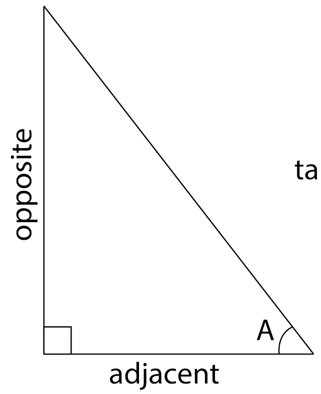
Wow! You just proved that the sum of the interior angles of a triangle equals: 180°

Example 8. Think of what the sum of the interior angles of a quadrilateral might be. Try to come up with a proof to support your hypothesis:

Any quadrilateral can be split into two triangles by drawing a diagonal between two opposite vertices. The interior angles of these two triangles align with the interior angles of the quadrilateral, so the sum of the interior angles of the quadrilateral = 2 × sum of the interior angles of a triangle = 2 × 180° = 360°.

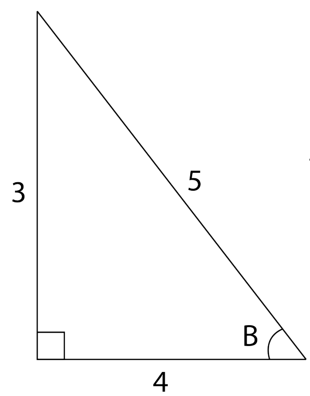
Trigonometry

Trigonometry is the study of the angles and sides of triangles. Let's look at one trigonometric function called the tangent (tan) function. It is used for right-angled triangles.



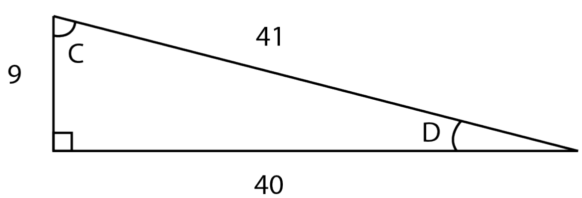
$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$

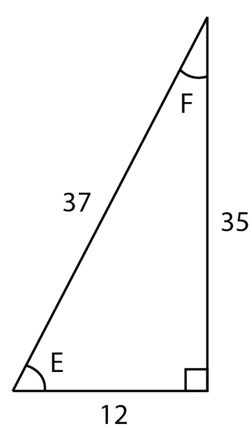
Example 9. Fill in the ratio for the triangle below:



$$\tan(B) = \frac{3}{4}$$

Example 10. Use the diagrams to find the following values of tan:





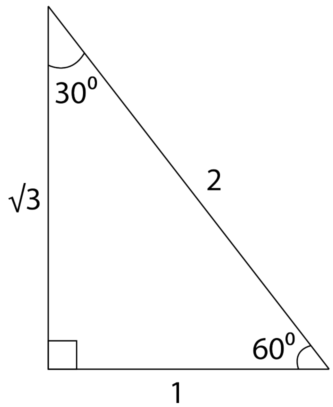
$\tan(E) = 35/12$

$\tan(F) = 12/35$

$\tan(C) = 40/9$

$\tan(D) = 9/40$

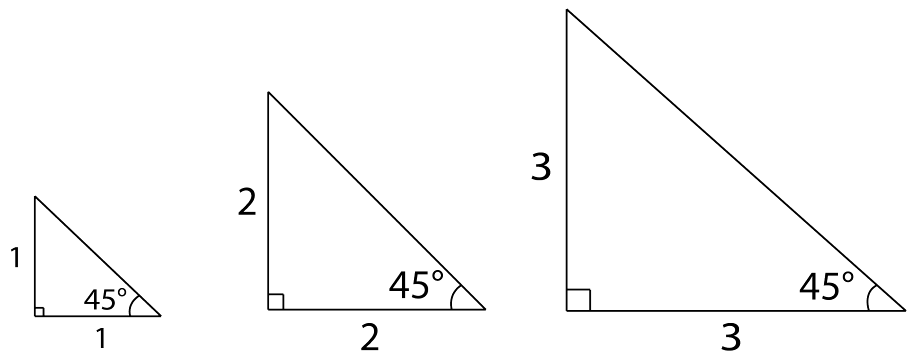
Example 11. Fill in the ratios:



$\tan(60^0) = \sqrt{3}$

$\tan(30^0) = \frac{1}{\sqrt{3}}$

Have a look at these three triangles. What do you notice?



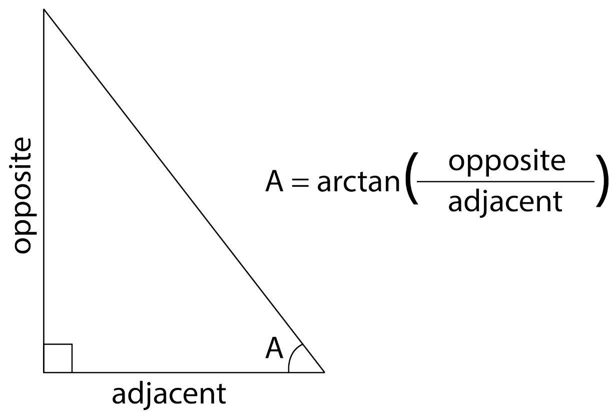
$$\tan(45°) = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = 1$$

So the value of tan(45°) is always 1, regardless of which triangle you use to calculate it with! This is true for the tan function in general. In the past, mathematicians would create large tables to record the value of tan for each angle. Thankfully, nowadays we can just use a calculator for this calculation.

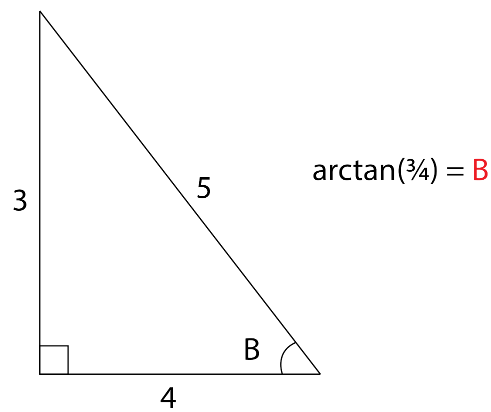
Example 12. Use a calculator to evaluate the following. Round your answer to two decimal places:

1. $\tan(50°) = 1.19$
2. $\tan(85°) = 11.4$
3. $\tan(20°) = 0.36$

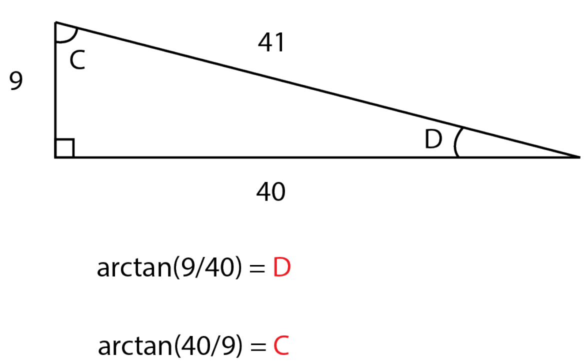
What if we want to do this backwards? What if we know the ratio, but we want to find the angle? To do this, we can use the opposite of the tan function, which we call **arctan**.

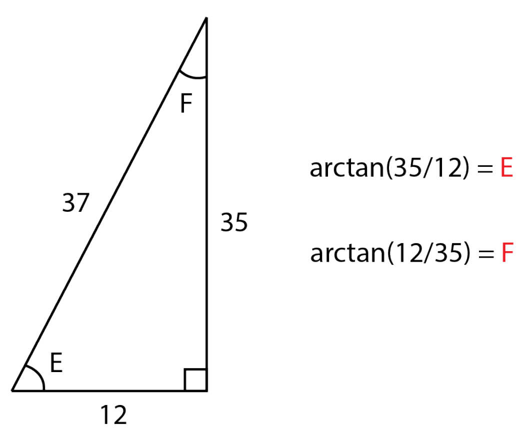


Example 13. Use the diagram to calculate the value of arctan:

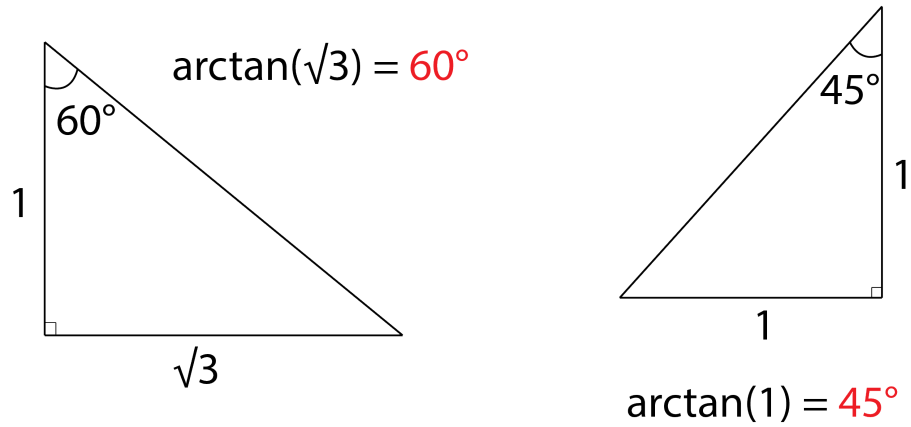


Example 14. Use the diagrams to calculate the following values of arctan:





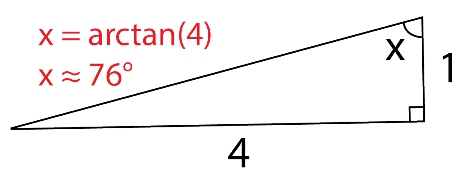
Example 15. Use the diagrams to calculate the following values of arctan:



Example 16. Use a calculator to calculate the following values of arctan to the nearest degree:

1. $\arctan(1) = 45°$
2. $\arctan(0.5) = 27°$
3. $\arctan(4) = 76°$

Example 17. What is the value of angle x in the triangle below? (You will need to use your calculator)

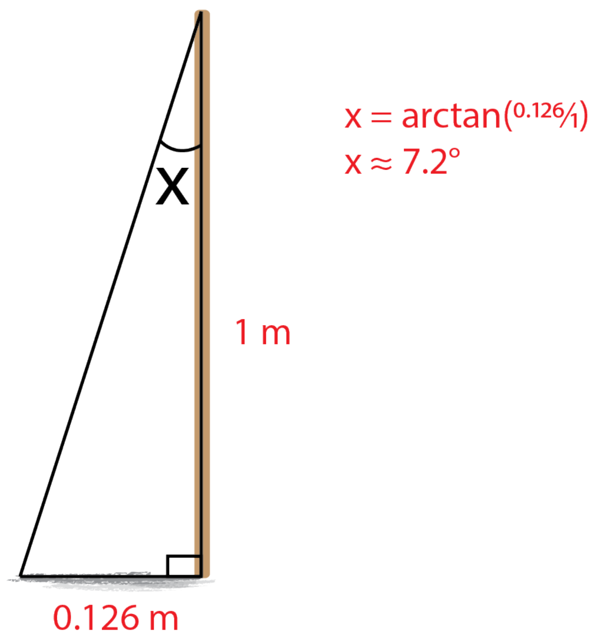


We now have all the tools we need to calculate the circumference of the Earth! Let's return to Eratosthenes' story.

Sticks and Shadows

Eratosthenes heard of a well in the city of Syene where, on the summer solstice, the sunlight would illuminate the bottom of the well, implying that the Sun was directly overhead. In his home city of Alexandria, Eratosthenes planted a stick into the ground on the solstice and observed that it cast a shadow.

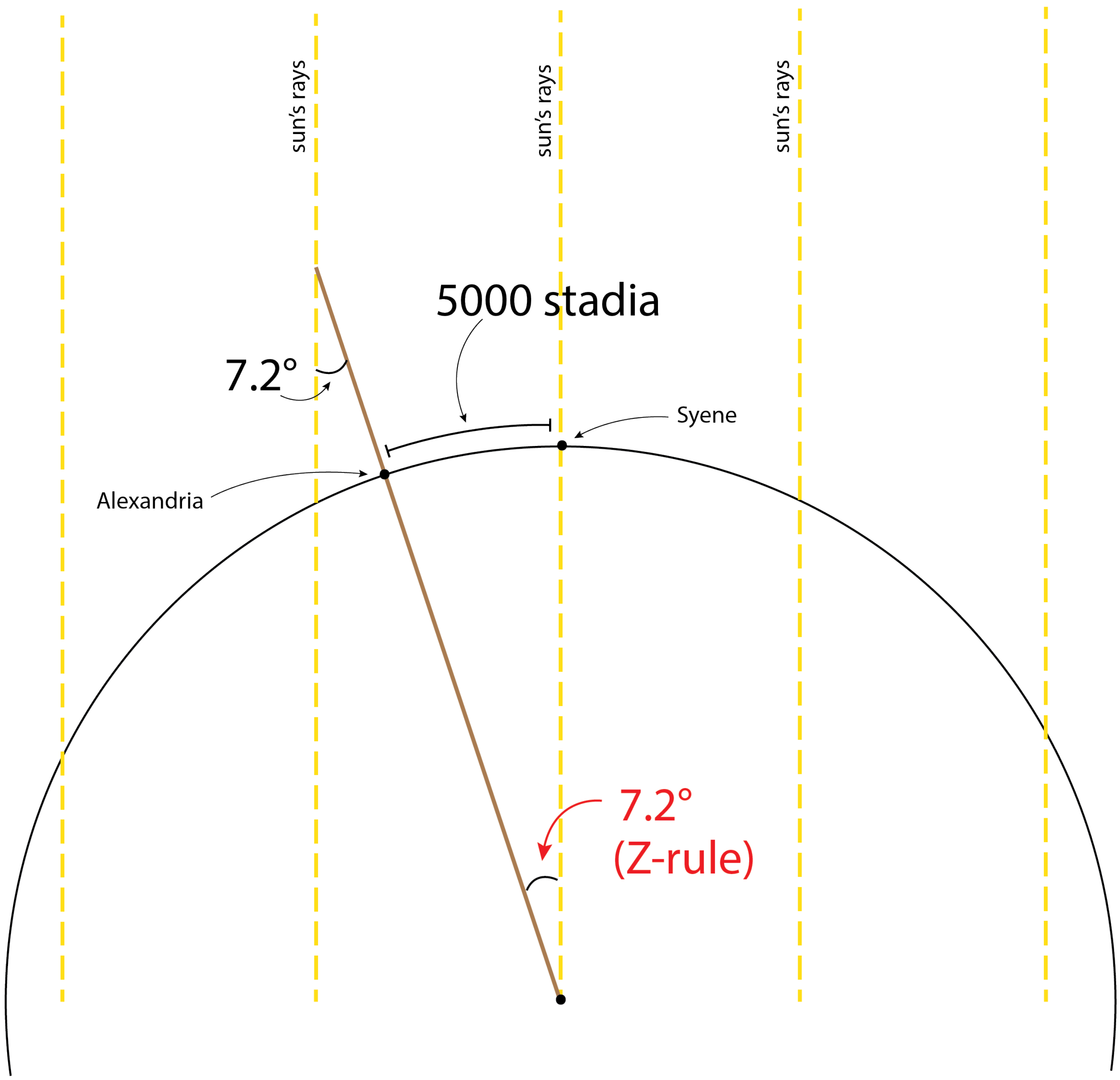
Let's assume that Eratosthenes's stick was 1 meter long, and the shadow he measured was 12.6 cm. What is angle x? *(watch out for units!)*



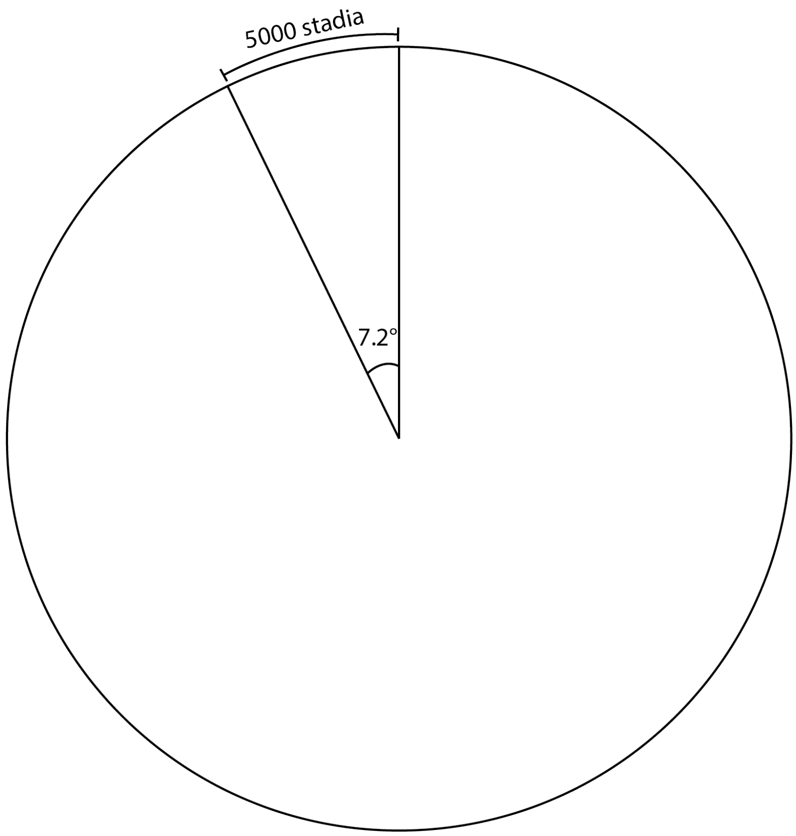
The ancient Greeks used a measurement called *stadia*, which was based on the circumference of a sports stadium. The length of a stadium varied between cities, but let's suppose that he used the Egyptian *stadia*, which is about 157.5 meters long. Eratosthenes determined that the distance between Alexandria and Syene was 5000 *stadia*.

When we deal with a calculation of this size, we need to make some assumptions to simplify our work. Eratosthenes assumed that the Sun was so far away that its rays were parallel by the time they reach Earth.

Let's summarize all of the information we have in a diagram (not to scale). Can you fill in the missing angle?



So now we have the following diagram, which shows a single "slice" of the Earth and its measurements:



How many of these "slices" would we need to form the entire Earth?

$$7.2° \times 50 \text{ slices} = 360°$$

Therefore the circumference of the earth in *stadia* is:

$$5000\ stadia \times 50 \text{ slices} = 250\ 000 \text{ stadia}$$

Converting *stadia* into kilometers, we get:

$$250\ 000\ stadia \times 157.5 \text{ m}/stadia \times 0.001 \text{ km/m} = 39\ 375 \text{ km}$$

The actual circumference of the Earth is 40 075 km, so our measurement is only off by 1.7%.

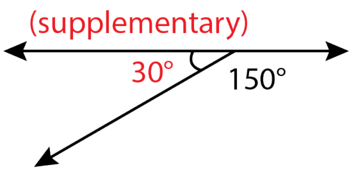
With a simple observation and a little bit of geometry, we have calculated the circumference of the Earth without leaving our desks!

Problem Set

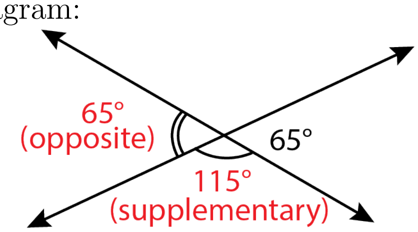
Problems marked with an asterisk (*) are challenge questions.

1. Calculate the indicated angle(s) in each diagram:

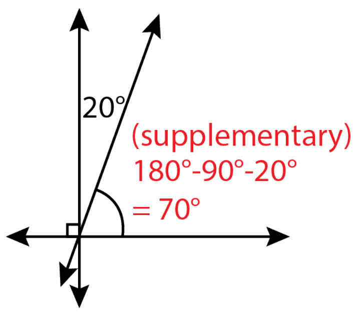
a)



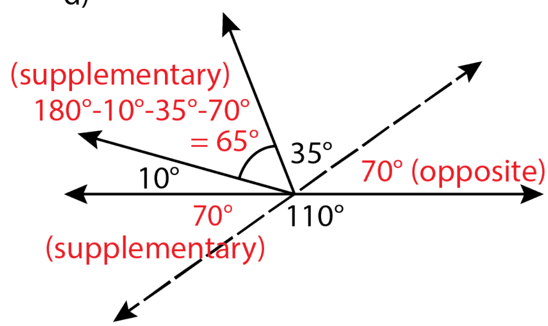
b)



c)

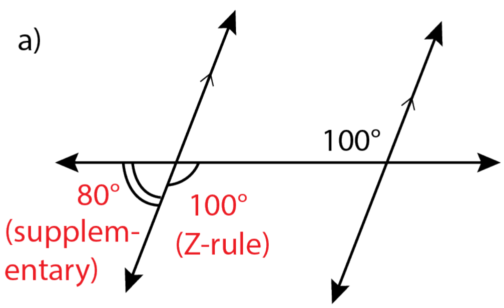


d)

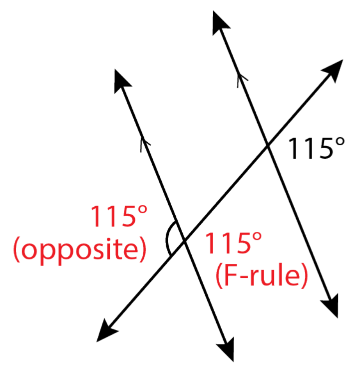


2. Calculate the indicated angle(s) in each diagram:

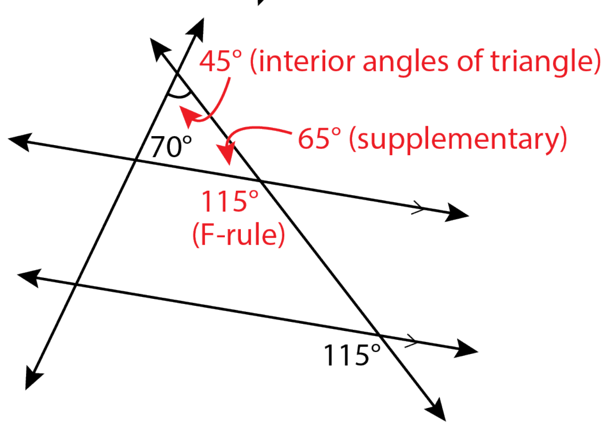
a)



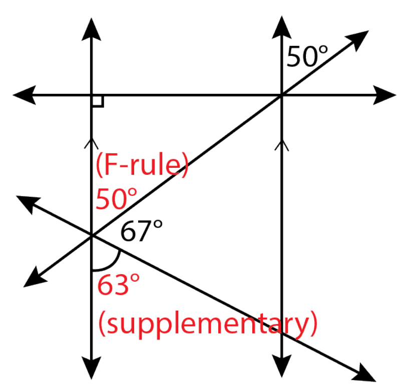
b)



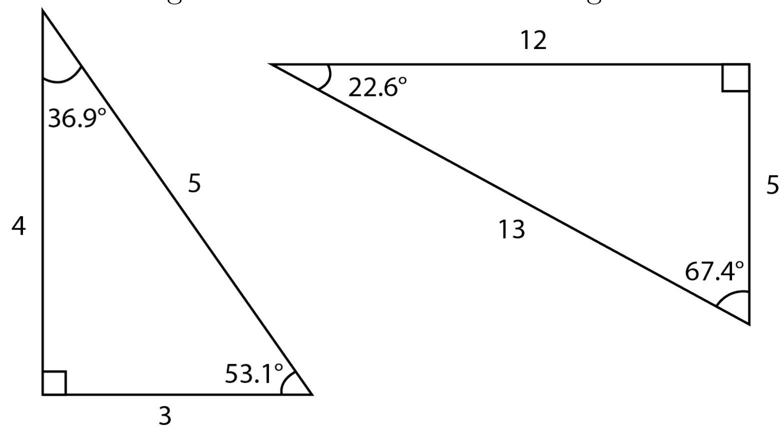
c)



d)



3. Use the diagrams to calculate the following values of tan:



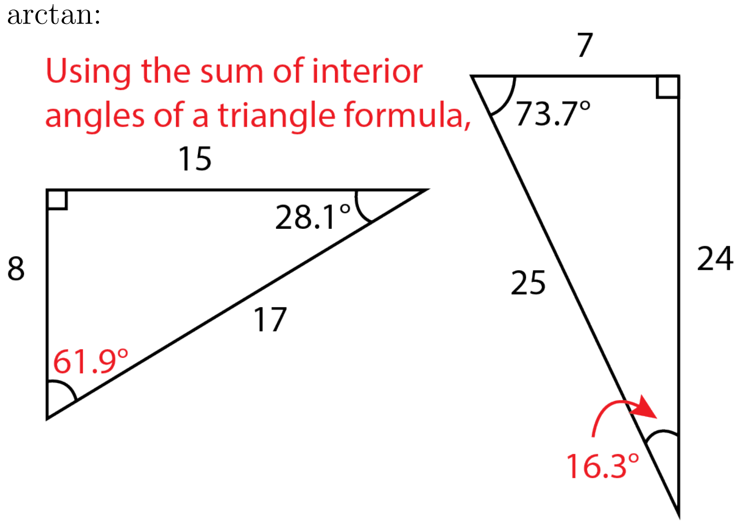
(a) $\tan(53.1°) = \frac{4}{3}$

(b) $\tan(36.9°) = \frac{3}{4}$

(c) $\tan(22.6°) = \frac{5}{12}$

(d) $\tan(67.4°) = \frac{12}{5}$

4. Fill in the missing angles, then use the diagrams to calculate the following values of arctan:



(a) $\arctan(24/7) = 73.7°$

(b) $\arctan(15/8) = 61.9°$

(c) $\arctan(8/15) = 28.1°$

(d) $\arctan(14/48) = \arctan(7/24) = 16.3°$

5. Use your calculator to evaluate the following. Round to two decimal places:

   (a) $\tan(23°) = 0.42$

   (b) $\tan(49°) = 1.15$

   (c) $\tan(189°) = 0.16$

   (d) $\tan(180°) = 0$

6. Use your calculator to evaluate the following. Round to two decimal places:

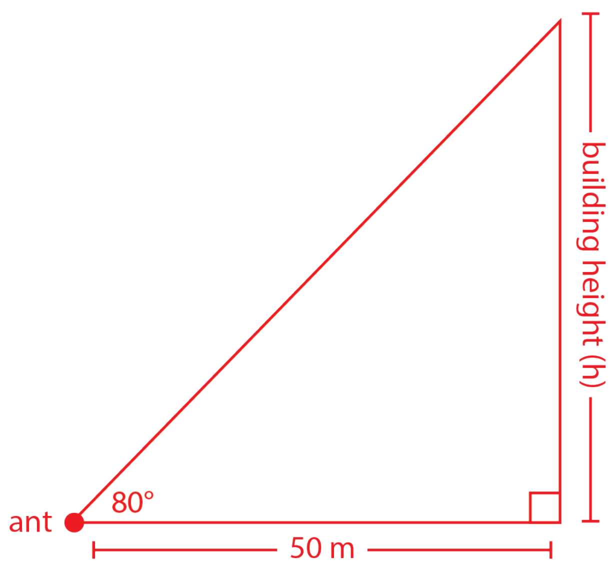
   (a) $\arctan(0.1) = 5.71°$

   (b) $\arctan(0.7) = 35.00°$

   (c) $\arctan(1.2) = 50.20°$

   (d) $\arctan(8) = 82.87°$

7. Suppose an ant has to look up at an angle of 80° to see the top of a building when it is 50 metres away from the bottom of the building. How tall is the building?



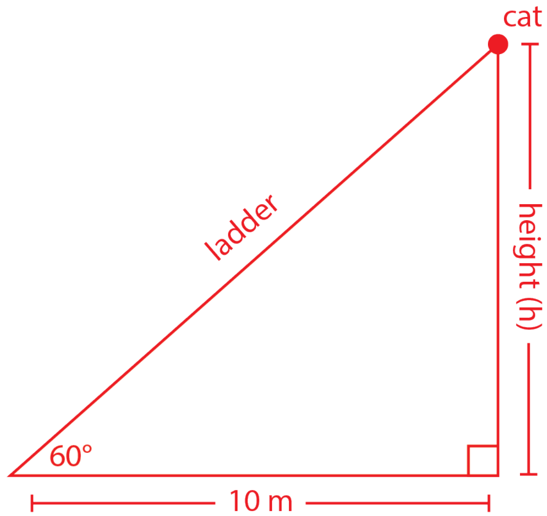
$$\tan(80°) = \frac{h}{50}$$

$$50 \times \tan(80°) = 50 \times \frac{h}{50}$$

$$50 \times \tan(80°) = h$$

$$h \approx 284 \text{ metres}$$

8. A firefighter is trying to rescue a cat from the top of a tree. To reach it, she positions her ladder 10 metres away from the tree, and tilts it up at an angle of 60° from the ground. How high up is the cat?



$$\tan(60°) = \frac{h}{10}$$

$$10 \times \tan(60°) = 10 \times \frac{h}{10}$$

$$10 \times \tan(60°) = h$$

$$h \approx 17 \text{ metres}$$

9. What is the radius of the Earth?

$$\text{Circumference} = 2\pi r$$

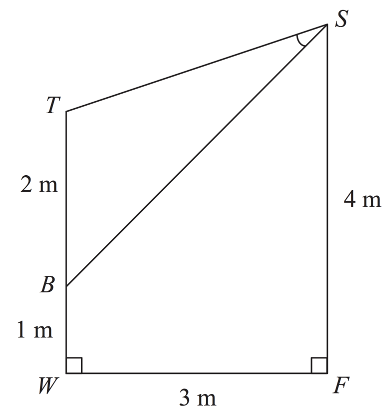
$$39375 = 2\pi r$$

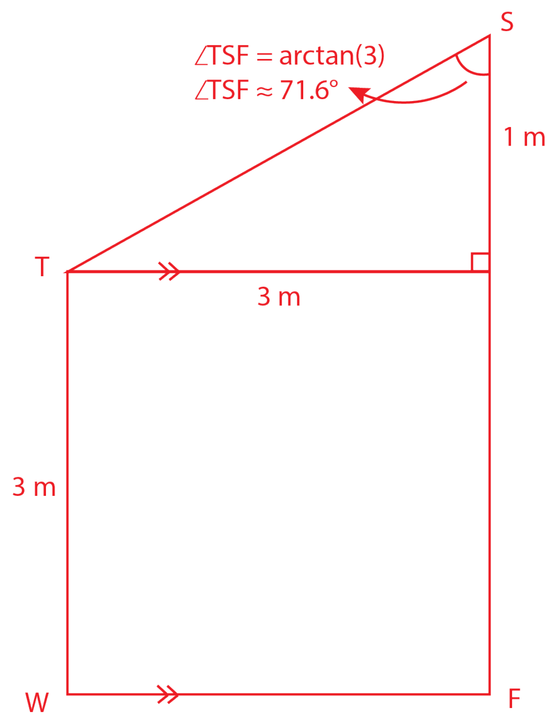
$$\frac{39375}{2\pi} = r$$

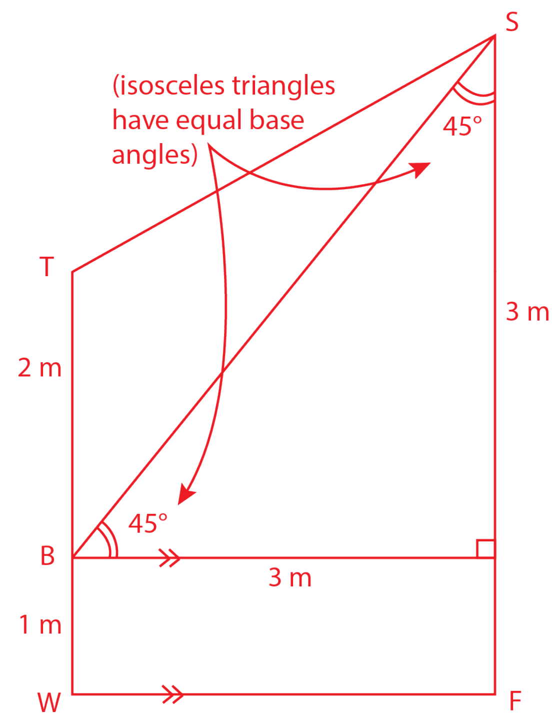
$$6266.7 \text{ km} \approx r$$

10. * In an art gallery, a 2 m high painting, BT, is mounted on a wall with its bottom edge 1 m above the floor. A spotlight is mounted at S, 3 m out from the wall and 4 m above the floor. Determine ∠TSB, accurate to 1 decimal place.

Taken from CEMC Problem of the Week (E), 2017-2018

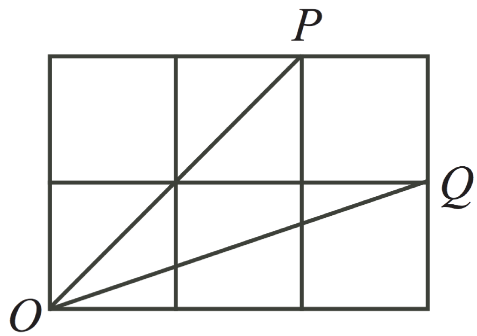




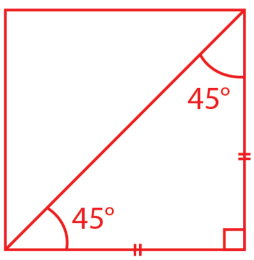


Therefore ∠TSB ≈ 71.6° - 45° ≈ **26.6°**

11. * Find, to one decimal place, the measure of $\angle POQ$ in the diagram below:

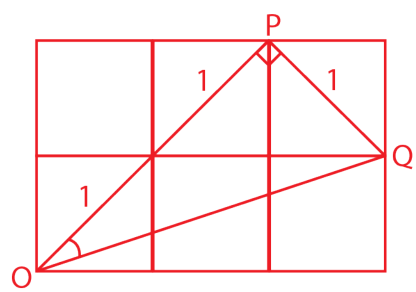


Note that the grid is composed of 6 identical squares. You do not need to know the side length of each square to complete this question! You will, however, need your calculator. Taken from 2008 Fermat Contest, #16



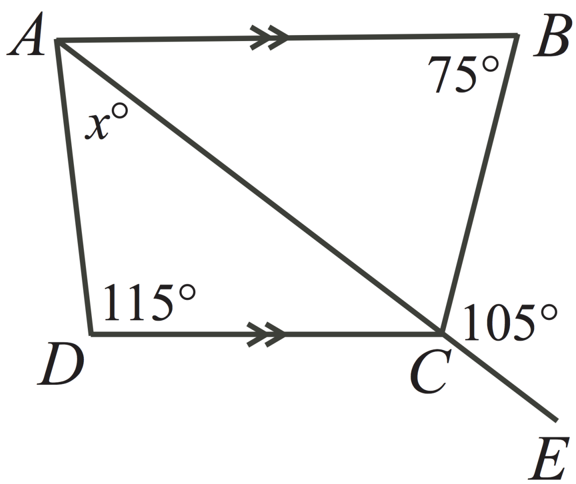
The diagonal of a square forms two isosceles right-angled triangles. Thus, the bottom two angles are equal, and each measures 45°.

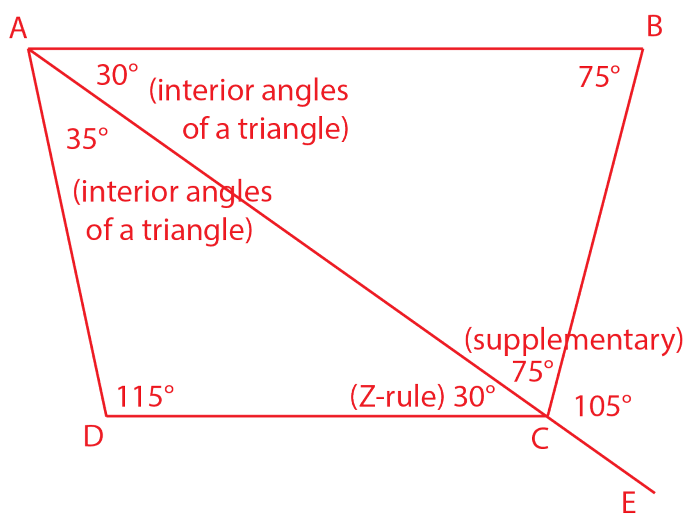
So $\angle OPQ = 45° + 45° = 90°$. Let the length of a diagonal equal 1 unit.



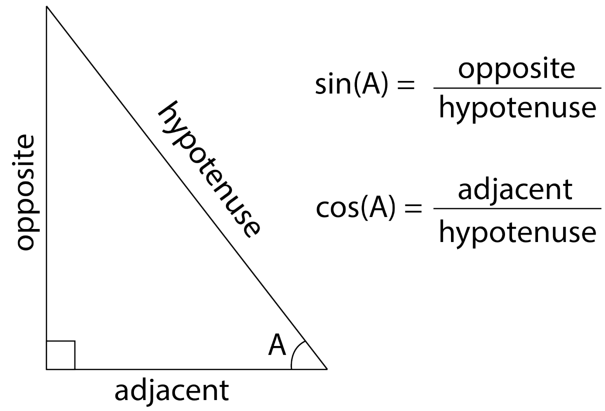
$\angle POQ = \arctan(\frac{1}{2})$
$\angle POQ \approx 26.6°$

12. * In the diagram, AB is parallel to DC and ACE is a straight line. What is the value of x? Taken from 2010 Gauss (Gr 8) Contest, #21





Extension: In today's lesson we learned about tan, which is the ratio of the length of the opposite side to the adjacent side. There are two more basic trigonometric ratios called **sine** (sin) and **cosine** (cos).



The remaining problems will involve sin and cos.

13. * Show that $\tan = \frac{\sin}{\cos}$

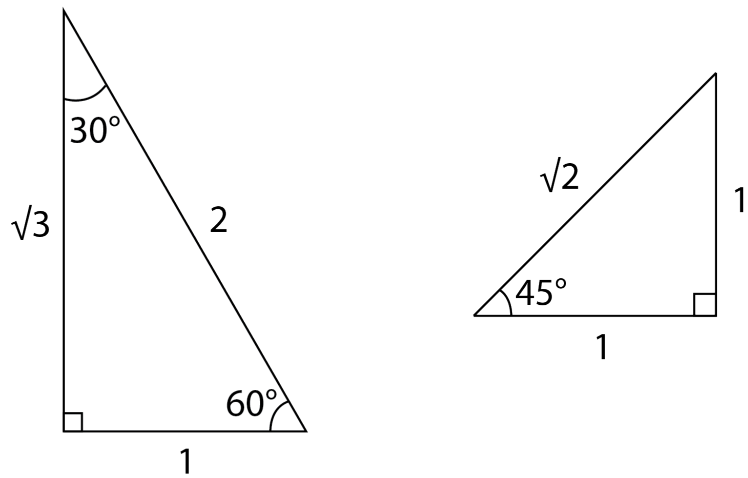
$$\begin{aligned}\frac{\sin}{\cos} &= \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}}\\ &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}}\\ &= \frac{\text{opposite}}{\text{adjacent}}\\ &= tan\end{aligned}$$

14. * If $\sin(30°) = \frac{1}{2}$ and $\cos(30°) = \frac{\sqrt{3}}{2}$, then what is $\tan(30°)$?

$$\begin{aligned}\tan(30°) &= \frac{\sin(30°)}{\cos(30°)}\\ &= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\\ &= \frac{1}{2} \times \frac{2}{\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

15. * Using the following diagrams, find the value of θ that completes this equation:

$$\cos 60^\circ = \cos 45^\circ \times \cos\theta$$



Taken from 2015 Fermat Contest, #12

$$\cos 60^\circ = \cos 45^\circ \times \cos\theta$$
$$\frac{1}{2} = \frac{1}{\sqrt{2}} \times \cos\theta$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = 45^\circ$$