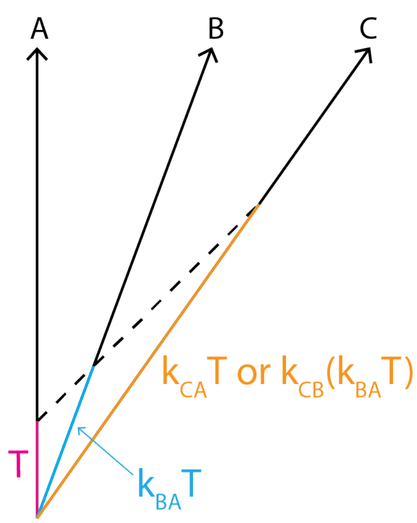
# Senior Math Circles
# March 4, 2020
# Special Relativity II

## Velocity "Addition" Law

Suppose Alice is at rest with Bob moving at speed $v_{BA}$ relative to Alice and Claire moving at speed $v_{CB}$ relative to Bob.

According to Galileo, Alice would measure Claire to be moving at speed $v_{CA} = v_{CB} + v_{BA}$ relative to her rest frame. However, this would allow for speeds in excess of lightspeed so must not be the correct velocity composition formula.

Instead, we can derive the correct formula using our $k$-factor.




The time that elapses on Claire's clock between passing Alice and receiving the light pulse can be determined in two ways:

1. Relating directly to Alice

$$\implies k_{CA}T$$

2. Relating to Bob's time and then relating Bob's time to Alice's

$$\implies k_{CB}(k_{BA}T) = k_{CB}k_{BA}T$$

Since both $k_{CA}T$ and $k_{CB}k_{BA}T$ should describe time that Claire measures, we have

$$k_{CA} = k_{CB}k_{BA} \qquad \implies \qquad \sqrt{\frac{1+\frac{v_{CA}}{c}}{1-\frac{v_{CA}}{c}}} = \sqrt{\frac{1+\frac{v_{CB}}{c}}{1-\frac{v_{CB}}{c}}}\sqrt{\frac{1+\frac{v_{BA}}{c}}{1-\frac{v_{BA}}{c}}}$$

**Exercise 1.** Solve for $v_{CA}$ in terms of $v_{CB}$ and $v_{BA}$ (and $c$). (Hint: To simplify calculation, set $c = 1$).

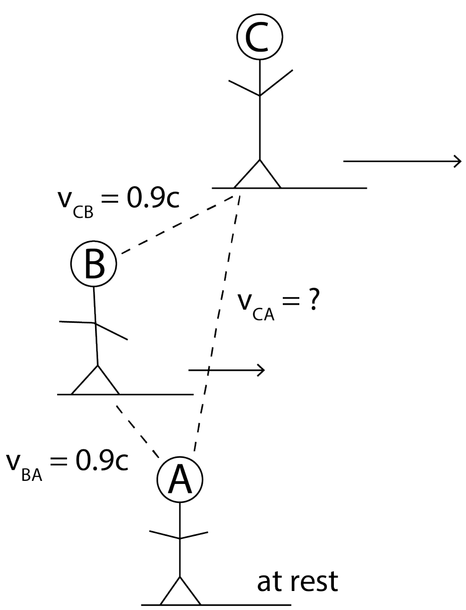
First set $c = 1$ (we'll restore it later) and then square both sides.

$$
\begin{aligned}
\frac{1+v_{CA}}{1-v_{CA}} &= \left(\frac{1+v_{CB}}{1-v_{CB}}\right)\left(\frac{1+v_{BA}}{1-v_{BA}}\right)(1-v_{CA}) \\
\implies \quad v_{CA} &= \frac{(\frac{1+v_{CB}}{1-v_{CB}})(\frac{1+v_{BA}}{1-v_{BA}}) - 1}{1 + (\frac{1+v_{CB}}{1-v_{CB}})(\frac{1+v_{BA}}{1-v_{BA}})} \\
&= \frac{(1+v_{CB})(1+v_{BA}) - (1-v_{CB})(1-v_{BA})}{(1-v_{CB})(1-v_{BA}) + (1+v_{CB})(1+v_{BA})} \\
&= \frac{\cancel{1} + v_{CB} + v_{BA} + \cancel{v_{CB}v_{BA}} - \cancel{1} + v_{CB} + v_{BA} - \cancel{v_{CB}v_{BA}}}{1 - \cancel{v_{CB}} - \cancel{v_{BA}} + v_{CB}v_{BA} + 1 + \cancel{v_{CB}} + \cancel{v_{BA}} + v_{CB}v_{BA}} \\
&= \frac{v_{CB} + v_{BA}}{1 + v_{CB}v_{BA}}
\end{aligned}
$$

Restoring $c$ gives $\boxed{v_{CA} = \dfrac{v_{CB} + v_{BA}}{1 + \frac{v_{CB}v_{BA}}{c^2}} \qquad \text{Velocity Composition}}$

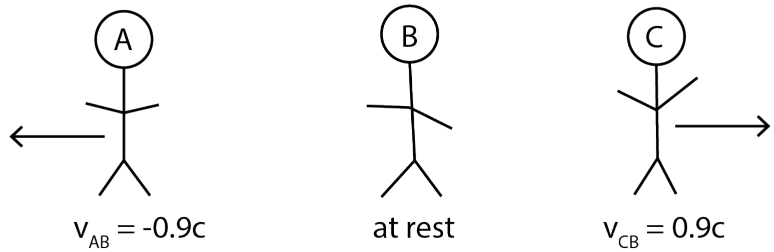
**Question.** Is this result consistent with Galilean relativity at low velocities?
Yes. If $|v_{CB}| \ll c$ and $|v_{BA}| \ll c$, then $|\frac{v_{CB}v_{BA}}{c^2}| \ll 1 \quad \implies \quad v_{CA} \simeq v_{CB} + v_{BA}$

**Example 1.** Suppose Alice observes Bob moving at $v_{BA} = 0.9c$ to the right and Bob observes Claire moving at $v_{CB} = 0.9c$ to the right. How fast does Claire move relative to Alice?




$$
\begin{aligned}
v_{CA} &= \frac{0.9c + 0.9c}{1 + \frac{(0.9c)^2}{c^2}} \\
&= \frac{1.8c}{1 + 0.81} \\
&= \frac{180}{181}c \quad < c \quad \checkmark
\end{aligned}
$$

It is interesting to think about this from Bob's perspective:



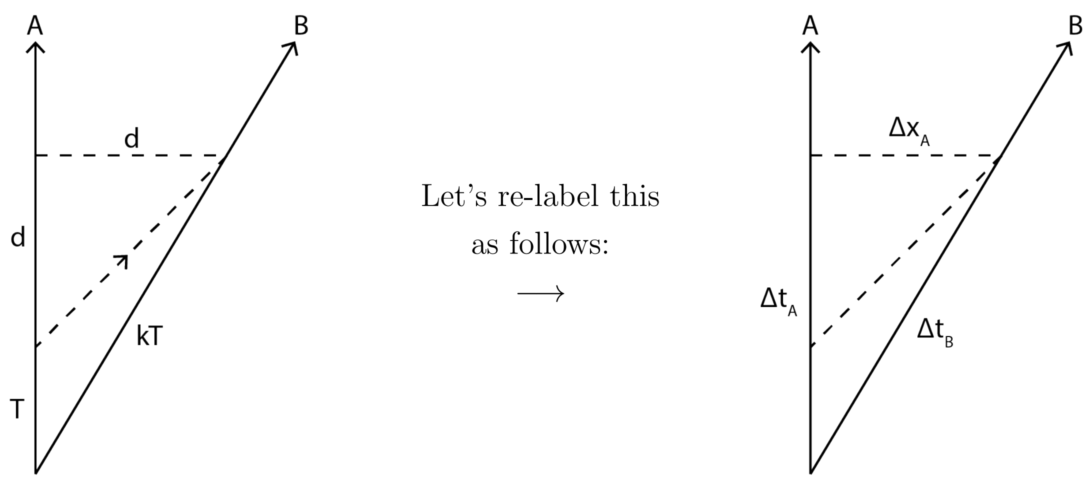

We still have $v_{CA} = \frac{180}{181}c$. So, even though Alice and Claire are moving away from Bob in opposite directions and he sees the distance separating them growing at a rate faster than the speed of light, Alice and Claire do not observe each other to be moving apart faster than light. How is this possible? Alice and Claire must not perceive time and space the same way Bob does.

**Question.** Suppose you're riding on a train at speed $v$ and turn on a flashlight. For an observer at rest with respect to the train, how fast does the light appear to move? What if you were riding on a beam of light?

$$\text{Riding on train: speed} = \frac{v+c}{1+\frac{vc}{c^2}} = \frac{c(1+\frac{v}{c})}{1+\frac{v}{c}} = c \quad \checkmark$$
$$\text{Riding on light: speed} = \frac{c+c}{1+\frac{c^2}{c^2}} = \frac{2c}{2} = c \quad \checkmark$$

# Time Dilation

Consider again the scenario where Alice sends a light pulse to Bob after time $T$ and Bob receives the pulse a time interval $kT$ after passing Alice.



In this new diagram:

- $\Delta t_A$ is the **time Alice measures** from when Bob passes her to when he receives the light pulse.
- $\Delta t_B$ is the **time Bob measures** from when he passes Alice to when he receives the light pulse.
- $\Delta x_A$ is the **distance** between Alice and Bob when Bob receives the light pulse as **measured by Alice**.

What we'd like to determine is how $\Delta t_A$ and $\Delta t_B$ are related.

From the diagrams, we observe:

(1) $\Delta t_A = T + d$

(2) $\Delta t_B = kT$

and we recall from earlier, (3) $d = \frac{v}{1-v}T$

Plugging (3) into (1) gives $\Delta t_A = T + \frac{v}{1-v}T = T(1 + \frac{v}{1-v}) = \frac{1}{1-v}T$

Therefore,

$$\frac{\Delta t_B}{\Delta t_A} = \frac{kT}{\frac{1}{1-v}T} = (1-v)k \qquad \text{Recall } k = \sqrt{\frac{1+v}{1-v}}$$

$$= \sqrt{(1-v)(1+v)}$$

$$= \sqrt{1-v^2}$$

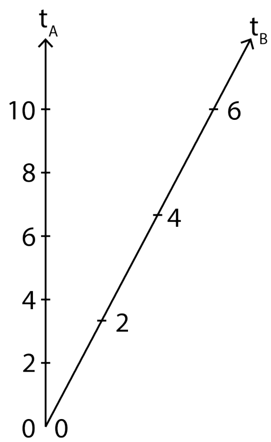
$$\to \sqrt{1-\frac{v^2}{c^2}} \qquad \text{Restoring } c = 3 \times 10^8 \text{ m/s}$$

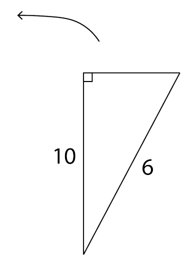
$$\boxed{\Delta t_B = \sqrt{1-\frac{v^2}{c^2}}\Delta t_A \qquad \text{Time Dilation}}$$

**Question.** Bob travels past Alice at speed $v = \frac{4}{5}c$ and both carry clocks. Alice watches Bob's clock while 10 seconds ticks by on her own clock. How much time does she determine elapses on Bob's clock over that 10 second interval?

If $v = \frac{4}{5}c$ and $\Delta t_A = 10$ s,

$$\begin{aligned} \implies \Delta t_B &= \sqrt{1 - \frac{v^2}{c^2}}\Delta t_A \\ &= \sqrt{1 - \left(\frac{4}{5}\right)^2}(10\text{ s}) \\ &= \left(\frac{3}{5}\right)(10s) \\ &= 6\text{ s} \end{aligned}$$




Something weird is going on with geometry here!

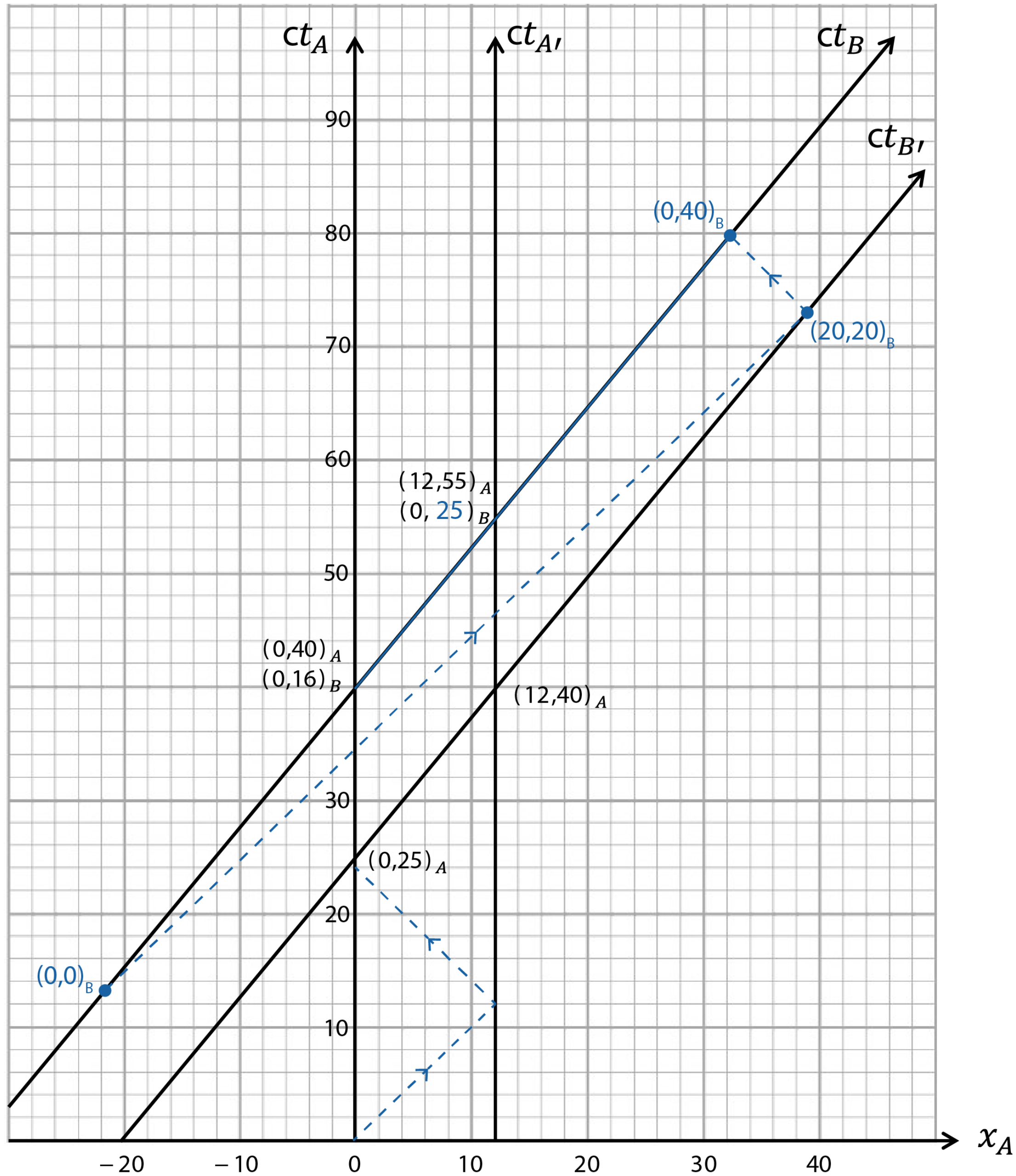
$\implies$ Alice sees Bob's clock ticking at 60% normal rate.

**Question.** At what rate does Bob see Alice's clock tick?
Also at 60% normal speed. Bob is equally entitled to view his frame as being at rest and from his frame, it is Alice that moves at speed $v = \frac{4}{5}c$.

It is important to keep in mind that time dilation is not a mechanical effect of some sort affecting only clocks. It is time itself that passes differently in different frames of reference. In the first example above, Alice would also see Bob moving slowly, his heart beating slowly, his voice slowed down, etc. Of course, Bob experiences none of these effects but instead observes them affecting Alice.

$\implies$ Special relativity is inconsistent with the notion of a "universal" Newtonian time by which all observers can synchronize their clocks.

**Question.** What happens to Bob's clock, as seen by Alice, as he approaches the speed of light?
As $v \to c$, $\sqrt{1 - \frac{v^2}{c^2}} \to 0$ so $\Delta t_B = \sqrt{1 - \frac{v^2}{c^2}}\Delta t_A \to 0$.
Bob's clock gets slower and slower as he approaches the speed of light. If he were able to travel at light speed, his clock would appear to stop!




1) How could A and A′ determine their separation distance? Radar ranging
2) At what speed are B and B′ travelling relative to A and A′? 4/5 c
3) At what time would B say he passes A′? $t_B = 25s$
4) Use a ruler to carefully locate t = 0 on Bob's worldline?

5) At t = 0 (in his frame) B sends a light pulse to B′ which B′ reflects back to B. Draw the worldline of this light ray.
6) Use a ruler to carefully determine at what time B receives the reflected light pulse. $t_B = 40$ s
7) What would B and B′ say is their separation distance? 20 m (compare to A and A′ w/o measure 12 m)

2) $v_B = \frac{\Delta x}{\Delta t}$
$= \frac{12}{15} c$
$= \frac{4}{5} c$

3) $\Delta t_B = \sqrt{1 - \frac{v^2}{c^2}} \Delta t_A$
$= \sqrt{1 - \frac{16}{25}} (15\ s)$
$= 9\ s$
$\Rightarrow t = 16\ s + 9\ s$
$= 25\ s$