Naïve set theory and the limits of axiomatization

The MU Puzzle[1]

(This exercise is taken from the book *Gödel, Escher, Bach: An Eternal Golden Braid* by Douglas Hofstadter.)

Consider the symbols **M, I, U** and all "words" created by writing these symbols, for example, the word **IMUIIIMI**. There are four rules for turning a word into another word. Whenever you see **x** or **y**, it means any "subword".

1. **xI** → **xIU** (If I is at the end, you can add a U after.)
2. **Mx** → **Mxx** (Any word starting in M can append a copy of its tail.)
3. **xIIIy** → **xUy** (Three neighboring I's can become a U.)
4. **xUUy** → **xy** (Two neighboring U's can be removed.)

Exercise 1

1. Start with the word **MI** and use the rules to derive the word **MUIUIU**. Use only one rule at a time, and write down which rule you used each time. How many steps did your derivation require?

   MI $\xrightarrow[2]{}$ MII $\xrightarrow[2]{}$ MIIII $\xrightarrow[2]{}$ MIIIIIIII $\xrightarrow[3]{}$ MUIIIII

   $\xrightarrow[3]{}$ MUIUI $\xrightarrow[1]{}$ MUIUIU

   6 steps.

2. Let's say you start with the word **MI** and use the rules to derive another word. How many **M**'s might be in that word? Is there any restriction on their positions?

   There will be exactly 1 M at the beginning of the word.

3. Using the above rules, can you start with the word **MI** and derive the word **MU**? If yes, write down your derivation as in part (1). If not, try to explain why not.

   No!! The number of Is can never be a multiple of 3 (i.e. 0). But proving this requires stepping "outside" the system.

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(Naïve) sets

Two ways to write down a set:

- We write down a collection of distinct **elements** separated by commas, between curly braces, and we just call that a set. For example:

  $$S = \{\text{Justin Trudeau}, \text{fork}, 7, \text{refrigerator}\}$$

  We may write fork $\in S$. This means that fork is an element of the set S. When we write the funny curvy $\in$ symbol, we mean "is an element of" or "belongs to".

- Or we specify a **universe** (for example, P = all people), and collect all **elements** in that universe ($x \in U$) satisfying a certain **property**. We use a colon (:) to separate the elements from the property they satisfy. For example:

  $$F = \{x \in P : x \text{ used a fork on February 11, 2020}\}$$

  is the set of all people who used a fork yesterday. Or

  $$T = \{y \text{ an integer} : y \text{ is divisible by } 3\}$$

- We can even make a set with all the above sets as elements. For example:

  $$Z = \{S, F, T\}$$

Exercise 2

1. When all the elements of a set A are also elements of another set B, we say A is **contained** in B, or A is a **subset** of B, and we write $A \subseteq B$. For example, {fork, refrigerator} $\subseteq S$ where S is the first example of a set.

   (a) Let $\mathbb{Z}$ represent the set of integers, let $\mathbb{Q}$ represent the set of rational numbers (fractions), let $\mathbb{N}$ represent the set of whole (natural) numbers, and let $\mathbb{R}$ represent the real numbers (number line). Write down all the containment relationships you can between these sets.

   $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

(b) Give an example of two sets A, B where neither set is contained in the other.

$A = \{1, 2\}$, $B = \{2, 3\}$

(c) Let $Z = \{S, F, T\}$ as in the example.

i. Is $S \in Z$? yes Is $S \subseteq Z$? no Is $\{S\} \subseteq Z$? yes

ii. Is fork $\in Z$? no Is $\{\text{fork}\} \in Z$? no

Is $\{\text{fork}\} \subseteq Z$? no

(d) Write down an example of sets A, B where $A \subseteq B$ and $B \subseteq A$. What is an appropriate name for this relationship and a symbol to denote it?

$A = \{6, 5\}$ $B = \{6, 5\}$

We will call this equality and write $A = B$.

2. We use the symbol $\emptyset$ to denote the set $\{\}$ containing no elements. We call this the **empty set**. For this exercise, let A be any set.

(a) Is $\emptyset \subseteq A$? Is $A \subseteq \emptyset$?

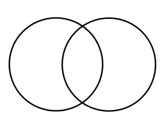
Yes, $\phi \subseteq A$. In general, $A \not\subseteq \phi$, unless $A = \phi$.

(b) Is $\{\emptyset\} = \emptyset$? Why or why not?

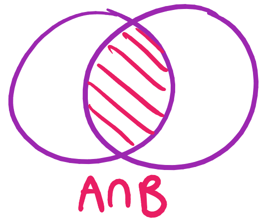
No! $\{\phi\}$ has one element, ϕ, but $\phi = \{\}$ has zero elements.

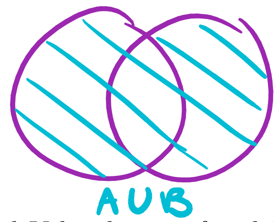
3. There are two common ways to combine two sets. The **intersection** of two sets, denoted $A \cap B$, is the set containing all elements that are in both A and B. The **union** of two sets, denoted $A \cup B$ is the set containing all elements that are in at least one of A, B.

This diagram is called a Venn diagram. Think of the left circle representing the set A and the right circle representing B.



(a) Draw a Venn diagram and highlight the region that corresponds to the intersection $A \cap B$. Draw a second Venn diagram and highlight the region that corresponds to the union $A \cup B$.





(b) Let X be the set of multiples of 5, and Y be the set of multiples of 3. Describe $X \cup Y$ and $X \cap Y$.

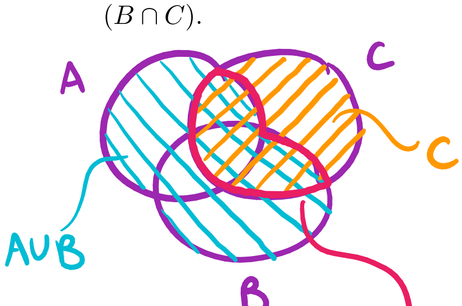
$X \cup Y$ = all multiples of 3 <u>or</u> 5

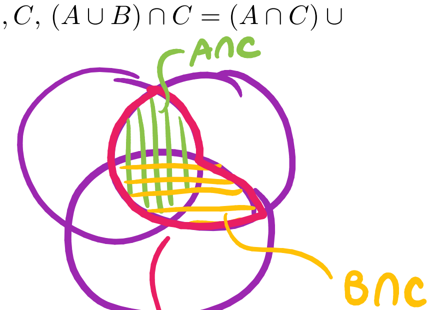
$X \cap Y$ = all multiples of 3 <u>and</u> 5, i.e. all multiples of 15

(c) What are the valid containment relationships between A, $A \cap B$, and $A \cup B$?

$$A \cap B \subseteq A \subseteq A \cup B$$

(d) **Bonus:** Prove that for three sets A, B, C, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.





$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

How should we axiomatize set theory?

Exercise 3

Get into a group of four. Throughout this exercise, we will invent some axioms that anything we call a set should satisfy. Our goal is that anything we previously thought was a set should satisfy these axioms, without introducing paradoxical situations.

1. What is one set we should always have? Write down a rule that declares this object to be a set.

   There is an empty set (denote it by $\emptyset$)

2. If our only objects are sets (imagine, the rest of math does not exist yet), what objects will we put inside our sets? That is, what kind of things can be elements of our sets?

   more sets inside sets!

3. Write down a rule that allows you to build a new set from a single existing set. This might be tricky. Think about your answer to (2).

   If A is a set then $\{A\}$ is also a set.

4. Write down a rule that allows you to combine two existing sets to form a new set. Be precise.

   There are many good answers. For example:

   A, B sets $\Rightarrow$ $\{A, B\}$ also a set

   or

   A, B sets $\Rightarrow$ $A \cup B$ also a set

5. What are some examples of sets that satisfy our first three axioms?

ϕ, $\{\phi\}$, $\{\{\{\phi\},\phi\},\phi\}$

6. How many elements can a set have (according to the rules so far)? How many elements would you like your sets to have? Can we add an axiom to allow this?

Depending on our rules: 2. or any finite whole number.
Perhaps we would like an infinite set.

7. Discuss with your group an axiom that will help prevent something like Russell's paradox from happening. After several minutes of discussion, write down your axiom.

There are several ways to go ... we will see a solution next week.