Grade 7/8 Math Circles
February 11/12/13 2020
Fractals

Introduction

Fractals are mathematical objects that are **self-similar**, meaning that they have the same structure when you zoom in on them. For example, this is a famous fractal called the Mandelbrot set:
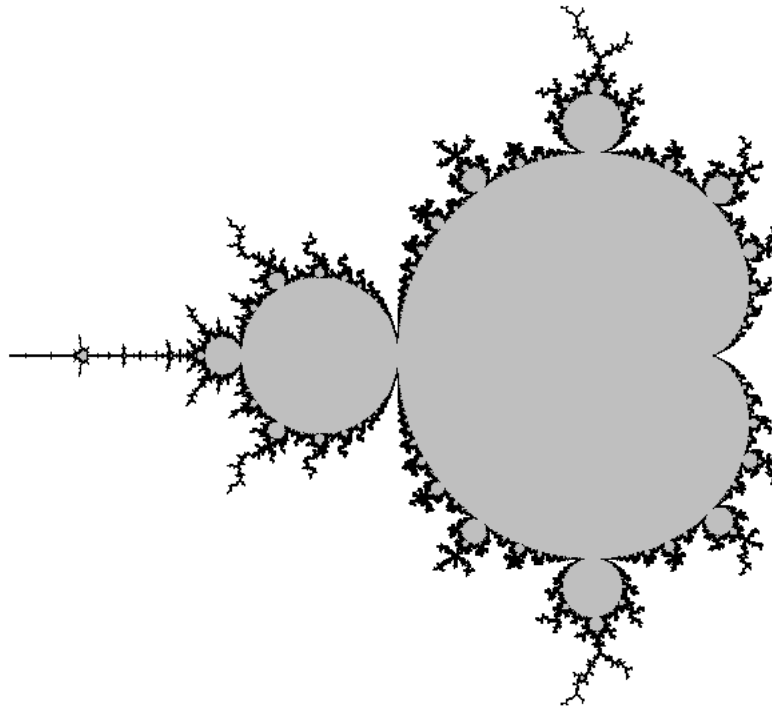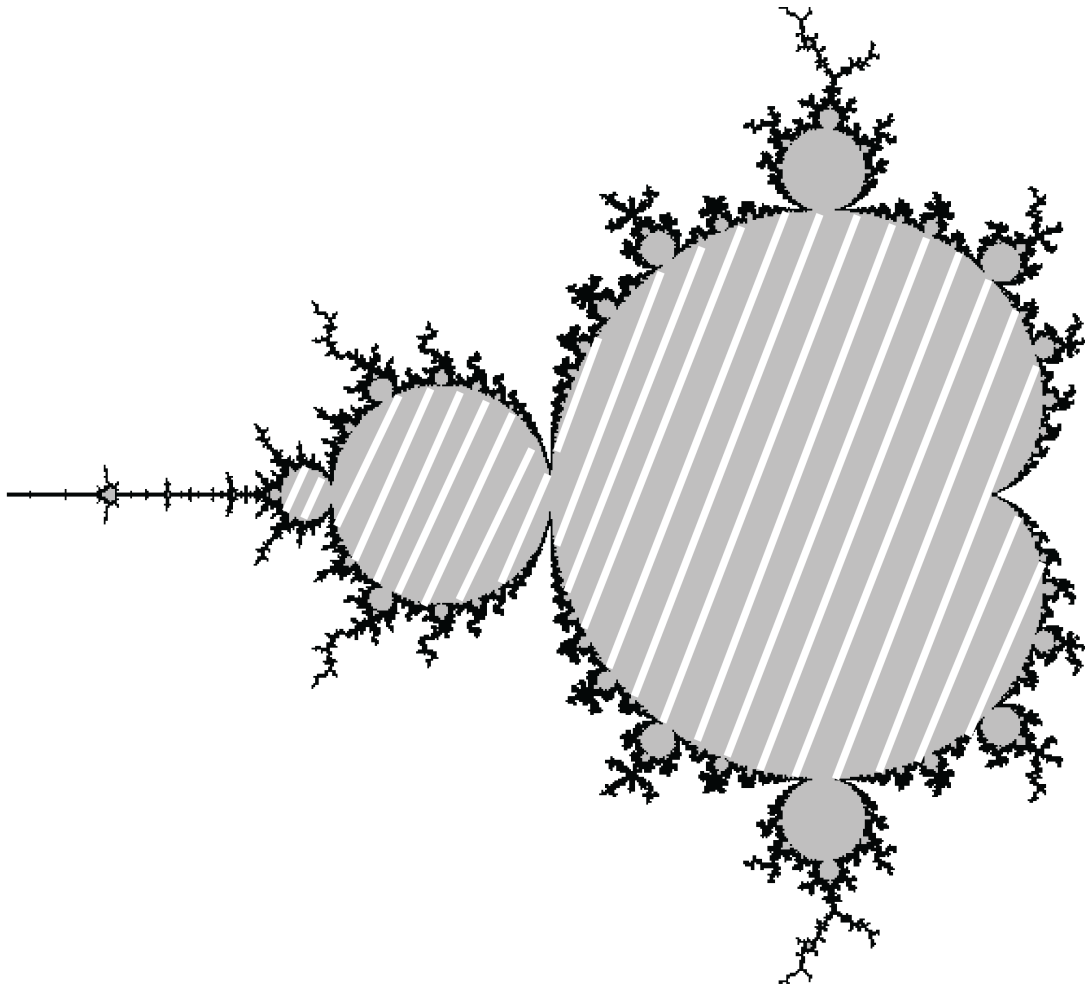

Image by Arnaud Cheritat, retrieved from:
https://www.math.univ-toulouse.fr/~cheritat/wiki-draw/index.php/File:MilMand_1000iter.png

The main shape, which looks a bit like a snowman, is shaded in the image below:

As an exercise, shade in some of the smaller snowmen that are all around the largest one in the image above. This is an example of self-similarity, and this pattern continues into infinity. Watch what happens when you zoom deeper and deeper into the Mandelbrot Set here: `https://www.youtube.com/watch?v=pCpLWbHVNhk`.

Drawing Fractals

Fractals like the Mandelbrot Set are generated using formulas and computers, but there are a number of simple fractals that we can draw ourselves.
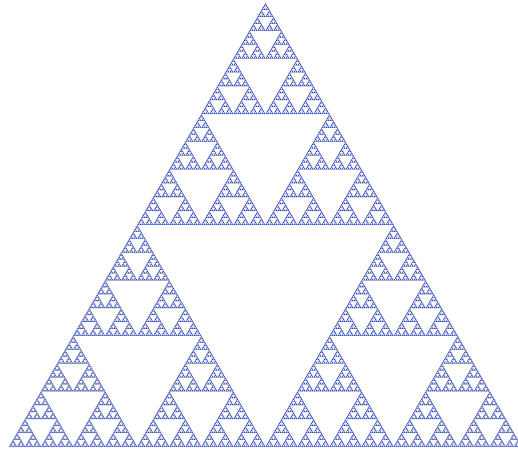
Sierpinski Triangle

Image by Beojan Stanislaus, retrieved from:
https://commons.wikimedia.org/wiki/File:Sierpinski_triangle.svg

Use the template and the steps below to draw this fractal:

1. Start with an equilateral triangle.

2. Connect the midpoints of all three sides. This will create one upside down triangle and three right-side up triangles.

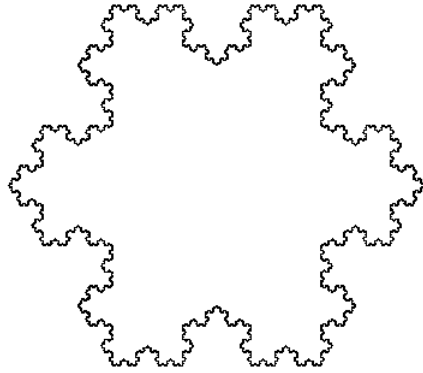3. Repeat from step 1 for each of the three right-side up triangles.
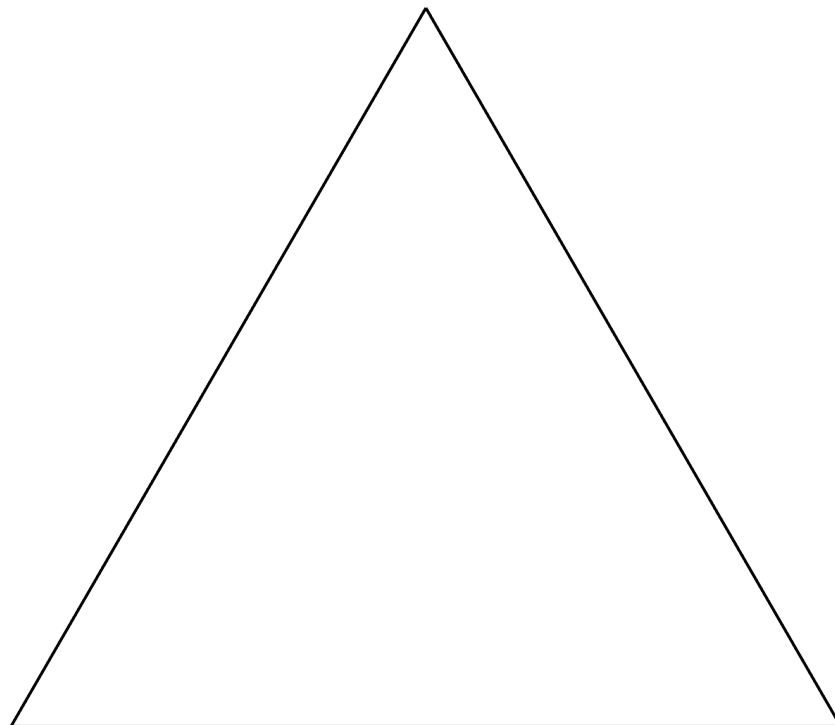
Apollonian Gasket

Retrieved from: `https://mathlesstraveled.com/2016/04/20/post-without-words-6/`
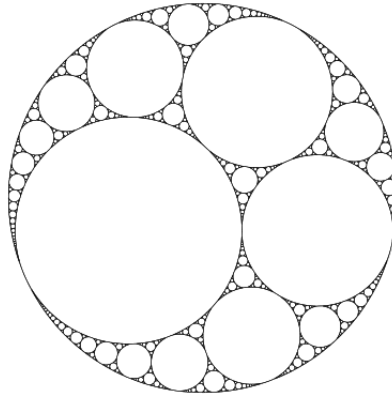
Use the template and the steps below to draw this fractal:

1. Start with a circle.

2. Draw three circles inside of the large circle that all touch each other once.

3. In each of the four gaps that are left around the circles, draw the largest possible circle that can fit.

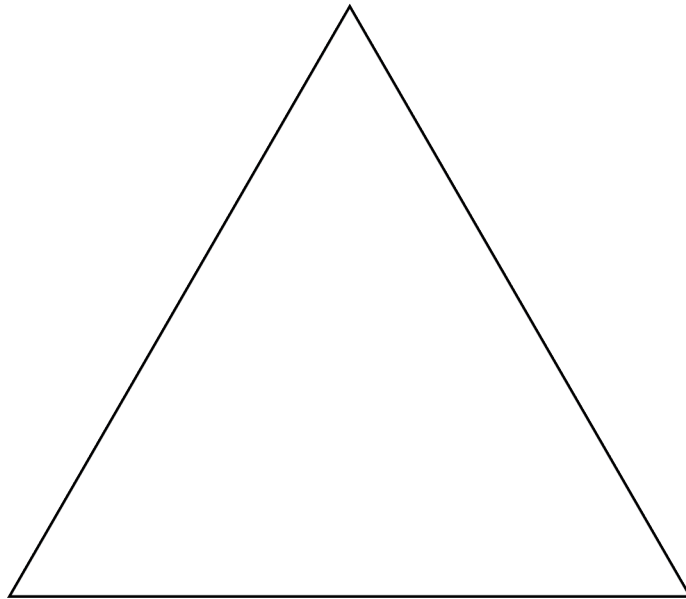4. Repeat from step 3 for each of the new gaps that are created.
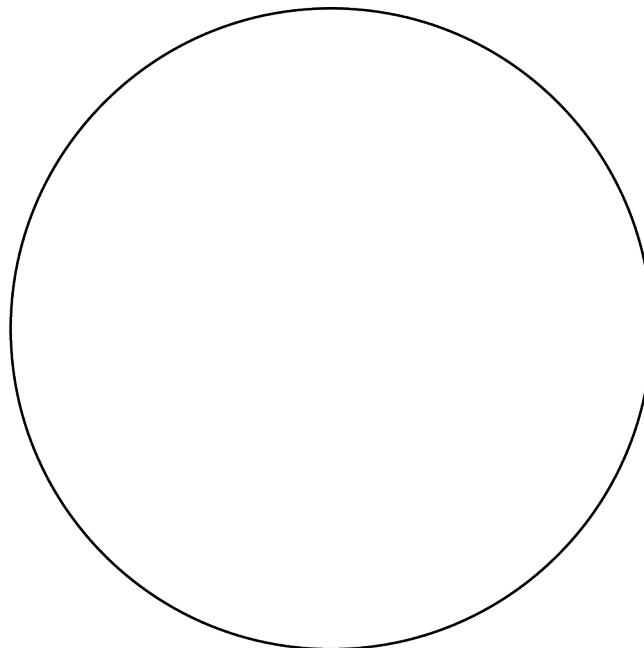
Koch Snowflake

Use the template and the steps below to draw this fractal:

1. Start with an equilateral triangle

2. Divide each edge into three equal pieces. In the middle piece, draw another equilateral triangle.

3. Repeat from step 2 for each of the sides of the new shape.

The Coastline Paradox

The Koch Snowflake has a very interesting property. Have a look at each of the iterations shown in the diagram, and answer the following questions:
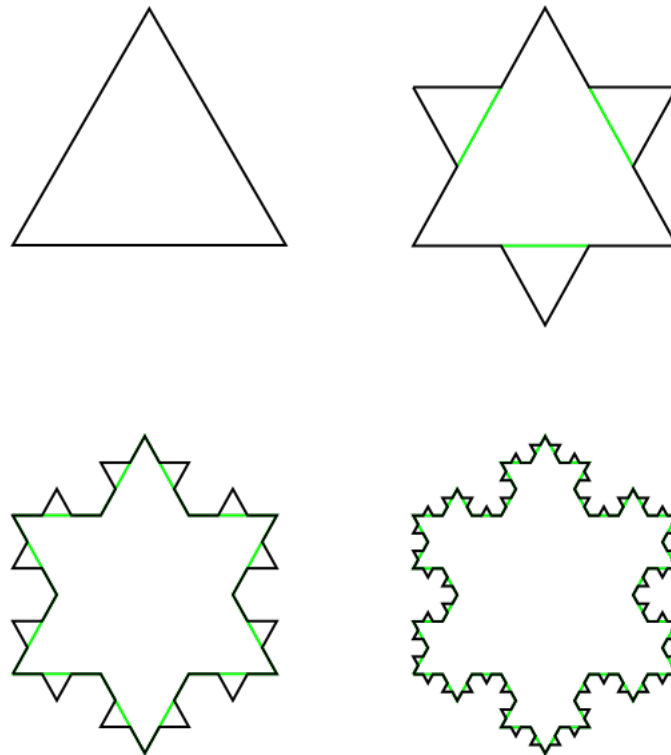
1. What is happening to the perimeter during each iteration?

2. What is happening to the area during each iteration?

The perimeter of the Koch snowflake keeps getting larger and larger, but the area is **bounded**. This means that the area stays within some finite boundary. To see this, we can simply draw a big square around the snowflake. No matter how many iterations of the snowflake we create, the area will always be contained within this square:
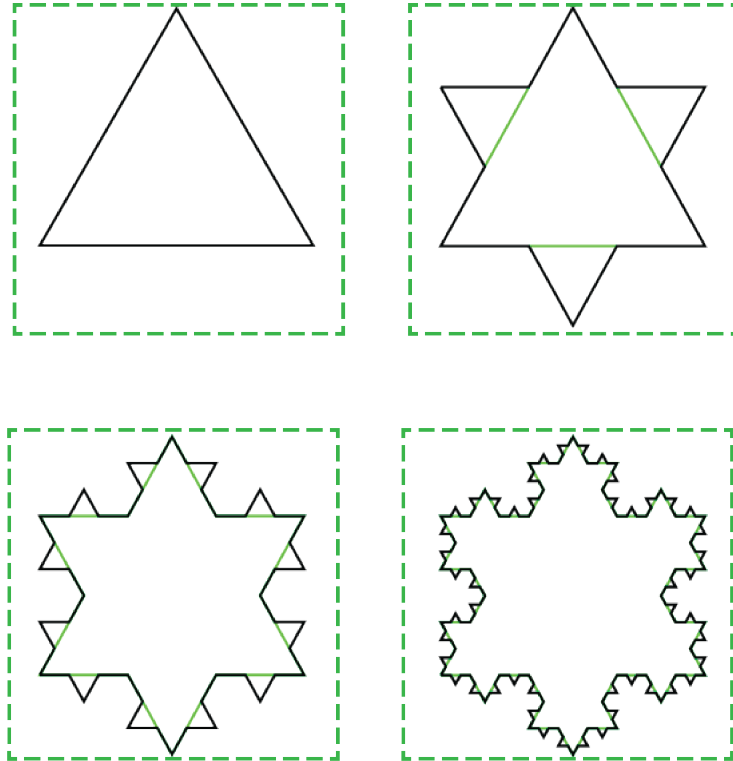
We can say that the Koch snowflake has **infinite perimeter** and **finite area**.

This concept is not limited to abstract mathematical objects like fractals. In fact, this problem appears very clearly when we try to measure the length of a coastline.

On the next page is a map of Great Britain (Image by Bruce Jones, retrieved from: `www.freeusandworldmaps.com/html/Countries/Europe%20Countries/UnitedKingdomPrint.html`). Let's try to measure the length of its coastline using various measuring sticks. For map 1, use lines that are 4 cm long. For map 2, use lines that are 2 cm long. For map 3, use lines that are 0.5 cm long.

1.

2.

3.

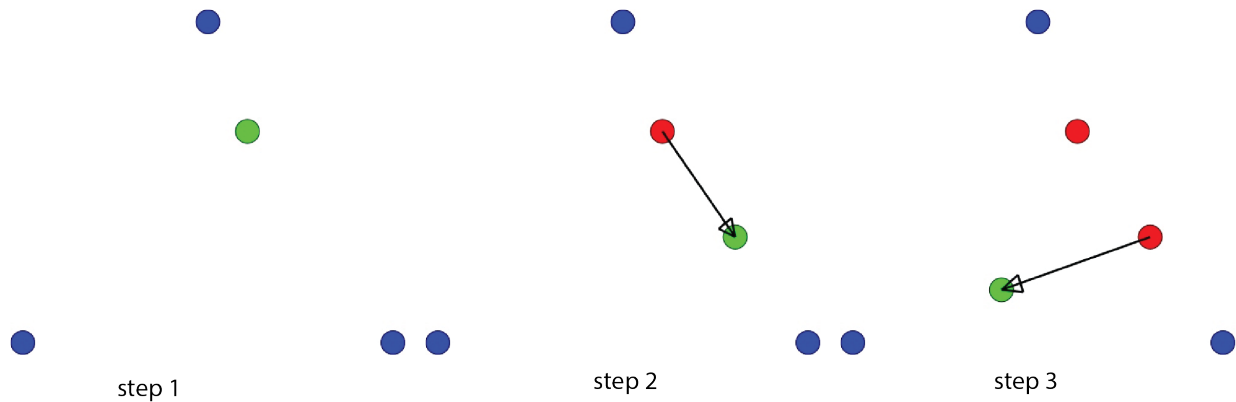| map	units	# of lines	coastline length (cm)
1	4 cm		
2	2 cm		
3	0.5 cm		

If you were to use smaller and smaller units, the coastline length would keep getting longer, just like the perimeter of the Koch snowflake! This is called the **coastline paradox**.

The Chaos Game

Looking at fractals leads us to an important field of math called **chaos theory**. This field studies the results of large systems that are very sensitive to initial conditions.
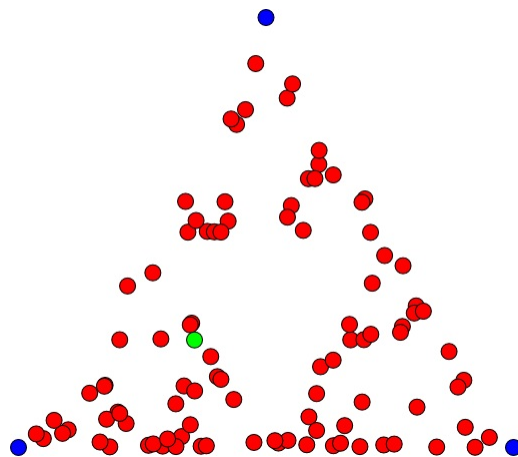
Let's play a game. Here are the steps:

1. Start with an equilateral triangle. Randomly place a dot within the triangle.

2. Randomly choose one of the three vertices, and move the dot halfway to the chosen vertex.
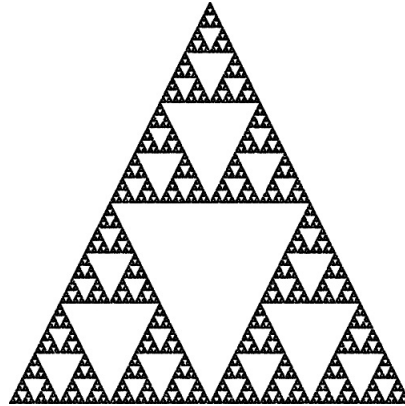
3. Repeat step 2!

step 1   step 2   step 3

Images by Ben Trube, retrieved from: `https://bentrubewriter.com/2012/04/23/` `fractals-you-can-draw-sierpinski-triangle-or-tri-force-on-steroids/`

After 100 steps, this is what it looks like:

Can anyone guess what this will end up becoming?

After 50 000 steps, this is what it looks like:

It's the Sierpinski triangle! This method of creating fractals was invented by British mathematician Michael Barnsley. The following diagram shows how variations of the chaos game can produce different fractals. In the image, n refers to the number of sides of the polygon picked in step 1 of the game, and r refers to the fractional distance travelled in step 2.
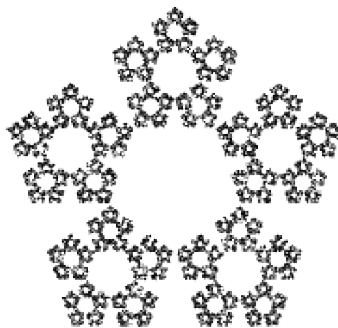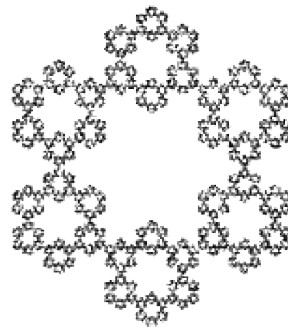

Retreived from http://mathworld.wolfram.com/ChaosGame.html
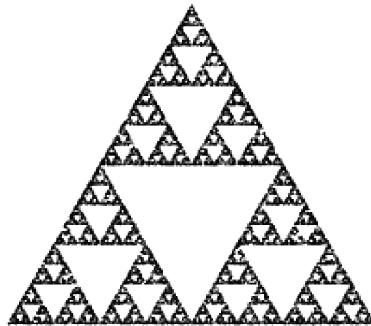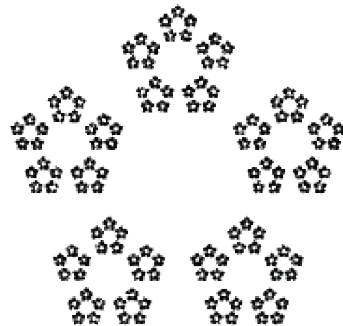
Let's watch this video to learn more about the chaos game: https://www.youtube.com/watch?v=kbKtFN71Lfs

Problem Set

Problems marked with an asterisk (*) are challenge questions.

1. Draw the dragon curve:

   (a) Start with a short vertical line.

   (b) Rotate it 90° clockwise and attach the new line to the end of the original.

   (c) Keep repeating step (b) with the shape that is created.
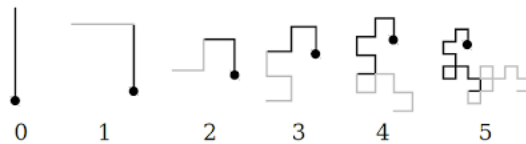
Retrieved from: `https://rosettacode.org/wiki/Dragon_curve`

The dragon curve can be complicated to draw after the first few iterations, but an easy way to create it is by repeatedly folding a strip of paper in half in the same direction, and then unfolding it. The view from above will show you the dragon curve!


Retrieved from: `https://www.cutoutfoldup.com/216-dragon-curve.php`
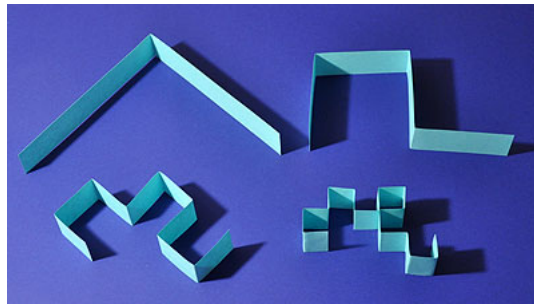
2. Draw the Cantor set:

   (A) Draw a line.

   (B) Remove the middle third.

   (C) Repeat step (b) for the remaining lines.

Retrieved from: https://cre8math.com/2017/02/12/a-new-cantor-set/
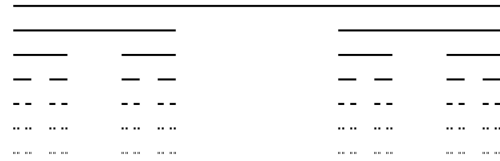
3. Draw the Sierpinski Carpet:

   (a) Start with a square.

   (b) Divide it into 9 equal squares and remove the middle square.

   (c) Repeat step (b) for each of the remaining 8 squares.
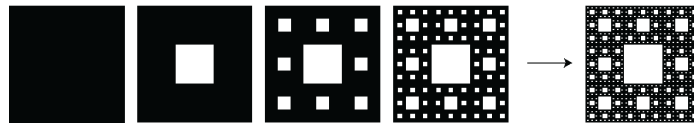
Retrieved from: https://www.encyclopediaofmath.org/index.php/Sierpi%C5%84ski_gasket

4. Draw the Pythagoras tree:

   (a) Start with a square.

   (b) Draw two squares on top of the first so that an isosceles right angled triangle is formed in the space between them.

   (c) Repeat step (b) for each of the two new squares.

Retrieved from: http://mathworld.wolfram.com/PythagorasTree.html
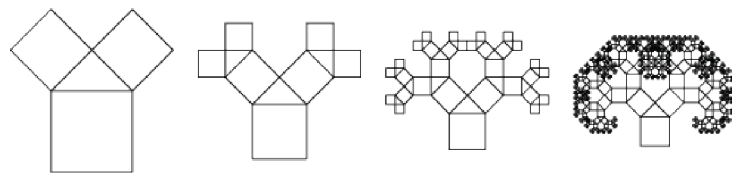
5. Try to create your own fractal!

6. Follow the instructions at the website below to create a 3-dimensional cutout of the Sierpinski Triangle.

http://fractalfoundation.org/resources/fractivities/fractal-cutout/

7. Identify a fractal object that can be found in real life.

8. Julia sets are complex fractals related to the Mandelbrot set from the beginning of the lesson. With your parents' permission, visit `https://math.berkeley.edu/~kmill/toys/julia/julia.html` and play around with this fractal generator to see what Julia sets can look like! Extra instructions for the special features of the generator can be found here: `https://math.berkeley.edu/~kmill/blog/blog_2014_02_02_julia.html`. Side note: Gaston Julia was a French mathematician who studied these sets in the early 1900s, well before computer modelling could develop these types of images. What a surprise it would be for him if he could see what his work looks like today!

9. These are the chaos game visualizers used in the video we watched in class:

https://www.geogebra.org/m/yr2XXPms

https://www.geogebra.org/m/bQ8ppzRj

Play around with them and see what sorts of results you get. For the first link, what happens if you start in a "cut out" part of the Sierpinski triangle? Or even outside of the triangle? What happens when you vary the rules of the game using the sliders on the left side?