The circle sequence game

The **circle sequence game** has the following rules:

- Start with an empty page.
- Draw a circle that does not intersect any other circle on the page, and write down the number of circles that you've just enclosed.
- Repeat as many times as you like.

The picture of circle that you get at the end is called **the circle picture** and the sequence of numbers you get is called the **circle sequence**.

For the first part of this session we will focus on circle sequences: If S is the collection of all possible circle sequences, we'd like to figure out what kinds of sequences live inside S. We'll call the sequences in S **good** and those outside of S **bad**.

Exercise 1: A theory of circle sequences

The goal of this exercise is to explore the statements that are true about the collection S of good sequences. We call these statements **theorems**. Try the following questions in any order.

1. Write down all the different circle pictures with exactly 3 circles. How many good circle sequences of length 3 are there?
2. Write down a proof of the theorem "Every (non-empty) circle sequence must start with a zero."
3. Come up with a sequence starting with a zero that is not in S.
4. Does your sequence from part 3 belong to a larger family of forbidden sequences? Can you describe those sequences?
5. Come up with at least one new theorem about circle sequences. Record your theorem(s), and compare with your neighbors. Can you convince your friends that your theorems are correct?

Exercise 2: Changing the rules

In this activity, we'll start to change the rules to the circle sequence game and see what happens.

You may change as many or as few rules as you like, in whatever ways you like, but it is probably a good idea to start small. Here are a few examples. Try them in any order.

1. What if circles are replaced with some other shape? What if they are replaced with "blobby potatoes" that are allowed to be any shape? (They are still not allowed to intersect).

2. What happens if you add a rule saying that each circle can only have at most 3 circles inside of it?

3. Think of (and write down) a rule set where the first entry in the corresponding sequence doesn't have to be a zero. In other words, where the "theorem" from Exercise 1, question 2 is false.

4. Write down a random sequence and give it to one of your neighbors, have them do the same for you. See if you can come up with a circle picture that generates your friend's sequence.

5. Come up with some rules of your own!

Exercise 3: Reverse engineering

Now we're going to see if we can recreate the rules of another group's game just by knowing which sequences are good and which ones are bad. Unlike the first two exercises, this one only makes sense in order.

1. Select your favorite rule set from Exercise 2.

2. Write down some of the good sequences and some of the bad sequences (and mark which are which!).

3. Exchange sequences with another group. Do not include your rule set!

4. Take the sample sequences that the other group has just given you, and see if you can figure out what their rules were. If you get stuck, ask for more samples. You can be explicit. i.e. it's fine to ask "is 01201" a good sequence?" (Hint: when asking for samples it's usually going to be more useful to learn about the bad sequences than the good ones.)

5. Whenever you think you know the rules, write down what you think the other group's rule set is.

6. When we tell you, both groups will get together and share their guesses about each other's rule sets.

7. If the other group guesses right, tell them!

8. If their guess is wrong, see if you can tell them a new sample sequence that is good in your rule set and bad in their guess, or bad in your rule set and good in their guess. That is, try to give them a sample sequence that distinguishes their guess from the true rules. (This might not be possible; it may be that two different rule sets define the same sequences S!)

A few additional exercises

Here are some more exercises to try if you're looking for more ideas.

1. Can you come up with a set of rules for sequences that constructs exactly the set S but doesn't talk about circles at all? (In other words, can you give a set of axioms that describes circle sequences from a different point of view).
2. How many different circle sequences are there of length n?
3. How many different circle pictures are there with n circles?
4. Come up with a rule set for which the previous two questions are easier (but not too easy).
5. Come up with better notation for the circle pictures. Can they be represented with strings (a sequence of letters/numbers/other characters) in a unique way?
6. How many "fundamentally different" rules sets can you think of where the first entry in the corresponding sequence doesn't have to be a zero?
7. If you know what a (combinatorial) graph is, come up with a rule set for the circle game where the circle pictures are graphs. If you don't know what a graph is, picture a constellation, a bunch of stars (points) connected by lines (lines). Alternatively, ask your friends!
8. Can you come up with a rule set where each circle picture corresponds to a *unique* circle sequence?
9. Come up with two "fundamentally different" rule sets for which the resulting collections of circle sequences are the same.
10. Pick your favorite collection of sequences. Maybe, palindromic sequences of length less than 5. Maybe binary strings of length at least 4. Maybe the intersection of the previous two suggestions, etc.

    Can you come up with a rule set that generates your chosen collection of sequence?