Senior Math Circles
February 26, 2020
Special Relativity I

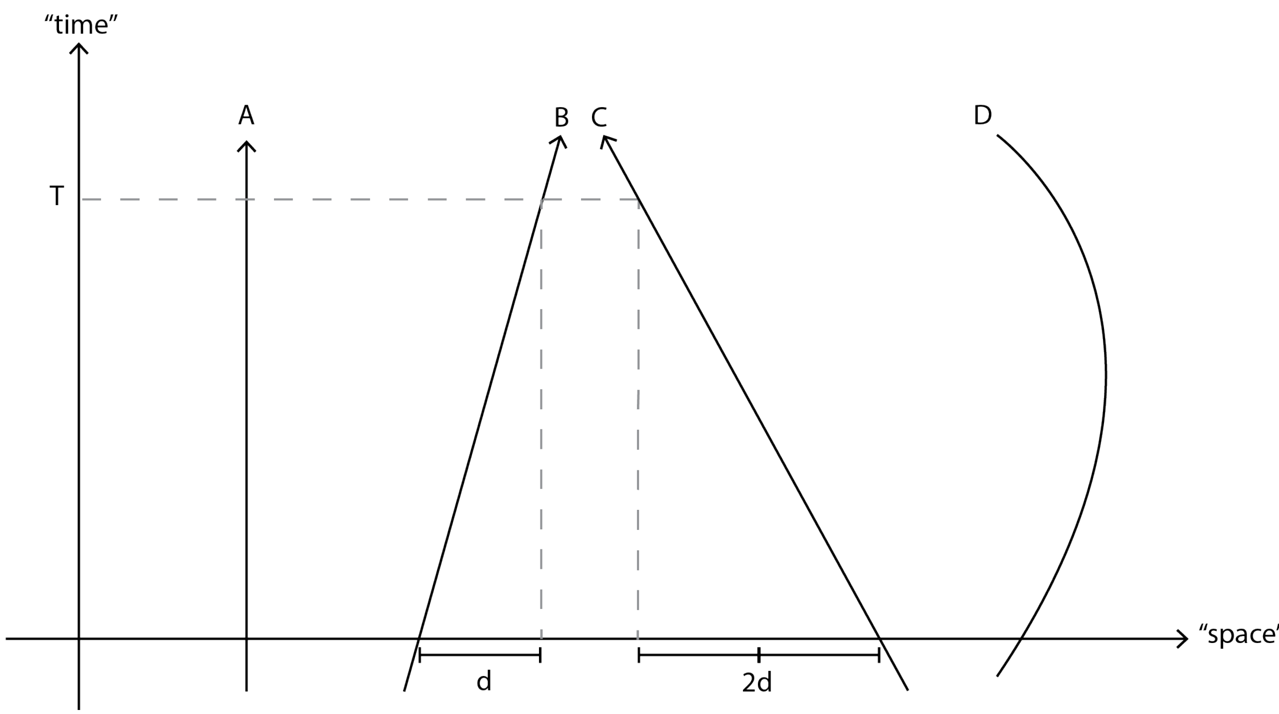
The theory of Special Relativity unites **time** and **space**. Space is not just a static theatre with a universal clock whose timing all observers can agree on. Instead, distances between points and intervals of time between events depend on the motion of an observer.

For this reason, it will be helpful to develop a framework for describing the **trajectories** of objects (i.e. observers or light) through **spacetime**.

We do this with **spacetime diagrams** and we call object trajectories **worldlines**.

A spacetime diagram is just a plot of “time” versus “spatial position” as viewed from a particular inertial reference frame.

Example 1. Describe the motions of the objects with the following worldlines moving in time and one spatial dimension.



A = This object moves through time but its position in space does not change. This is an object **at rest** (i.e. not moving) with respect to the reference frame.

B = As time increases, the position of this object increases at a constant rate. This is an object moving with a constant velocity (to the right) relative to the reference frame.
Note: use variable x to denote spatial coordinate (position) and t to denote temporal coordinate (time)

$$x_b(t) = vt + b$$

C = An object moving at a constant velocity to the left. Notice, it covers twice the distance that B covers in the same time. This means it has twice the speed.

$$x_c(t) = -2vt + c$$

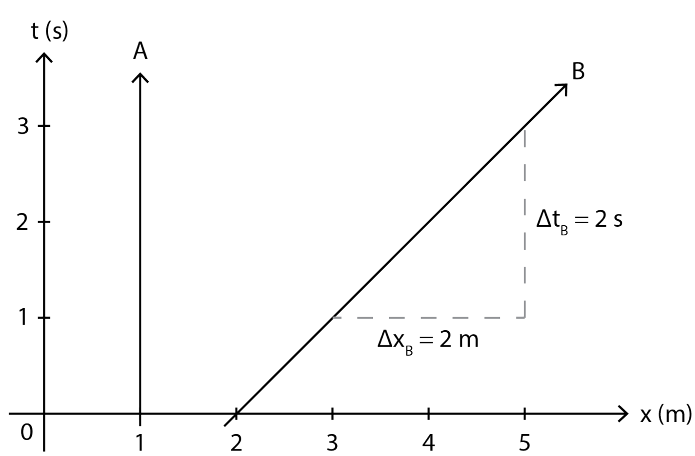
D = This object is initially moving to the right but slows down and changes direction. It is undergoing **acceleration** so we say it is in a **non-inertial** reference frame (i.e. the observer can "feel" their acceleration or measure it with a mass on a spring).

Example 2. Given the following spacetime diagram illustrating the worldlines of observers Alice and Bob as seen from an inertial frame in which Alice is at rest:

a) Determine Bob's velocity as seen by Alice.

b) Construct a spacetime diagram illustrating the same worldlines but from a frame in which Bob is at rest at $x = 5$ m.

c) How would Bob describe his and Alice's trajectories from the point of view of his frame?

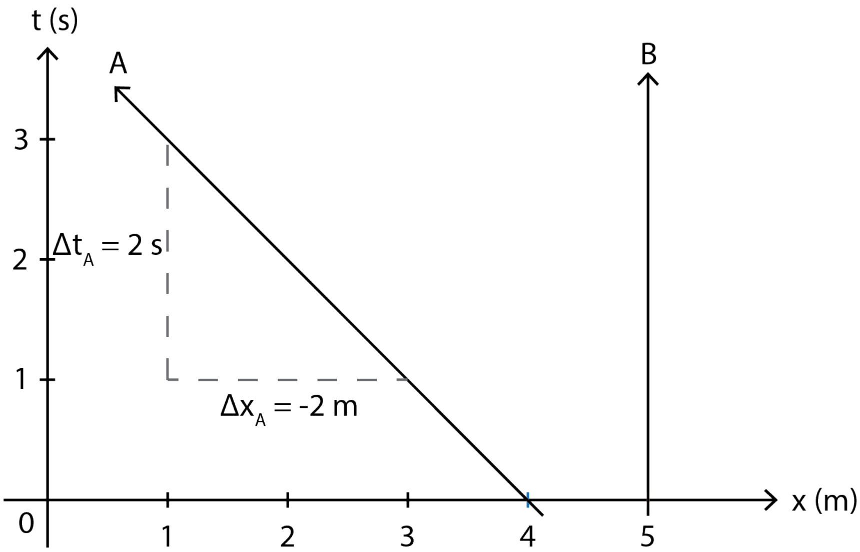


a) Bob's velocity can be determined by looking at how much distance he covers in a given interval of time.

$$v_B = \frac{\triangle\, x_B}{\triangle\, t_B} = \frac{2 \text{ m}}{2 \text{ s}} = 1 \text{ m/s}$$

Notice Bob's velocity is the reciprocal of the slope of his worldline.

b)



c) In Bob's frame, he sees Alice moving with velocity

$$v_A = \frac{\triangle\, x_A}{\triangle\, t_A} = \frac{-2 \text{ m}}{2 \text{ s}} = -1 \text{ m/s}$$

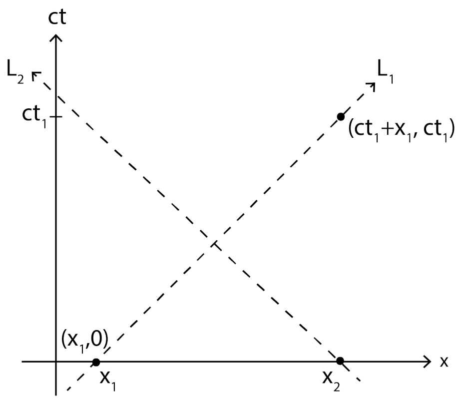
The negative sign indicates movement to the left.

This example illustrates an important idea in physics - **motion is defined relative to a particular frame** (which is often tied to a particular observer).

Moreover, observe that we needed to introduce a new spatial coordinate system when shifting from Alice's rest frame to Bob's rest frame.

It appears that we did not, however, need to introduce a new time coordinate. That is, 2 seconds elapsed for Alice is the same as 2 seconds for Bob. As we'll soon see, this is not entirely accurate — the calculations above are only correct if we take our $\triangle$t values to be those measured in the rest frame.

The effects of Special Relativity (like disagreements in measurements of time intervals) are really only apparent for objects with speeds comparable to the **speed of light**, $c \approx 3 \times 10^8$ m/s. For this reason, it is common to plot spacetime diagrams as ct (instead of just t) versus x or to set $c = 1$ m/s, which we will do later. In a spacetime diagram scaled in this fashion, the worldline of a light ray makes a $\pm 45°$ angle with the axes.



Observe, ct has units of distance. For example, if $t = 1$ s, $ct = (3{\times}10^8 \text{ m/s})(1 \text{ s}) = 3{\times}10^8$ m

For example, consider light ray L_1 with equation

$$x_{L_1}(t) = ct + x_1$$

(where c is velocity and x_1 is the position at $t = 0$)

The slope can be calculated using two points on the line, say

$$(x_1, 0) \text{ and } (ct_1 + x_1, ct_1)$$
$$\implies \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{ct_1 - 0}{ct_1 + \cancel{x_1} - \cancel{x_1}} = 1$$

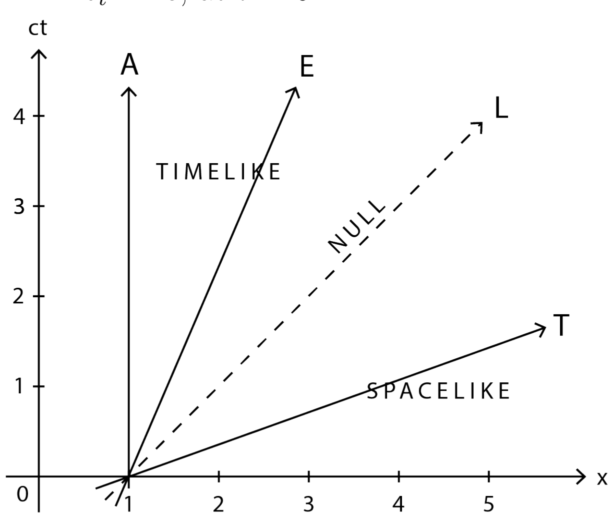
Example 3. Draw a spacetime diagram of ct vs x and sketch the following worldlines.

a) Alice standing at $x = 1$.

b) A pulse of light sent in the positive x-direction by Alice at $t = 0$.

c) An electron whizzing past Alice in the positive x-direction at half the speed of light, $v_e = \frac{c}{2}$, at $t = 0$.

d) A tachyon travelling past Alice in the positive x-direction at twice the speed of light, $v_t = 2c$, at $t = 0$.



We say objects moving at less than the speed of light are on **timelike** worldlines.

$$v < c \implies \text{timelike}$$

Objects moving faster than light are on **spacelike** worldlines.

$$v > c \implies \text{spacelike}$$

And objects moving at lightspeed are on **null** or **lightlike** worldlines.

$$v = c \implies \text{null}$$

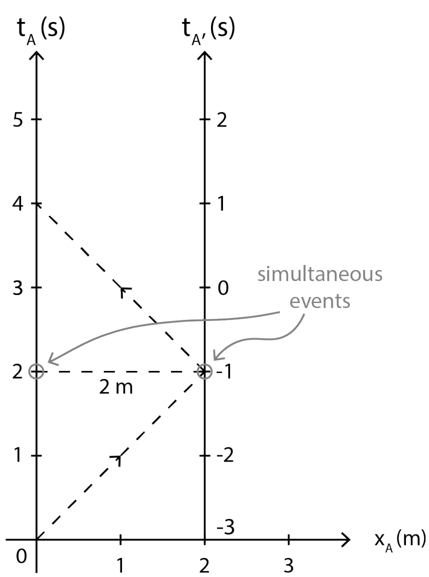
Radar Ranging

As we'll see soon, time elapses at different rates for observers moving relative to one another but to demonstrate this we'll need some way of doing two things:

1. **Synchronizing clocks** between two observers at rest with respect to one another
2. **Measuring the velocity** of a passing observer.

Both of these tasks can be achieved by bouncing around light signals similar to the way a radar gun is used to detect the velocity of a speeding car.

Let's look at clock synchronization first. We'll pretend $c = 1$ m/s to simplify our discussion.



With $c = 1$ m/s, a plot of t vs x has light at $\pm 45°$.

Suppose Alice (A) and Alice-prime (A′) are at rest with one another but spatially separated with clocks out of sync.

Alice can't just yell "set your clock to 12 PM now" to achieve synchronization because it takes time for her sound to reach Alice-prime, so Alice-prime must account for a **time delay**.

This time delay is equal to the distance separating A and A′ multiplied by the speed of sound. Or, if A instead uses a flash of light, it is equal to the distance multiplied by the speed of light.

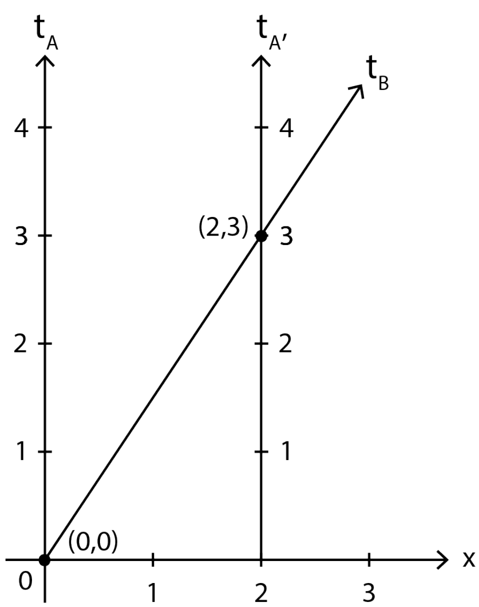
So here's how Alice and Alice-prime can synchronize their clocks:

1. At $t_A = 0$ on Alice's clock, Alice sends a light pulse to Alice-prime.
2. At $t_{A'} = -1$ s on Alice-prime's clock, Alice-prime detects and reflects the light pulse back to Alice.
3. At $t_{A'} = 4$ s on Alice's clock, Alice receives the reflected light pulse.

Alice can now infer that Alice-prime is 2 m away and received her light pulse when her own clock read $t_A = 2$ s. She can now yell to Alice-prime this observation and Alice-prime can move her clock forward or backward accordingly. In this case, she would move it 3 s forward.

Question. Why can't Alice-prime just walk over to Alice, synchronize clocks, and walk back?
In the process of walking back, Alice-prime moves with velocity relative to Alice. As we'll see, this would cause their clocks to immediately fall out of sync.

Now that we can synchronize clocks between observers at different locations within an inertial reference frame, it is simple to determine the velocity of an object moving relative to that frame.

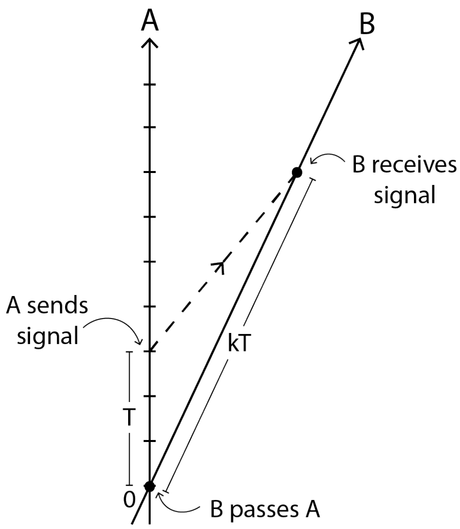


$$v_B = \frac{\text{distance}}{\text{time}} = \frac{(2-0)\text{ m}}{(3-0)\text{ s}} = \frac{2}{3}\text{ m/s}$$

Recall, we're pretending $c = 1$ m/s so we actually have $v_B = \frac{2}{3}c$.

Now let's start trying to answer the big question. In Alice and Alice-prime's frame, 3 seconds elapse between Bob passing Alice and Alice-prime. How much time elapses for Bob between these two events? Galileo and Newton would say 3 seconds, but this turns out to be wrong!

The k-Calculus



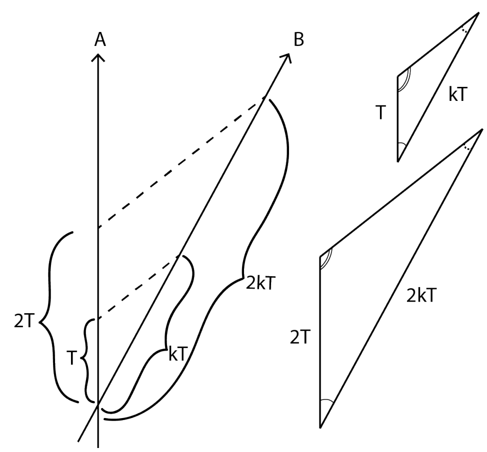
Suppose Bob travels at constant velocity v relative to Alice. At a time T seconds after Bob passes Alice, Alice sends a light pulse to Bob.

Since Bob travels with $v < c$, the light pulse eventually catches up to him.

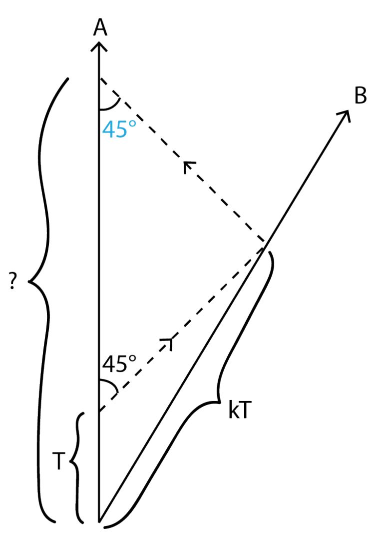
We don't know how much time elapses on Bob's clock between him passing Alice and receiving the light pulse but we know it is proportional to T. Let's say it is kT for some unknown k.

Question. How do we know the time Bob measures is **proportional** to the time Alice measures?

If we scale T, the time Bob measures scales by the same amount by similar triangles (angle-angle).



Next, suppose Bob reflects Alice's light pulse back to her.



Question.

1. How do we know the angle in blue is 45°? (Hint: How does the motion of a source affect the speed of a signal?)
2. How much time elapses between Bob passing Alice and Alice receiving the reflected pulse?

1. The **second postulate** of Special Relativity says:

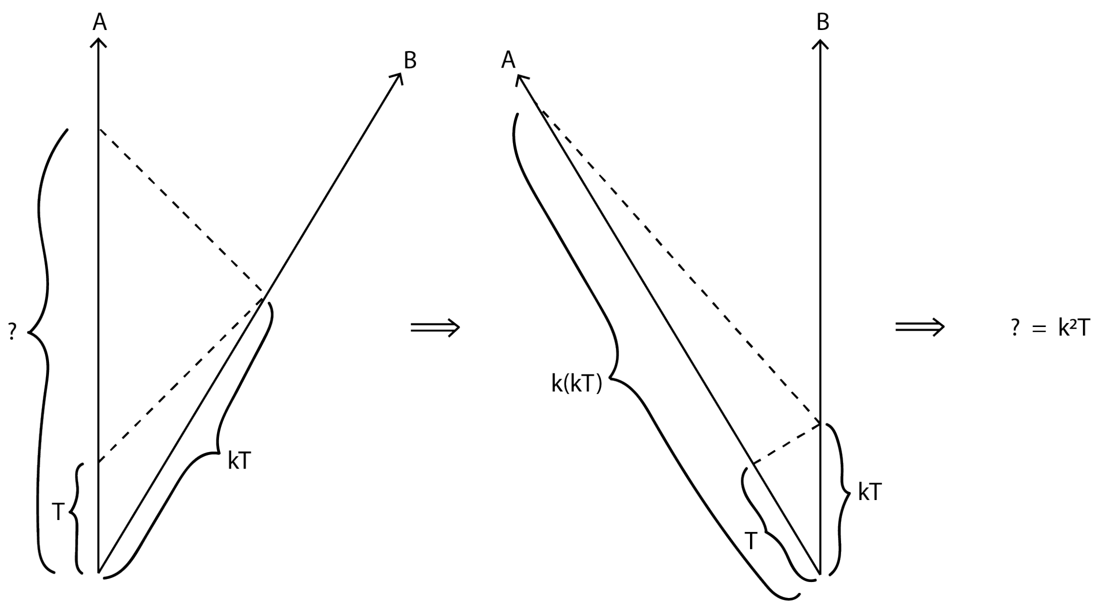
   "The speed of light is independent of the motion of the source."

   so, even though Bob is moving when he reflects the light pulse, Alice still sees it moving at speed c. $\implies$ Angle is 45°

2. The **first postulate** of Special Relativity says:

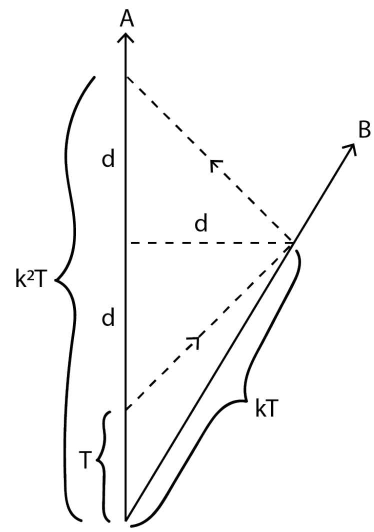
   "The laws of physics are the same in all inertial reference frames."

   So we can redraw our diagram from Bob's perspective (with same k-factor):



   Bob sees Alice moving at speed v. He receives the light signal after time kT in his frame and Alice can therefore apply the same k-factor to this time interval to figure out how much time elapses in her frame before she receives the light pulse.

Let's update our diagram with our newly discovered results.



Let's also denote the distance between A and B when Bob receives the light pulse d. Note, d will be equal to Bob's speed multiplied by the time elapsed since he passed Alice.

$$d = v(T + d)$$

With c = 1 m/s, we can treat d as if it has time or space units.

Observe also that

$$k^2 T = T + 2d$$

This gives us two equations:

$$d = v(T + d) \tag{1}$$
$$k^2 T = T + 2d \tag{2}$$

Question. Solve equations (1) and (2) for k in terms of v. (Hint: First use (1) to solve for d in terms of T.)

By (1) we have,

$$\begin{aligned} d - vd &= vT \\ \implies d(1 - v) &= vT \\ \implies d &= \frac{v}{1 - v} T \end{aligned}$$

Plugging this into (2) gives

$$k^2 \cancel{T} = \cancel{T} + \frac{2v}{1 - v} \cancel{T} \quad \implies \quad k^2 = 1 + \frac{2v}{1 - v}$$

$$\implies k^2 = \frac{1 - v + 2v}{1 - v} = \frac{1 + v}{1 - v}$$

$$\implies k = \sqrt{\frac{1 + v}{1 - v}} \quad \xrightarrow[c=3\times 10^8 m/s]{\text{restoring}} \quad \boxed{k = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}}$$

Now that we have k, we can show all sorts of cool results!