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Grade 7/8 Math Circles

February 25/26/27 2020

Unsolved Problems

Introduction

What do mathematicians do?

The way in which math is taught in schools seems to imply that we know everything there is to know about math. However, this cannot be further from the truth! There are many questions in math that we do not have the answers to. This is what mathematicians work on every day.

Scientists observe things in their discipline, make hypotheses, and then design tests to find the answers to those questions. Mathematicians work in a similar way. They observe patterns and formulate hypotheses (which in math are called **conjectures**) based on their observations.

The difference between scientific research and mathematical research is the way we reach conclusions. In science, to show that something is true, you must follow the scientific method. You test the hypothesis rigorously and look for consistent results. However, in math, it is not enough to show that a statement is true for 100 numbers, 1,000 numbers, or even 1,000,000 numbers. We have to show that it is true for **all numbers**.

Proofs

A **proof** is a way of showing that a mathematical statement is true. Let's explore a few statements to see how we might prove or disprove them.

Example 1. The sum of two even integers is even.

This seems like an obvious statement, but how do we know it is true? Before we start, let's establish the definition of even. An integer s is even if it is divisible by 2. In other words, there exists some integer t such that $s = 2t$.

Let a and b be two even integers.

$\Rightarrow a = 2x$ and $b = 2y$ for some integers x and y.

$\Rightarrow a + b = 2x + 2y$

$\Rightarrow a + b = 2(x + y)$

$\Rightarrow a + b$ is even by definition.

□

That square at the end is called a halmos or a tombstone, and it is used to indicate that a proof is finished. Alternatively, you can write "Q.E.D," which stands for *quod erat demonstrandum*, Latin for "that which was to be shown."

Exercise 1. The sum of two odd integers is even.

Like in the previous example, we should start by defining what an odd number is. An integer s is odd if it is 1 more than an even integer. In other words, there exists some integer t such that $s = 2t + 1$.

Let a and b be two odd integers.

$\Rightarrow a = 2x + 1$ and $b = 2y + 1$ for some integers x and y.

$\Rightarrow a + b = 2x + 1 + 2y + 1$

$\Rightarrow a + b = 2x + 2y + 2$

$\Rightarrow a + b = 2(x + y + 1)$

$\Rightarrow a + b$ is even by definition.

□

The next type of proof we will try is proof by **contradiction**. For this type of proof, we assume that the statement is false, and then prove that this leads to some nonsensical result. Therefore, the statement must be true.

Example 2. There exist no integers x and y such that 8x + 4y = 1.

Assume that the statement is false. That is, there exist some integers x and y such that 8x + 4y = 1.

$$\begin{aligned}
&8x + 4y = 1 \\
\Rightarrow\ &\frac{8x + 4y}{4} = \frac{1}{4} \\
\Rightarrow\ &2x + y = \frac{1}{4}
\end{aligned}$$

A whole number multiplied by 2 is still a whole number, and a whole number plus another whole number is still a whole number, therefore the left-hand side of this equation is a whole number. However, the right-hand side is not! This is a contradiction, so our assumption must be wrong. Therefore the statement is true. □

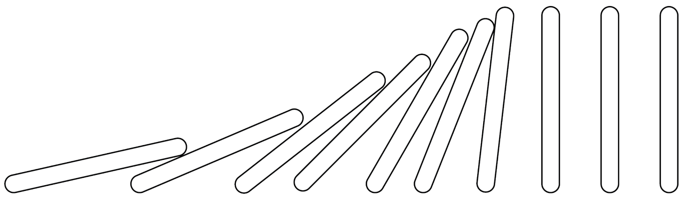
Exercise 2. Let n be a positive whole number. If n is odd, then n^2 is odd.

Let's start by assuming the statement is false. That is, that if n is odd, then n^2 is *even*. Since n is odd, by the definition of being odd, let $n = 2x + 1$ for some integer x.

$$\begin{aligned}
n^2 &= n \times n \\
&= (2x + 1)(2x + 1) \\
&= 4x^2 + 2x + 2x + 1 \\
&= 4x^2 + 4x + 1 \\
&= 2(2x^2 + 2x) + 1
\end{aligned}$$

So n^2 is odd by definition. But we already assumed that n^2 is even! A number cannot be both even and odd, so we have a contradiction. Therefore our original assumption was wrong, and the statement is true. □

The last type of proof we will look at today is proof by **induction**. Let's start with a very basic example of the principle of induction. Let's suppose that you have a long line of dominoes which you want to topple. When you topple the first one, it will topple the second one. When you topple the second it will topple the third. This will continue forever until you run out of dominoes. `https://www.youtube.com/watch?v=6bMRCOiWxyc`



This is the essence of induction. Let's try an example to illustrate the process.

Example 3. If n is a positive whole number, $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

We can easily calculate this for some example values of n. Let's try:

- n = 1

  $1 = 1$ $\qquad \frac{1(1+1)}{2} = \frac{1\times 2}{2} = \frac{2}{2} = 1$

- n = 6

  $1 + 2 + 3 + 4 + 5 + 6 = 21$ $\qquad \frac{6(6+1)}{2} = \frac{6\times 7}{2} = \frac{42}{2} = 21$

- n = 9

  $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ $\qquad \frac{9(9+1)}{2} = \frac{9\times 10}{2} = \frac{90}{2} = 45$

How do we prove this for all values of n? If we think of each value of n as a domino, we want to be able to topple all of them.

Let's suppose that the statement is true for some number k. That is,

$$1 + 2 + ... + k = \frac{k(k+1)}{2}$$

Then what is $1 + 2 + ... + k + (k+1)$? Well based on our assumption above, we can say that

$$\begin{aligned} 1 + 2 + ... + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)(\frac{k}{2} + 1) \\ &= (k+1)(\frac{k}{2} + \frac{2}{2}) \\ &= (k+1)(\frac{k+2}{2}) \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

Which is what our statement predicted! This shows that **the previous value of n being true will cause the next value of n to be true as well.**

We know that the statement is true for n=1 from the examples we tried. Now n=1 being true will cause n=2 to be true. Then n=2 being true will cause n=3 to be true. And this will continue forever, causing the statement to be true for all values of n! The values of n are like a long line of dominoes that are now being toppled.

These three proof techniques are not a complete list. These are just a few ways for you to get started!

Fermat's Last Theorem

The generalization of the problem that we worked on before class is known as Fermat's Last Theorem. It states that there do not exist three positive integers a, b, and c which satisfy the equation

$$a^n + b^n = c^n$$

for integer values of n greater than 2. Fermat made this observation in 1637 as a margin note in a book. He stated that he had a general proof for this theorem, but that it was too large to fit in the margin. It remained unsolved for over three centuries, until it was proven by British mathematician Andrew Wiles in 1995.

Wiles' proof unified two different areas of mathematics: elliptic curves and modular forms. This is what we aim for when we work on problems in mathematics. Sometimes, the result is useful. Other times, the proof itself can be useful if it shows us a new technique or builds connections between seemingly unrelated branches of math.

Millenium Problems

In the year 2000, the Clay Mathematics Institute established a list of 7 "Millenium Prize Problems," which are some of the most difficult unsolved problems. Whoever solves any of these problems will be awarded $1 million!

Most of these problems involve very difficult math, but the Institute has published accessible introductions to each of the problems on their website: `https://www.claymath.org/millennium-problems`. One of the problems, the Poincaré conjecture, was solved in 2006 by Russian mathematician Grigori Perelman. The other six problems remain unsolved.

Tips for approaching problems

1. Test the statement using some examples. This can help us to understand why a statement is true for ourselves.
2. Draw diagrams and try to visualize the problem.
3. Try to prove a modified version of the statement. Simplify it in some way to see if that gives you any clues as to what works and what doesn't work.
4. Work together! Talk with your neighbour, brainstorm different ways of approaching the problem, and try working on examples together.

Problem Set

The following problem set contains a mix of both solved and unsolved problems.

Have fun!

1. Pick a positive whole number. If it is even, divide it by 2. If it is odd, multiply it by 3 and add 1. Repeat this process indefinitely. Will the sequence always eventually reach 1?

   This is a very famous unsolved problem in mathematics called the Collatz Conjecture. As simple as the problem sounds, mathematicians have made very little progress on it.

2. Can every even integer greater than 2 be written as the sum of two primes?

   Goldbach's conjecture - Unsolved

3. There is no greatest even whole number.

   Assume that there is a greatest even whole number. Call it x.

   $\Rightarrow x = 2a$ [for some integer a, by definition of being even]

   $\Rightarrow x + 2 = 2a + 2$

   $\Rightarrow x + 2 = 2(a + 1)$

   $\Rightarrow x + 2$ is greater than x and it is even by definition.

   But x was assumed to be the greatest even whole number. This is a contradiction, so our original assumption was wrong and the statement is true. $\square$

4. Are there infinitely many primes p such that p+2 is prime?

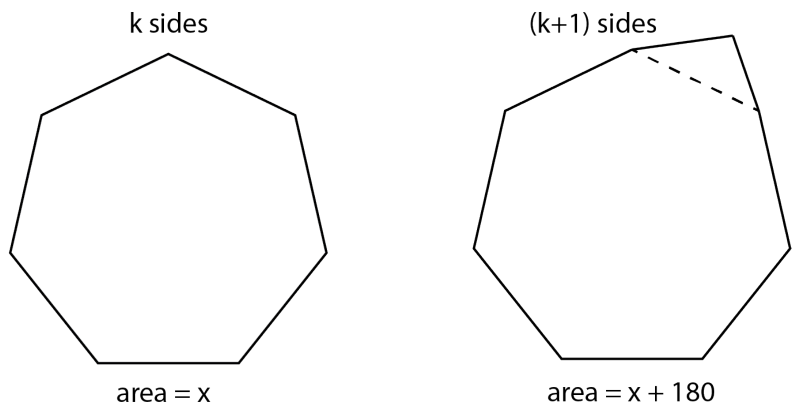
   Twin prime conjecture - Unsolved

5. The sum of the interior angles of a polygon with n sides is $(n-2) \times 180°$.

We can solve this problem using induction.

Base Case: We know that sum of the interior angles of a triangle (3 sides) is 180°. $(3-2) \times 180° = 1 \times 180° = 180°$. Therefore the statement is true for n=3.

Inductive Step: Assume that the statement is true for a polygon with k sides, that is, the sum of the interior angles of a polygon with k sides is $(k-2) \times 180°$. If we have a k-sided polygon, we can create a (k+1)-sided polygon by attaching a triangle to the k-sided polygon like so:

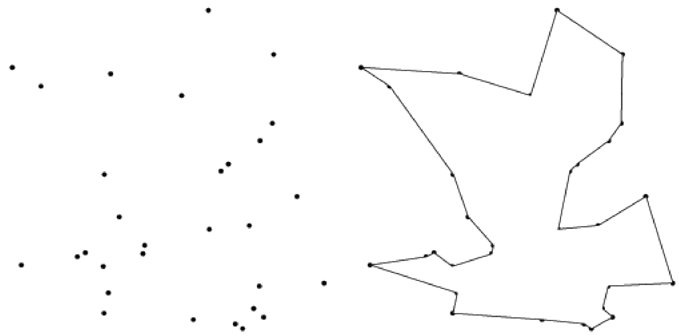


Therefore, the sum of the interior angles of the (k+1) sided polygon is:

$$\begin{aligned}
&\text{sum of interior angles of k-sided polygon} + 180° \\
&= (k-2) \times 180° + 180° \\
&= 180° \times (k-2+1) \\
&= 180° \times ((k+1)-2)
\end{aligned}$$

Which is what we predicted! Therefore, by induction, the statement is true. $\square$

6. Given a list of cities and the distances between each pair of cities, you want to find the shortest possible route that visits each city exactly once before returning to the city you started in. Can you find a general method to solve a problem like this?



Retrieved from: `http://mathworld.wolfram.com/TravelingSalesmanProblem.html`

Travelling Salesman Problem - Unsolved. This problem is closely related to the "P vs NP" Millenium Problem.

7. There is no smallest rational number greater than 0.

Note: A rational number is a number that can be expressed as a fraction of integers in lowest terms. An irrational number cannot. For example, $\frac{6}{5}$ and 19 are rational numbers, but π is irrational.

Assume that there is a smallest rational number greater than 0. Call it a. Then $0 < \frac{a}{2} < a$ and $\frac{a}{2}$ is rational, so $\frac{a}{2}$ is the smallest rational number greater than 0. But we assumed that a was the smallest rational number greater than 0. This is a contradiction, therefore our assumption was wrong and the statement is true. $\square$

8. $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational. Therefore, there exist some integers p and q in lowest terms such that:

$$\begin{aligned}\sqrt{2} &= \frac{p}{q}\\ (\sqrt{2})^2 &= (\frac{p}{q})^2\\ 2 &= \frac{p}{q} \times \frac{p}{q}\\ 2 &= \frac{p^2}{q^2}\\ 2q^2 &= p^2\end{aligned}$$

Therefore p^2 is even, and so p is even. Also, since p and q are a fraction in lowest terms, q must be odd (or else they would have a common factor of 2 and not be in lowest terms).

Since p is even, by definition, there exists an integer x such that $p = 2x$. Therefore:

$$\begin{aligned} 2q^2 &= (2x)^2 \\ 2q^2 &= 2x \times 2x \\ 2q^2 &= 4x^2 \\ q^2 &= 2x^2 \end{aligned}$$

Therefore q^2 is even, and so q is even. But we already said that q must be odd. This is is a contradiction, therefore our assumption was wrong and the statement is true. □

9. What is the least number of colours required to color a 2D plane such that no two points at distance 1 from each other have the same color? (This number is called the "chromatic number of the plane")

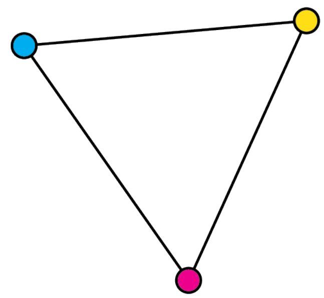
   Hadwiger-Nelson Problem - Unsolved. It is either 5, 6, or 7. Here is a brief history of how we know:

   The following image shows a successful colouring of the plane using 7 colours. The diameter of each hexagon is slightly smaller than the unit distance. Therefore, the chromatic number of the plane is at most 7.

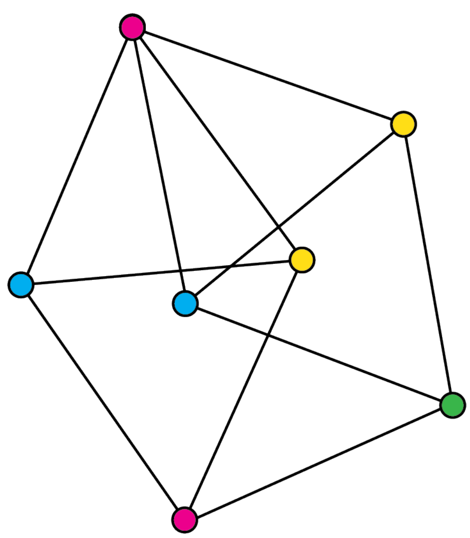
Retrieved from `http://mrhonner.com/wp-content/uploads/2018/06/hexagonal-tiling.png`

Now let's find a lower bound. The shapes below are called "graphs," and each dot is called a "vertex." Two vertices are 1 unit apart if they are connected by a line.

The graph above shows two vertices that are 1 unit apart, so they have to be different colours. Thus, the chromatic number of the plane is at least 2.



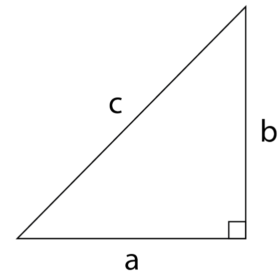
The graph above shows three vertices arranged in an equilateral triangle with side lengths of 1 unit, so they each have to be different colours. Thus, the chromatic number of the plane is at least 3.



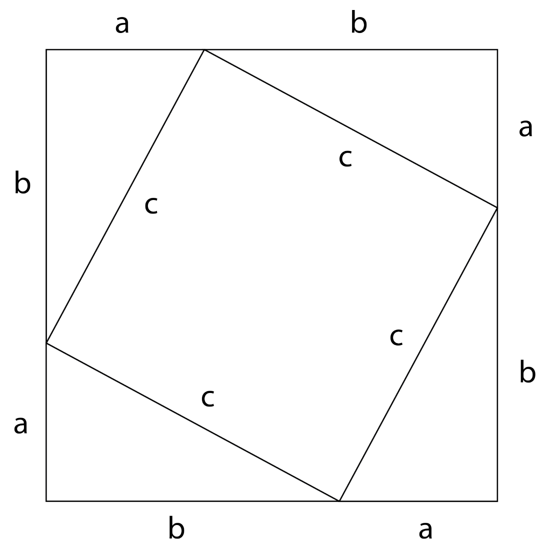
The graph above is called the "Moser spindle," and it shows 8 vertices arranged in a way that we require 4 colours to colour them! Thus, the chromatic number of the plane is at least 4.

In 2018, Aubrey de Grey found a 1581-vertex graph that requires 5 colours to colour, so the chromatic number of the plane is at least 5!

10. For every right-angled triangle with legs of length a and b and hypotenuse (longest side) of length c, prove that $a^2 + b^2 = c^2$.



You may have seen this equation before. It is known as the Pythagorean Theorem. We can prove it using an area argument as follows:



If we arrange four copies of the same triangle like in the image above, we form a larger square. This large square has area $(a+b)^2$, but its area is also equal to the sum of the areas of the four small triangles and the small square. The small square has area c^2, and each triangle has area $\frac{1}{2}ab$. Therefore,

$$\begin{aligned}(a+b)^2 &= c^2 + 4 \times \frac{1}{2}ab \\ a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + 2ab + b^2 - 2ab &= c^2 + 2ab - 2ab \\ a^2 + b^2 &= c^2\end{aligned}$$

11. Does there exist a rectangular prism whose length, width, height, diagonals of each face, and main diagonal (connecting the bottom back left corner to the top front right corner) are all integers?

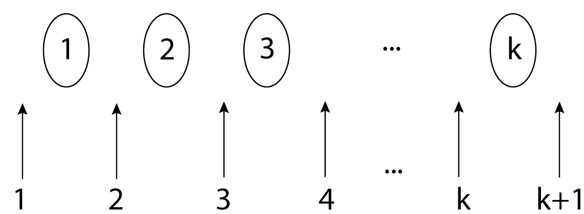
    Integer Brick Problem - Surprisingly unsolved!

12. There are $1 \times 2 \times 3 \times ... \times n$ ways to arrange n people in a line.
    (Order matters! Anna, Bob, Carla is a different arrangement from Carla, Bob, Anna, which is a different arrangement from Bob, Anna, Carla, etc.)

    We can solve this problem using induction:

    Base Case: There is 1 way to arrange 1 person in a line, so the statement is true for n=1.

    Inductive Step: Assume that there are $1 \times 2 \times 3 \times ... \times k$ ways to arrange k people in a line. As illustrated in the diagram below, there are $k + 1$ ways to add the $k + 1$st person into a line of k people.

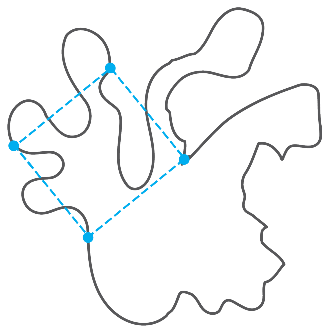


$$\begin{aligned}
&\text{Therefore, the number of ways to arrange k+1 people} \\
&= (k+1) \times \text{ ways to arrange k people} \\
&= (k+1) \times 1 \times 2 \times 3 \times ... \times k \text{ [by the inductive step]} \\
&= 1 \times 2 \times 3 \times ... \times k \times (k+1), \text{ as required!}
\end{aligned}$$

    Therefore, by induction, the statement is true. □

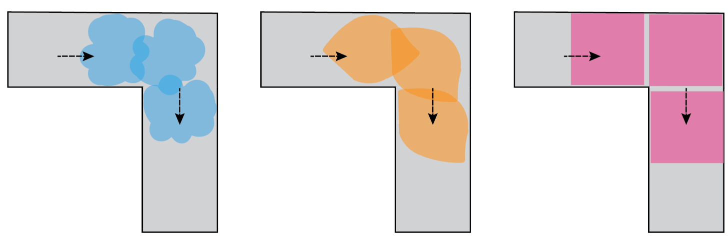
    An alternate way to think about this problem is as follows: Imagine there are n seats to be filled by n people. There are n people who can be placed in the first seat. Then, there are n-1 people who can be placed in the second seat (since 1 person is seated in the first seat). There are n-2 people who can be placed in the third seat, and so on. Therefore in total, there are $n \times (n-1) \times (n-2) \times ... \times 1$ ways to arrange the n people.

13. For any Jordan curve (a closed loop that doesn't cross itself), can you draw a square whose vertices are on the curve?



Inscribed Square Problem - Unsolved in the general case. It has actually been proved for most "normal" types of curves, but the proof is much too complex to write out here.

14. What is the shape of largest area that can be moved through an L-shaped corridor?



Moving Sofa Problem - Unsolved. The largest shape that has been found so far is the one below, but we have not proved that there are none larger, so the problem remains open.

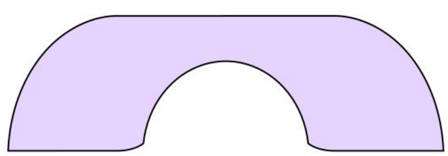


Image by Dan Romik. Retrieved from: `https://www.math.ucdavis.edu/~romik/movingsofa/`