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Centre for Education in
Mathematics and Computing

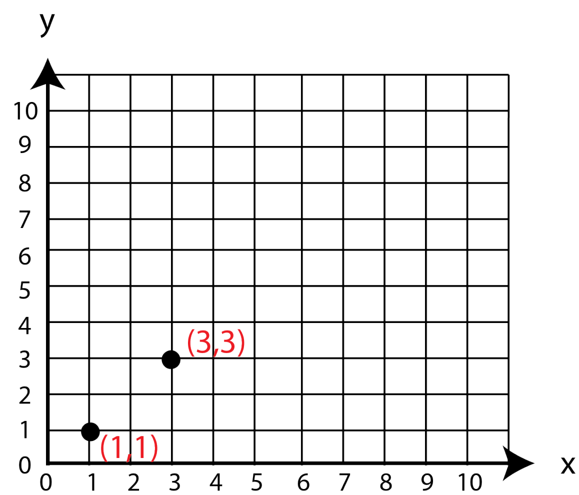
Grade 7/8 Math Circles

March 24/25/26 2020

General Relativity & Black Holes

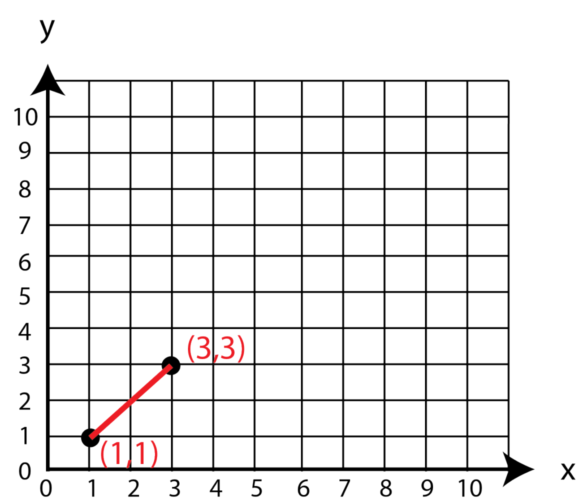
Finding the Shortest Distance Between Two Points

Let's say we plot the following two points on a coordinate grid:



What is the shortest distance between these two points?

..... A straight line! Like this:



Pythagorean Theorem and the Distance Between Points

What's Pythagorean Theorem Again?

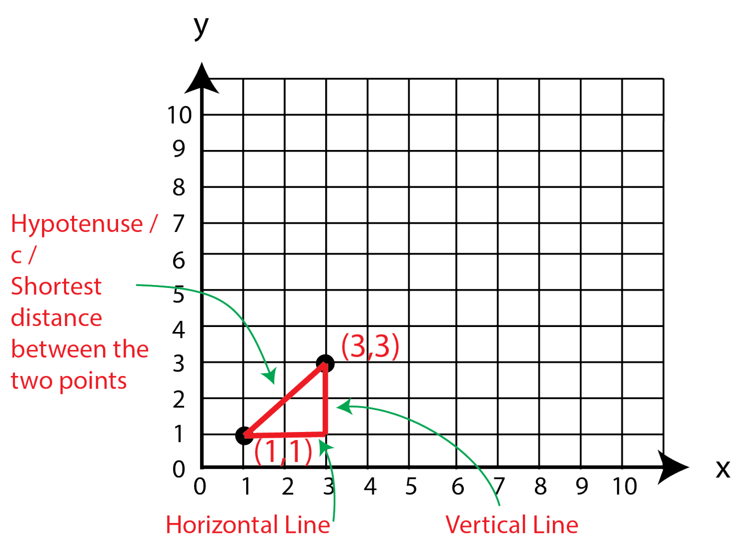
Pythagorean theorem says that if we have a right angled triangle, we can find the length of the hypotenuse using the formula,

$$c^2 = a^2 + b^2$$

Where c is the length of the hypotenuse and a and b are the lengths of the other two sides.

So how can we use this formula to find the shortest distance between the two points on the graph?

Let's make a right angled triangle out of our points and the line that we already have:



We need to know the horizontal and vertical side lengths of the triangle.

The horizontal line in our graph goes from the point (1,1) to the point (3,1) and since this line is horizontal and the y-values of the two points are the same, we only need to look at our two x-coordinates (as this is the axis in the horizontal direction) to find the distance between these two points. The distance between these two points is the result of subtracting our two x-coordinates, 3 - 1 = 2. So our horizontal side length has a value of 2 units.

The vertical line goes from the point (3,1) to the point (3,3). So here, our x-values are the same and we have a vertical line, so we only need to look at the y-values to determine the length of this line. The distance between these two points is the result of subtracting our two y-coordinates, 3 - 1 = 2. So our vertical side length is 2 units.

Now that we have two of our side lengths, we can use the Pythagorean theorem to find the third side length, which is the shortest distance between these two points:

$$
\begin{gathered}
c^2 = a^2 + b^2 \\
c^2 = 2^2 + 2^2 \\
c^2 = 4 + 4 \\
c^2 = 8 \\
c = \sqrt{8} \\
c \approx 2.83
\end{gathered}
$$

So, the shortest distance between these two points is approximately 2.83 units.

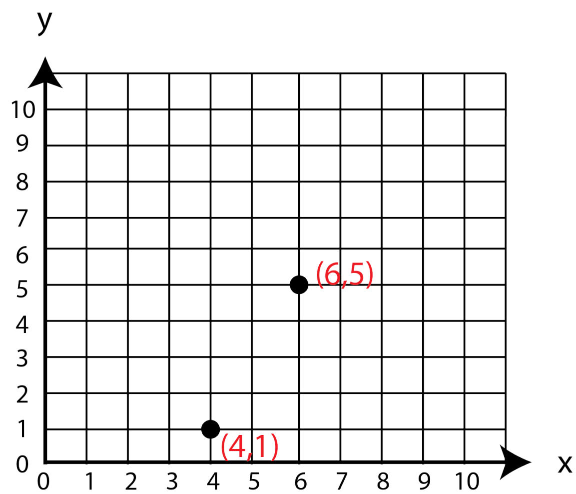
Instead of first finding the horizontal and vertical side lengths and then using Pythagorean Theorem, we can condense this all into a new equation for finding the shortest distance between two points on a regular graph:

$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Where,
D is the distance between the points,
x_2 and y_2 are the bigger x and y values, respectively,
x_1 and y_1 are the smaller x and y values, respectively.

Exercise:

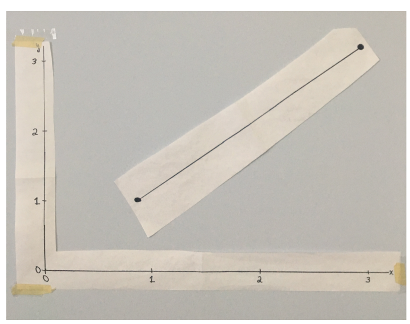
Use this formula to find the shortest distance between the two points on the following graph:



*Your answer should be 4.47 (If you got this but rounded to 4, 4.5, or added extra decimals that is fine).

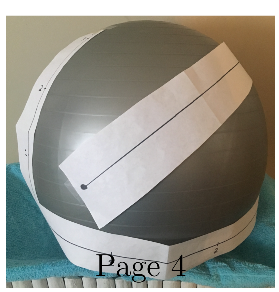
What if we have a curved graph?

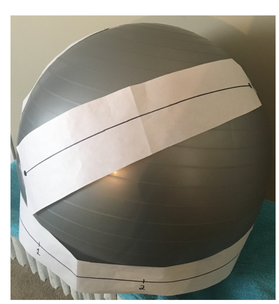
Let's take our original graph:



We have already discussed that the shortest distance between points (1,1) and (3,3) here is a straight line. We see this straight line drawn on the graph.

But now, lets take this graph and tape it onto a ball:



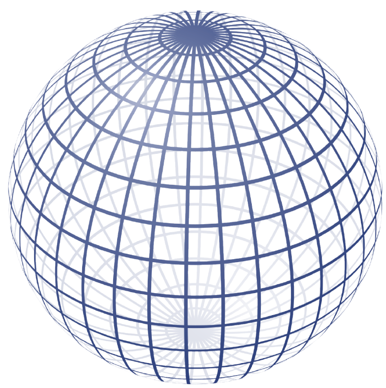


You may also be able to get a better view of the graph here: Curved Graph Video

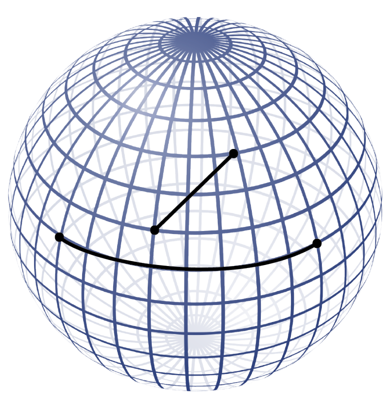
All I have done is taken the same graph and just taped it to the ball. Now you will notice that this line connecting our two points isn't really a straight line at all. It was straight before, but now it is curved. The line must follow the curve of the ball. So now the shortest distance between these two points on the ball is this curve that we see in the above pictures. For anything that I graph on this ball, the shortest distance between two points will be a curve, not a straight line.

We know that a curved path between two points is always longer than a straight path between the same two points, so now our distance between (1,1) and (3,3) will be longer on our curved graph (on the ball).

We can also think of a curved graph with a grid like this:



Again, if I try to draw lines of shortest distance between two points on this graph, I will get a curve:

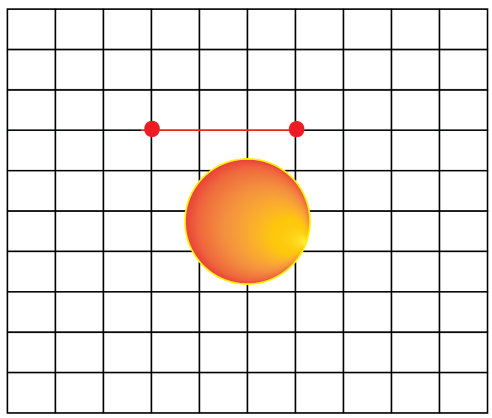


Once again, we know that the distance between two points on a curved graph is longer than the distance between those same two points on a normal, flat graph.

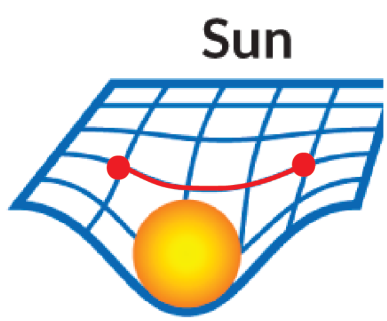
Curved Space

This idea of a curved graph is how general relativity views space. In everyday life, we always graph things on a normal graph like the very first one we saw in this lesson. Before general relativity, we assumed that space and everything in the universe exists on a flat graph. General relativity says that everything is really on a curved graph (ie. Space is curved).

For example, if you wanted to fly your rocketship in space from one point to another near the sun, you would think that you are just moving on a straight line like this:



But really, the space around the sun is *curved*, so you would be moving on a curved path like so:



Why is space curved?

Anything that has mass will create a "dent" in space. Space must work around these objects. The more massive an object, the more it curves space.

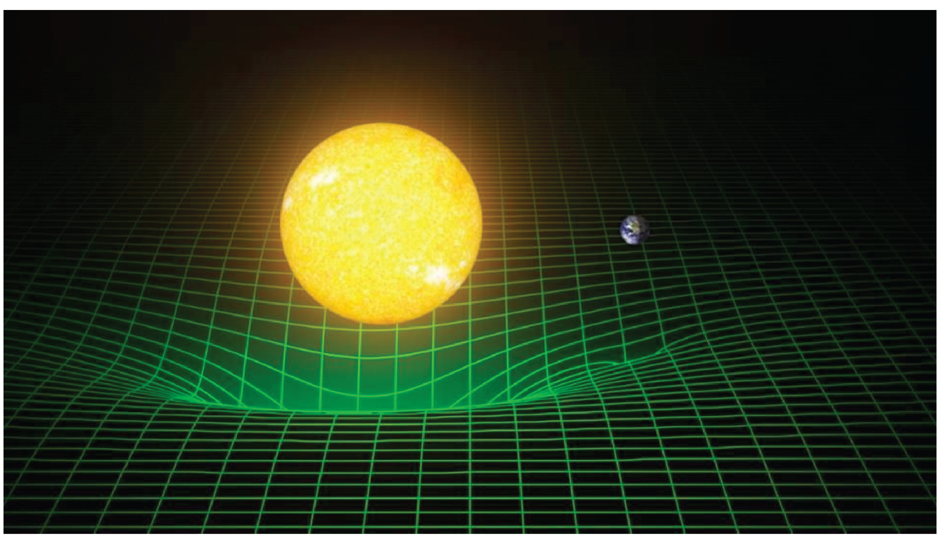
Let's imagine that the white table cloth in the following images represents space. When I put different objects on the table cloth, they each create their own dent or bending of the table cloth.



The tennis ball, which is small and has little mass, barely creates a dent in the cloth. The yoga ball, which is bigger and has a little more mass, creates a wide and shallow dent in the cloth. The weight, which has a greater mass, creates a deeper dent in the cloth.

The way that the table cloth bends around these objects is similar to how space bends around celestial objects.

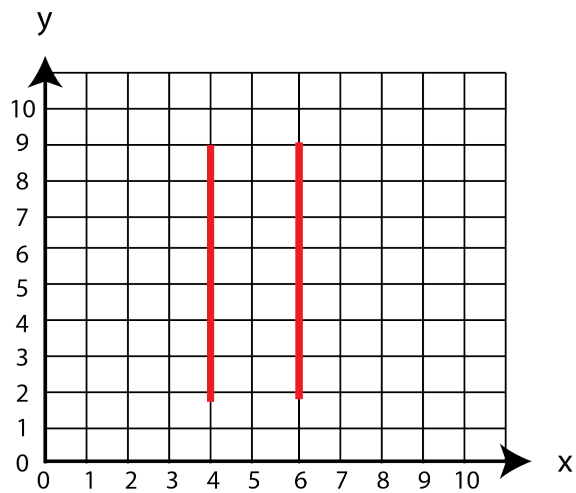
For example, the sun's mass causes space to bend around it, like so:



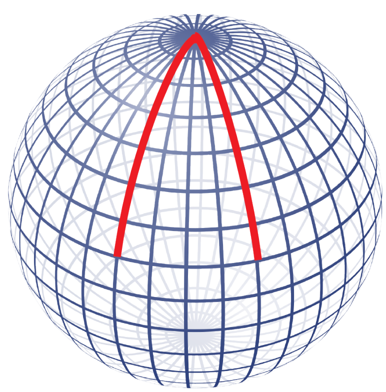
Consequences of curved space

Parallel lines are not so parallel

Here we have two parallel lines:



We know that parallel lines are lines that will never meet/intersect. As we can see, if we extend these two lines, they will never intersect each other. But this is in flat space, what if we tried to draw parallel lines on a curved surface?

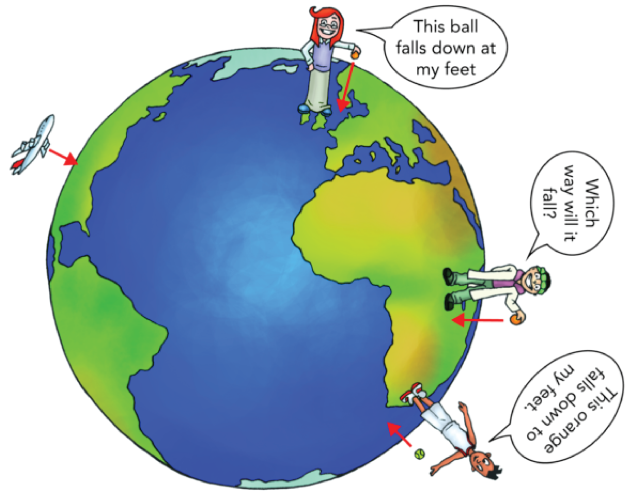


These two lines start off looking like regular parallel lines. However, they must follow the curves of the curved graph. This curving causes the lines to eventually meet/intersect. Hence, parallel lines cannot be drawn here, because the lines will eventually meet.

How might a black hole curve space?

What is gravity?

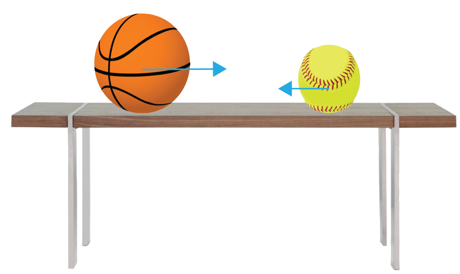
When we talk about the Earth's gravity, we describe it as a force of attraction where the Earth pulls everything towards its centre. This is how we are able to stay on the ground rather than float, and it is also why when we throw a ball into the air it does not just keep going up forever, but rather it falls back down toward the ground. (Keep in mind that this is *Newton's* explanation of gravity, which is how we typically think of it. However, later in this lesson we will talk about how Einstein explained gravity).



Retrieved from kaiserscience.wordpress.com

Is the Earth the only object that has gravity?

No!! All objects have gravity!! That is, every object has a force of attraction that pulls other objects towards its centre. The more mass an object has, the stronger its force of gravity.



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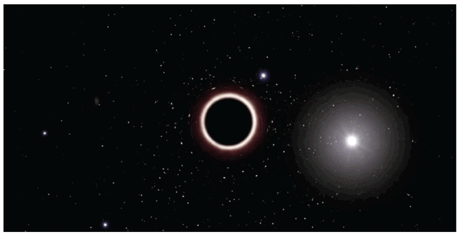
What is a black hole?

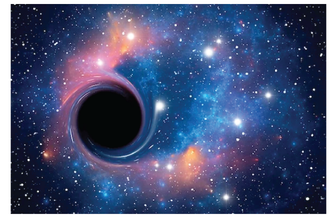
A black hole is an object in outer space whose gravity is so strong that once another object gets close enough to it, it is trapped inside the black hole forever and can never escape. This is because black holes have so much mass that they pull everything towards it with a huge amount of force.

What does a black hole look like?



Retrieved from www.nbcnews.com

Retrieved from ny-times.com



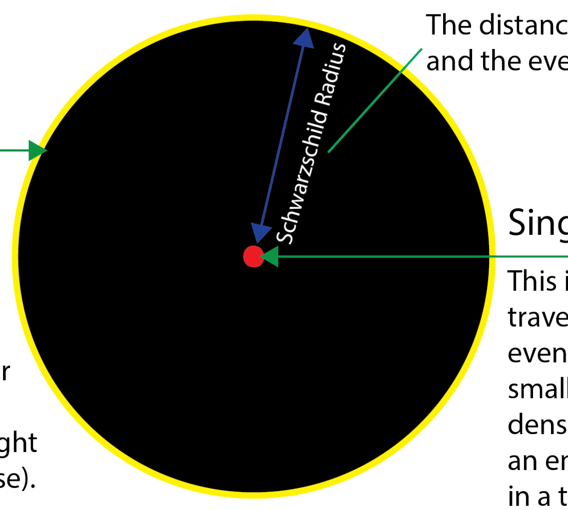
Retrieved from blogs.scientificamerican.com

Diagram of a black hole:

The distance between the singularity and the event horizon.

Event Horizon

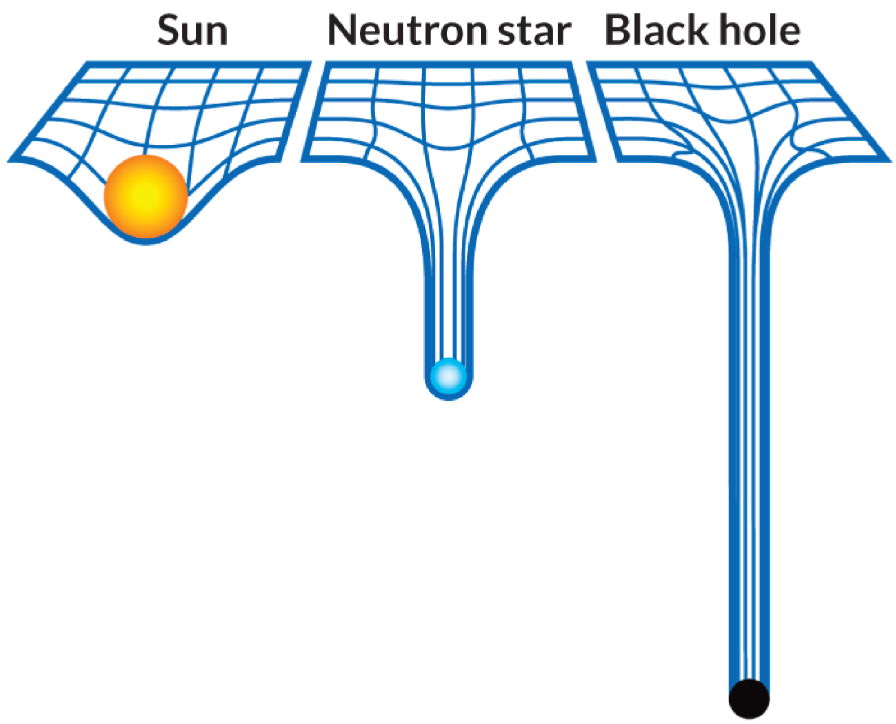
The edge of the circle around the black hole marks the point of no return. Once something crosses the event horizon, it can never get back out. To do so it would need to travel faster than the speed of light (which is impossible as the speed of light is the speed limit in the universe).



Singularity

This is the place that all matter travels to once it is inside the event horizon. It is extremely small and has an enormous density (this means there is an enormous amount of mass in a tiny space).

Since black holes are the densest objects in the universe (have the most enormous masses in tiny spaces), we can guess that they will curve space A LOT (remember we said that the more mass an object has, the more it will curve space). Black holes curve space more than any other object, they curve space to the extreme.



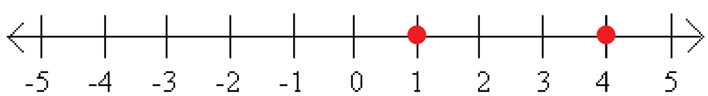
Distance Between Two Points Near A Black Hole

When dealing with a Schwardschild black hole, we only consider the 1 - dimensional distance between the points. If the space near the black hole was flat space (was not curved), our equation for the distance between two points would be:

$$D^2 = (R_2 - R_1)^2$$

Where,
D is the distance between the points,
R_2 is the distance from the point further from the black hole to the event horizon and R_1 is the distance from the point closer to the black hole to the event horizon.
Note: Both R_1 and R_2 are measured in flat space and are the shortest distances from the event horizon to their corresponding points, which are outside of the black hole.

Think of it like the distance between two points on a number line:



The distance between the point 1 and 4 here is simply 4 - 1 = 3 (Or D = $R_2 - R_1$). All we have done in our equation is square both sides.

But space is not flat near a black hole, it is very curved. So our shortest distances between two points will not be simple straight lines, but rather curves, which will cause the distances between two points to be larger than we would expect. Our equation for the distance between two points near a Schwardzchild black hole, where space is curved, is

$$D^2 = \frac{(R_2 - R_1)^2}{1 - \frac{R_s}{R}}$$

Where,
D is the distance between the two points,

R_2 is the point further from the black hole and R_1 is the distance closer to the black hole,
R_s is the Schwardzchild radius of the black hole,
R is $\frac{R_1+R_2}{2}$.
Note: Both R_1 and R_2 are measured in flat space and are the shortest distances from the event horizon to their corresponding points, which are outside of the black hole.

Example:

1. Suppose we have a black hole with a Schwardzschild radius of 10 km. Calculate the distance between two points outside the black hole that are at distances $R_1 = 250{,}000$ and $R_2 = 250{,}001$ from the event horizon, where R_1 and R_2 are measured in flat space and are the shortest distances from the event horizon to the corresponding points.

   Well, if we were dealing with flat space, we would use the equation,

$$D^2 = (R_2 - R_1)^2$$

to get

$$D^2 = (250,001 - 250,000)^2$$
$$D = \sqrt{(1)^2}$$
$$\text{D} = 1$$

   We would get that the distance between these two points is 1.

   However, as we are dealing with curved space near a black hole, we will need to use the equation,

$$D^2 = \frac{(R_2-R_1)^2}{1-\frac{R_s}{R}},$$

Where $R_s = 10$ km and R $= \frac{R_1+R_2}{2} = \frac{250{,}000+250{,}001}{2} = 250{,}000.5$

to get

$$D^2 = \frac{(250{,}001-250{,}000)^2}{1-\frac{10}{250{,}000.5}}$$

$$D = \sqrt{\frac{(250{,}001-250{,}000)^2}{1-\frac{10}{250{,}000.5}}}$$

$$\text{D} \approx 1.00002$$

So the distance between these two points due to the curved space around the black hole is approximately 1.00002, which is very close to the distance that they would have been apart in flat space, 1.

This shows us that the distance between two points far away from a black hole is very similar to the distance between two points in flat space. What if we moved closer to the black hole?

2. Suppose we have the same black hole with a Schwardzschild radius of 10 km. Calculate the distance between two points that are at distances $R_1 = 10$ and $R_2 = 11$ from the event horizon.

   If we were dealing with flat space, we would use the equation,

$$D^2 = (R_2 - R_1)^2$$

to get

$$D^2 = (11 - 10)^2$$

$$D = \sqrt{(1)^2}$$

$$\mathrm{D} = 1$$

   We would get that the distance between these two points is 1.

   But we are dealing with curved space near a black hole, so we use the equation,

$$D^2 = \frac{(R_2 - R_1)^2}{1 - \frac{R_s}{R}},$$

Where $R_s = 10$ km and R $= \frac{R_1 + R_2}{2} = \frac{10+11}{2} = 10.5$

to get

$$D^2 = \frac{(11-10)^2}{1 - \frac{10}{10.5}}$$

$$D = \sqrt{\frac{(11-10)^2}{1 - \frac{10}{10.5}}}$$

$$\mathrm{D} \approx 4.58$$

   So the distance between these two points due to the curved space around the black hole is approximately 4.58, which is significantly different than the distance that they would have been apart in flat space, 1.

   What does this tell us?

It tells us that the distance between two points as we move closer to the black hole will vary more greatly from the distance that the two points would have been apart in flat space, because space gets more and more curved as we move toward the black hole.

Exercise:

1. Let's say we have the same black hole with a Schwardzschild radius of 10 km. Calculate the distance between two points that are at distances $R_1 = 390$ and $R_2 = 400$ from the event horizon in both flat space and curved space, just as we did in the above examples.

   *You should get that the distance is 10 in flat space, but approximately 10.13 in curved space.

2. Now we have the same black hole with a Schwardzschild radius of 10 km. Calculate the distance between two points that are at distances $R_1 = 10$ and $R_2 = 20$ from the event horizon in both flat space and curved space.

   *You should get that the distance is 10 in flat space, but approximately 17.3 in curved space.

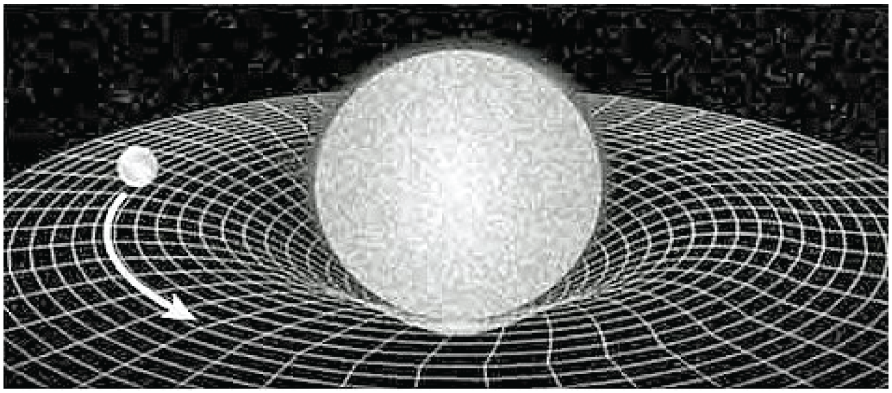
What is gravity, really?

We have already talked about what gravity is according to Newton, but Einstein provides us with a more accurate description.

Einstein's theory of general relativity and the idea of curved space is of importance to us because it finally explained what gravity actually is; it is not just an invisible force at a distance. The more massive an object is, the more it curves space. The more it curves space, the more effect it has on the objects around it, because it causes the objects to move on a curved path around it. This is why the more massive an object is, the more gravity it appears to have.

Inertia: The property all objects have of wanting to remain unchanged. All objects want to keep moving the same way that they are moving, at the same speed. If an object is not moving, it wants to continue to not move. In order to make an object move differently than how it is moving, you must apply some force to it.

Why do the planets orbit the sun?



We know that planets orbit the sun due to gravity, but what is really going on? Let's think of this using the idea of curved space:

Due to inertia, all the planets in our solar system want to keep moving in a straight line, at the same velocity. However, we know that the sun curves space. This means that all of the planets in our solar system are moving in a region of space that has been curved by the sun. So, the planets just continue moving in the straightest line possible, but due to these lines being curved by the sun, they end up circling around the sun.

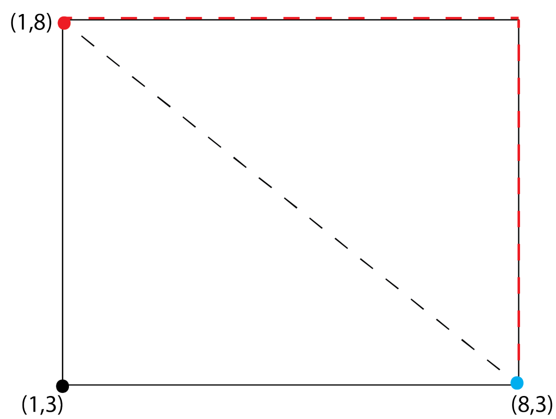
Watch this video for a visualization of this: General Relativity and Orbits

We can more thoroughly and accurately describe gravity and other phenomena with the concept of Einstein's curved space - time rather than just curved space, however this concept is more difficult and we will therefore not touch on it in this lesson.

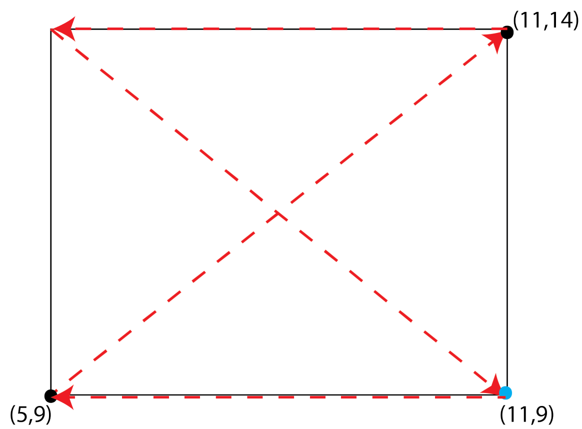
Problem Set

(When working with decimal numbers, default to rounding to two decimal places.)

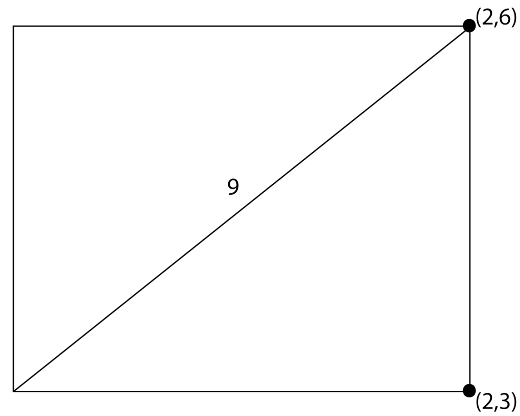
1. You are standing at the blue dot in the following diagram and want to cross a rectangular field to meet your friend at the red dot. Prove that it is shorter to take the black dotted path than than red dotted path by calculating the distance of each path.



2. A dog walks along their rectangular backyard, starting at the blue dot. They walk along the edge to point (5,9), along the diagonal to point (11,14), along the opposite edge and back across the other diagonal to its starting point, which is at point (11,9). The dotted red line in the diagram represents their path taken. What is the total distance travelled by the dog?

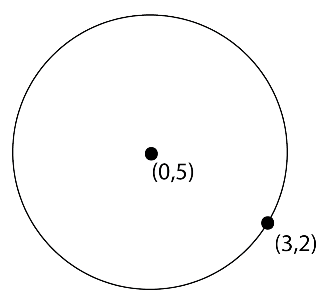


3. The following diagram represents the fence that runs along the perimeter of Harry Potter's backyard. The diagonal of the backyard is 9 metres long and two of the corners are represented with coordinates. What is the area of Harry's backyard?



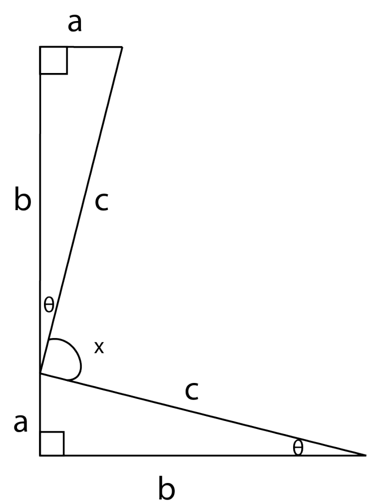
4. Find the value of n such that the distance between points (2.5,3) and (6,n) is approximately 10 units.

5. Given the point at which the following cirlce is centred on and a point on the circle, find the radius of this circle.



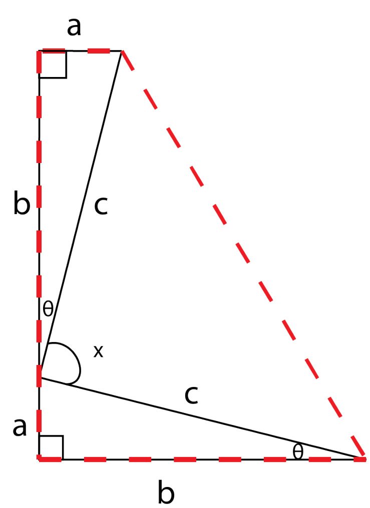
6. We talked about one proof of the pythagorean theorem in our unsolved problems lesson, but there are actually multiple proofs of this theorem. Let's look at Garfield's proof:

   (a) Let's take two of the exact same triangles and arrange them like so:

   What is the value of angle x?

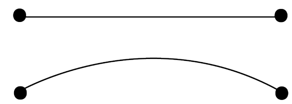
   (b) Now let's create the following trapezoid outlined in a red, dotted line:

   The formula for the area of a trapezoid is $\frac{1}{2}$(top base + bottom base)(height). Write an equation for the area of this trapezoid.

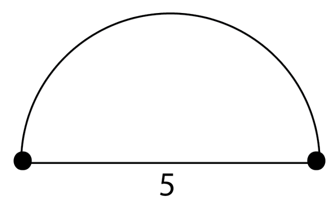
   (c) Now, set your equation for the area of the trapezoid equal to the area of the three individual triangles that make up the trapezoid, all added together. With some manipulating of this equation, you should be able to come up with the result, $a^2 + b^2 = c^2$.

7. The two sets of two points are the same distances apart. Which of these paths between the two points is shortest?



8. Rank the following objects in order of the one that will curve space the most to the one that will curve space the least: Our moon, our Sun, Jupiter, Earth. (Notice that this ranking will also be the order of the planet that has the most amount of gravity to the one that has the least).

9. The shortest distance between the following two points is 5 in normal space. The curved line represents the distance between the two points on some curved graph. The diagram forms a perfect semicircle. What is the distance between the two points according to the curved graph?



10. We have a black hole with a Schwardzschild radius of 10 km. Calculate the distance between two points that are at distances $R_1 = 50$ and $R_2 = 70$ from the event horizon in both flat space and curved space.

11. We have a black hole with a Schwardzschild radius of 10 km. If $R = 20$ and $R_2 - R_1 = 10$, find the values of R_1, R_2 and D.