Introduction to ZFC and Infinity

We're going to spend some time looking at the most prevalent system of axioms for set theory. They are called the Zermelo-Frankel Axioms, and many of them look like the axioms we were looking at last week. We're not going to look at every single axiom, and we're also not going to state all of them the way you would see them in a book. If you would like to see them in more detail, a first place to look is the Wikipedia page.

Some of the ZFC axioms

1. **The axiom of extensionality**: If two sets have the same elements, then then they are the same set.
2. **The axiom of pairing**: Given two elements a and b, there exists a set containing exactly those two elements.
3. **The axiom of specification**: Given a set X, and a property that elements of X may or may not satisfy, the collection "the elements of X with the property" is a set. In particular, this gives us the sets from the set-builder notation that we discussed last time: $\{x \in X : \phi(x)\}$ where $\phi(x)$ is some statement that can be true or false for any given set x.
4. **The axiom of union**: There exists a set Y consisting the union of all of the elements of X.
5. **The axiom of power set**: The power set of X exists. The *power set* of a set X is the set $\mathcal{P}(X)$ consisting of all of the subsets of X.
6. **The axiom of infinity**: There exists an infinite set. (We'll make this more precise in Exercise 2.)

Exercise 1: Playing with ZFC

1. The whole point of axiomatizing set theory was that we didn't want to allow things like Russell's paradox. But the axiom of specification looks like it can create one. Is this a problem? What sort of restrictions would we want to put on the statement $\phi()$ so that $x \in X : \phi(x)$ is not paradoxical? (Hint: Think of this more as a philosophical question than a mathematical one. In order to make this question formal, we would need to get into the mathematical theory of logic, which is a topic for another session!).
2. We said last week that we should have an empty set, but we haven't explicitly included that as an axiom of ZFC. How can we construct the empty set from the axioms above? (Hint: Use the axiom of specification with a property that is impossible to satisfy.)
3. Check that if x is a set, $\{x\}$ is a set. (Hint: Use pairing and extensionality).

Exercise 2: Building objects from sets

Part A: Natural numbers

Now that we've played a bit with our proposed axioms of set theory, it's time to put them to the test: We'll start by building the natural numbers.

1. What set should we consider to be equal to the number zero?
2. What set should we consider to be equal to the number one?
3. The first two questions didn't leave much room for creative answers, however, now we have a choice to make, there are two reasonable candidates for the number two. They are,

$$2 := \{\{\{\}\}\} = \{1\} \qquad \text{or} \qquad 2 := \{\{\{\}\}, \{\}\} = \{1, 0\}.$$

   Which of these is a more appealing notion of "two" and why?

4. Given a set x, we define the **successor of** x to be the set $S(x) := x \cup \{x\}$. In a recursive manner, we define $0 := \emptyset$ and for $n > 0$, $n := S(n-1)$. By this definition, $1 = S(0) = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$.
   (a) By the same logic as above, write 2 as the successor of some number.
   (b) Use the definition of successor to write 2 using only the symbols: $\{ \quad \} \quad \emptyset$.
   (c) Use what you've just done to write 2 using only the symbols: $\{ \quad \} \quad 0 \quad 1$. Does this resemble one of the choices from question 3?
5. **Bonus:** Suppose x is a set, prove (from the axioms of ZFC) that $S(x)$ is also a set. (union, and Ex 1).

At this point, we can construct any particular natural number that we like, but we haven't argued that there is a set which contains all of them: This is what the axiom of infinity is for, it states that "there exists a set X with the empty set as a member, and the property that, for all elements of $x \in X$, $S(x) \in X$. The smallest infinite subset of any set that we obtain from this axiom is the natural numbers, it can be extracted using the specification axiom.

Part B: Ordered Pairs

An **ordered pair** is usually written (x, y) for some numbers x and y. We use ordered pairs in math all the time! Can we use sets to build ordered pairs?

1. What's the problem with representing the ordered pair (x, y) with the notation $\{x, y\}$?
2. A popular way to build an ordered pair from sets is $(x, y) := \{\{x\}, \{x, y\}\}$. Use this definition to check when it's the case that $(x, y) = (y, x)$.

3. Using the above definition, and our definition of the number 0 from Part A, what set represents the ordered pair $(0, 1)$? How about $(0, 0)$? Simplify your answers, noting that a set does not "remember" how many copies of a given element it has.

Part C: Functions

Normally we think of a function as a rule that assigns some output (i.e. $f(x)$) to every input (i.e. x) let's encode this notion in the language of sets: A **function** f with domain D and range R is defined as a set $F = \{(d, r) : d \in D, r \in R\}$ with the additional property that if $(d, r) \in F$ and $(d, s) \in F$ then $r = s$.

1. Explain what the above "additional property" is doing.
2. Consider the function $f = \{(0, 0), (1, 2), (2, 4), (3, 6)\}$. What do we mean when we write $f(2)$? Describe f in "normal language" (what is it doing)?
3. What is the set theoretic definition of the identity function from $\mathbb{N}$ to $\mathbb{N}$ (that is, the function which does nothing to each input). What about the function from some set S to $\{0\}$ that sends each element to zero?
4. In high school, we say a function f is **one-to-one** if whenever $f(x) = f(y)$, we have that $x = y$. In other words, there can never be two different inputs that map to the same output. Define one-to-one formally, using ordered pairs and the set-theoretic definition of a function.
5. We also say a function f from D to R is **onto** if for every r in R, there is some d in D satisfying $r = f(d)$. In other words, the function "hits" every point in the range set. Define onto formally, using ordered pairs and the set-theoretic definition of a function.
6. Which of the following functions from $\mathbb{N}$ to $\mathbb{N}$ is one-to-one? Which is onto?
   (a) $\{(0, 1), (1, 2), (2, 3), (3, 0)\}$
   (b) $\{(0, 1), (1, 3), (2, 3), (3, 1)\}$
   (c) $\{(n, 7) : n \in \mathbb{N}\}$
   (d) $\{(n, n + 1) : n \in \mathbb{N}\}$
   (e) $\{(n, n + 1) : n \text{ even}\} \cup \{(n, n - 1) : n \text{ odd}\}$
7. A function that is both one-to-one and onto is called a **bijection**.

   As you may know, if f is a bijection from D to R then there is an inverse bijection g from R to D so that the composition of g and f is the identity map. Give a set theoretic description of the inverse of a bijection.
8. Call a set **two-like** if there is a bijection from that set to the set $2 = \{0, 1\}$. What can you say about all two-like sets?

9. Let A and B be finite sets and let $|A|$ and $|B|$ denote the number of elements of A and B respectively.
   (a) Let f be a one-to-one function from A to B. Explain why $|A| \leq |B|$.
   (b) Let g be an onto function from A to B explain why $|A| \geq |B|$.
   (c) Let h be a bijection from A to B. What can we conclude about $|A|$ and $|B|$?

Exercise 3: Infinite sizes

As suggested by the exercises at the end of the previous section, bijective maps are useful for understanding the sizes of finite sets. In fact, they are actually used to define infinite sizes! This can lead to some surprising results.

1. Construct a bijection between the natural numbers and the integers. (It takes a bit more work to give a set-theoretic definition of the integers, so just work with the definition that you know and love, but rest assured, we CAN construct the integers from the axioms of ZFC.)
2. From the previous exercise, we see that the integers and the natural numbers have, in some sense, the same size. What about the natural numbers and the real numbers? (Again, with a bit more work we could construct the real numbers set-theoretically. If you're interested look at the Wikipedia page for "Dedekind Cut").

The punchline to this exercise is that you cannot construct a bijection between the natural numbers and the real numbers. Look up "Cantor's Diagonal argument" to see the proof (it's a very pretty proof that we intended to present in class).

So, the natural numbers are one size of infinity, and the real numbers are a larger size. One question we might ask is, are there any sizes of infinity between these two?

This question is called the Continuum Hypothesis, and it is not answerable from within ZFC! It is impossible to prove that it is true AND it is impossible to prove that it is false. This illustrates, in the language of last week, that ZFC is incomplete.

Further reading

If your interest is piqued, here are some books to look at:

- Logicomix - Apostolos K. Doxiadis. A graphic novel covering most of the topics of the last three weeks!
- Satan, Cantor, Infinity - Raymond Smullyan. A collection of logic puzzles designed to teach the core concepts of Gödel's incompleteness theorems.

- Everything and More - David Foster Wallace. A tour of infinity with lovely prose.
- Gödel, Escher, Bach: An Eternal Golden Braid - Douglas Hofstadter. A gigantic treatise on logic, visual art, and music, tied together by themes of self reference and paradox.