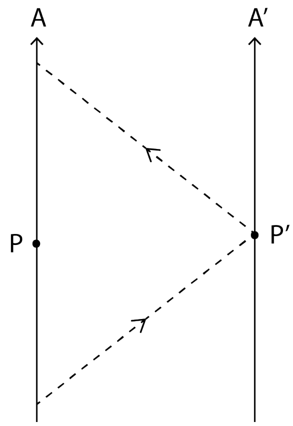
Senior Math Circles
March 11, 2020
Special Relativity III

Relativity of Simultaneity

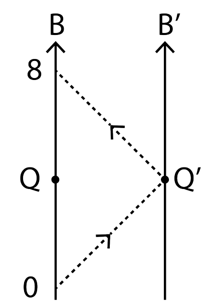
Recall the process for a pair of observers to synchronize their clocks.



Alice sends a light signal to Alice-prime who reflects it back to Alice. Alice can divide the transit time in half to establish that events P and P′ are **simultaneous**.

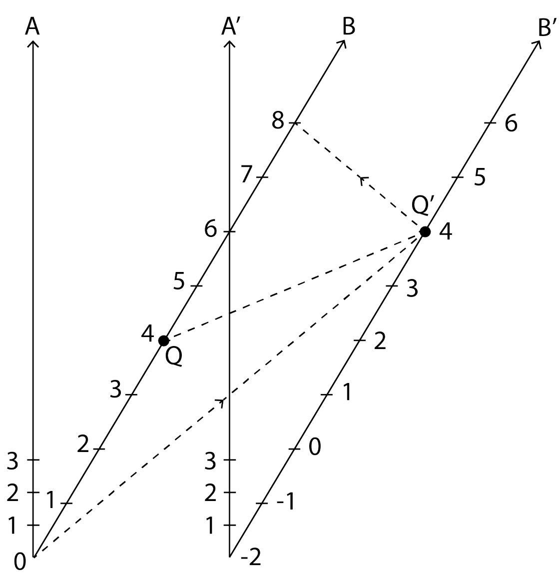
Now suppose Bob and Bob-prime are moving at a speed v relative to Alice and Alice-prime such that B and B′ pass A and A′, respectively, at $t_A = 0$.

How would B and B′ synchronize their clocks?



They could use the same procedure as A and A′ (but in their frame) to find events Q and Q′ which they deem to be simultaneous.

But what would this look like to A and A′?



Suppose 8 seconds elapse for Bob while waiting for the light pulse to return.

He would note $t_B = 4$ s as the time of event Q and instruct Bob-prime to do the same for event Q′ but observe what this implies:

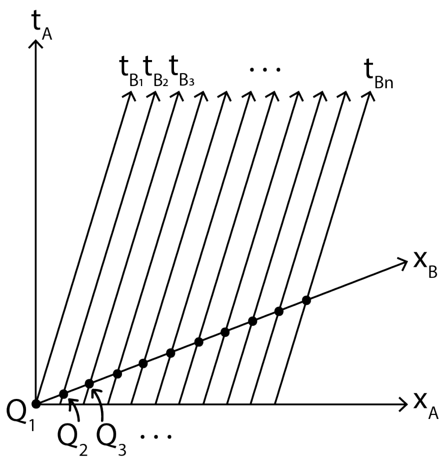
Events Q and Q′ are **simultaneous** for B and B′ but **not simultaneous** for A and A′.

Specifically, A and A′ would say event Q happens *before* event Q′.

We call this the **relativity of simultaneity**. That is, whether or not an observer deems two events to be simultaneous depends on their frame of reference (i.e., the velocity).

Question. What are the implications of relativity of simultaneity on how different observers will measure the distance between fixed points in space?

Imagine the same experiment but with a line up of observers in the Bob frame.



The Bobs would find that they can make a continuous line of events they would call simultaneous.

Just like a lineup of Alices would call all events taking place at $t_A = 0$ their spatial axis (i.e. the x_A axis), the Bobs would agree on a set of simultaneous events, call it $t_B = 0$ and treat it as their spatial axis (i.e. an x_B axis).

$\implies$ The spatial axis of a moving observer gets "tilted" relative to the rest frame.

Let's do an exercise to see how distances between points vary between frames as a consequence of this effect. (See exercise on separate sheet).

In the exercise, we observe that not only will observers in different reference frames disagree on the times between events but also the distances between points in space.

A careful analysis using radar ranging (or other techniques) shows the following result:

> Suppose an object would have length L_o when viewed in its rest frame. If the same object moves at speed v, its length appears **contracted**. The apparent length, L, is given by

$$\boxed{L = \sqrt{1 - \frac{v^2}{c^2}} L_o \quad \text{Length Contraction}}$$

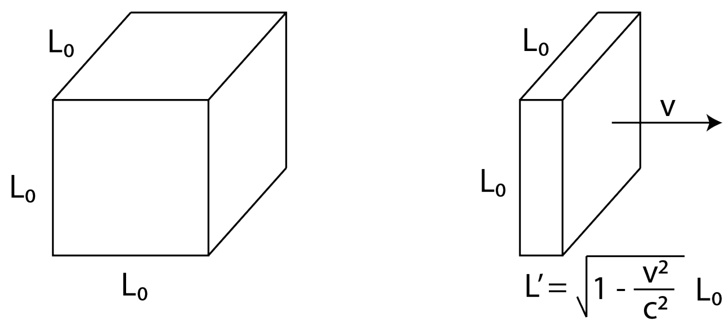
*Notice that the $\sqrt{1 - \frac{v^2}{c^2}}$ term is the same term as in time dilation.

$$\sqrt{1 - \frac{v^2}{c^2}} < 1 \implies L < L_o.$$

Example 1. A meter stick moves at 60% lightspeed past Alice. How long does the meter stick appear to be to Alice?

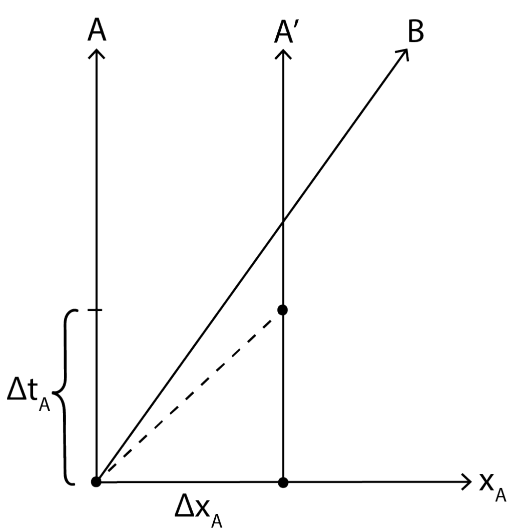
$$L_o = 1\text{ m},\ v = \frac{3}{5}c$$
$$\implies \sqrt{1-\frac{v^2}{c^2}} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$
$$\implies L = \sqrt{1-\frac{v^2}{c^2}}L_o = \left(\frac{4}{5}\right)(1\text{ m}) = 0.80\text{ m}$$

Note, length contraction only has an effect along the direction of motion.



Length contraction and time dilation go hand-in-hand to maintain consistency between what observers "see" and the consequence of our postulates which is that the speed of light is constant for all observers.

For example, consider a pulse of light travelling between two points in Alice's rest frame. Suppose she measures these points to be a distance L apart. The time for light to travel between the points would be $\Delta t_A = \frac{L}{c}$.

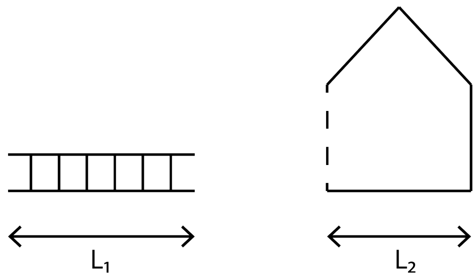


Moving relative to Alice, Bob would see the distance between these two points as less than what Alice says since he sees it length-contracted.

But Bob still sees the light travelling at speed c so must also measure less time passing during the trip the light takes.

Ladder Paradox

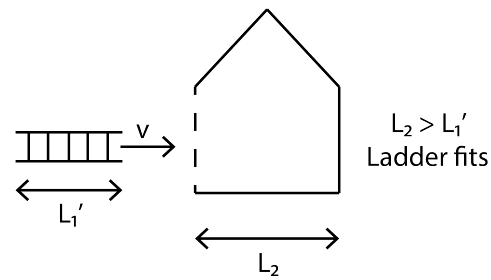
Consider a ladder whose length at rest is slightly longer than the depth of a barn in which we'd like to store the ladder.



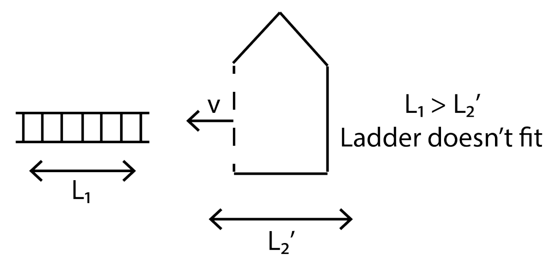
At rest, the ladder won't fit into the barn since $L_1 > L_2$.

Can we use relativity to make it fit?

Suppose the ladder is moving toward the barn at a very high speed. From the Barn-frame, the ladder will appear shortened so it should now fit.

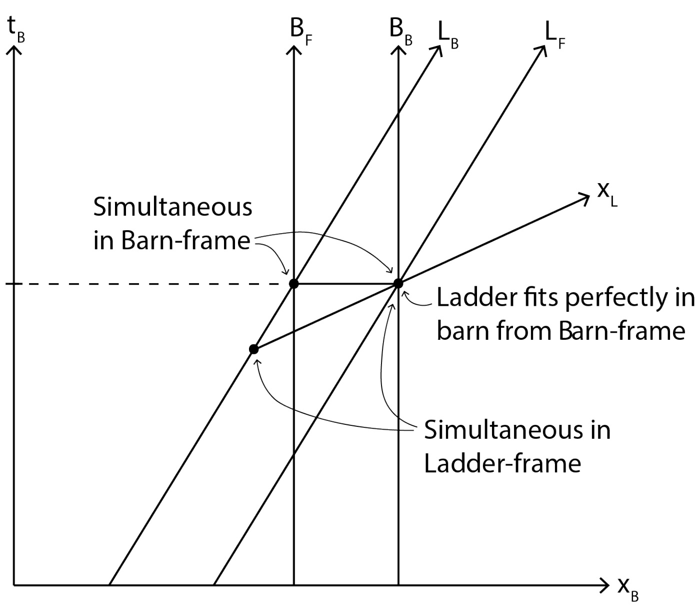


Is this reasoning correct? It seems to fall apart when we look at the situation from the Ladder-frame. This is because from the ladder's point of view, it is the barn that gets length contracted.



This apparent inconsistency is the paradox!

To help us resolve the paradox, let's look at a spacetime diagram from the Barn-frame of the ladder entering the barn.



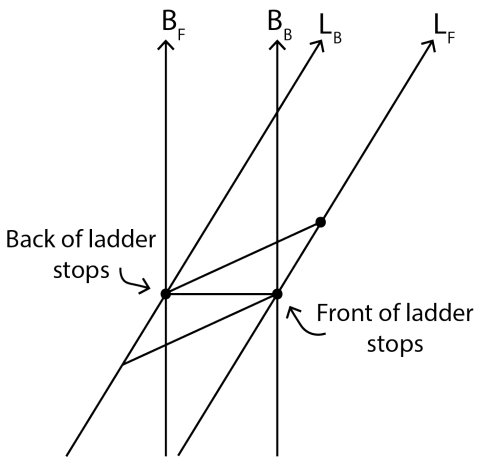
$B_{F,B}$ = Front/back of barn
$L_{F,B}$ = Front/back of ladder

Due to the relativity of simultaneity, when the front of the ladder hits the back of the barn, the back of the ladder would still be outside the barn, as seen from the Ladder-frame which is consistent with pictures above.

So what actually happens?

There is nothing wrong with what we see happening in the Barn-frame. The ladder *does* fit in the barn. How do we reconcile this with what happens in the Ladder-frame?

Consider from both frames how the ladder decelerates once the front of the ladder reaches the back of the barn.



From the Barn-frame, suppose the ladder stops instantaneously once it is in the barn.

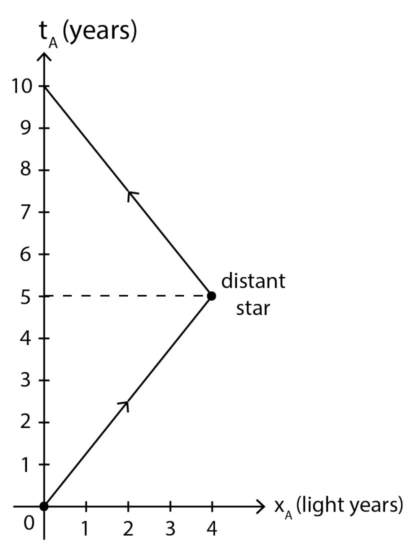
From the Ladder-frame, the front of the ladder stops before the back of the ladder. In other words, the back keeps moving after the front stops.

$\implies$ Ladder gets compressed (or breaks) gradually from the front to back in the Ladder-frame.

In the Barn-frame, this compression (or breakage) takes place simultaneously across the full length of the ladder as soon as the ladder stops and therefore begins returning to its rest length.

Twin Paradox

Consider the following spacetime diagram illustrating a round trip Bob takes to a nearby star 4 lightyears away while Alice remains on Earth.



Suppose $t_A = 0$ corresponds to the present year, 2020.

(a) According to Alice, in what year does Bob return to Earth?

(b) According to Bob, how long does each leg of his trip take? What about the full trip?

(c) Why are these two times different? Doesn't Bob see Alice moving at the same speed (with just one quick change in direction)?

(a) According to Alice, Bob returns 10 years after leaving which would be 2030.

(b) Bob is always moving at velocity $v = \frac{4}{5}c$ relative to Alice so his clock runs slow. For each leg we have

$$\Delta t_B = \sqrt{1 - (\tfrac{4}{5})^2}\Delta t_A = \tfrac{3}{5}(5 \text{ years}) = 3 \text{ years}.$$

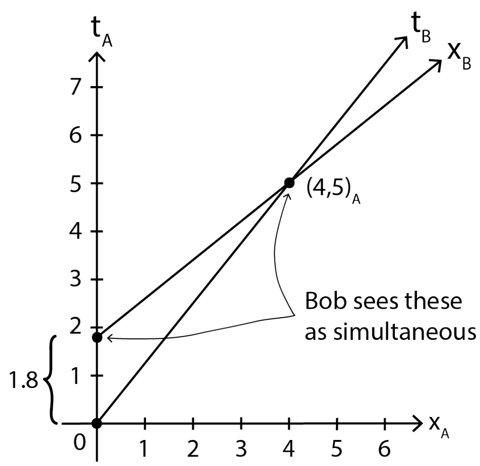
Therefore, Bob would say his trip takes a total of 6 years.

(c) Recall, Bob's spatial axis will be tilted relative to Alice's. We can figure out by how much by determining how much time he sees elapse for Alice on each leg of his trip.

On the way out, Bob sees Alice's clock running slow. When he arrives at the star, he would say Alice has experienced

$$\Delta t_A = \tfrac{3}{5}(3 \text{ years}) = \tfrac{9}{5} \text{ years} = 1.8 \text{ years}.$$

This means that Bob sees (0,1.8) and (4,5) as simultaneous events.

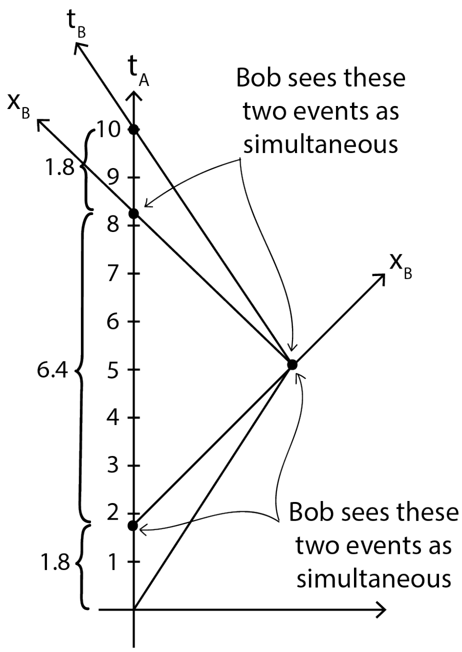


The set of all simultaneous events occurs on a line parallel to the spatial axis, so another way of putting this is that these events are both on Bob's spatial axis tilted in time.

So Bob does indeed see Alice's clock running slow while he's heading toward the star, and we'll find the same is true for his return trip. How is it possibly true then that his clock ticks less time than Alice's at the end of the trip?

The answer lies in what happens when Bob turns around!

When Bob turns around, his spatial axis — signifying events he sees as simultaneous — changes orientation dramatically.



As soon as Bob turns around, his spatial axis flips and he believes $(4, 5)_A$ and $(8.2, 0)_A$ are simulataneous events.

This makes it possible for him to experience a 3 year trip home while only seeing 1.8 years elapse on Alice's clock.

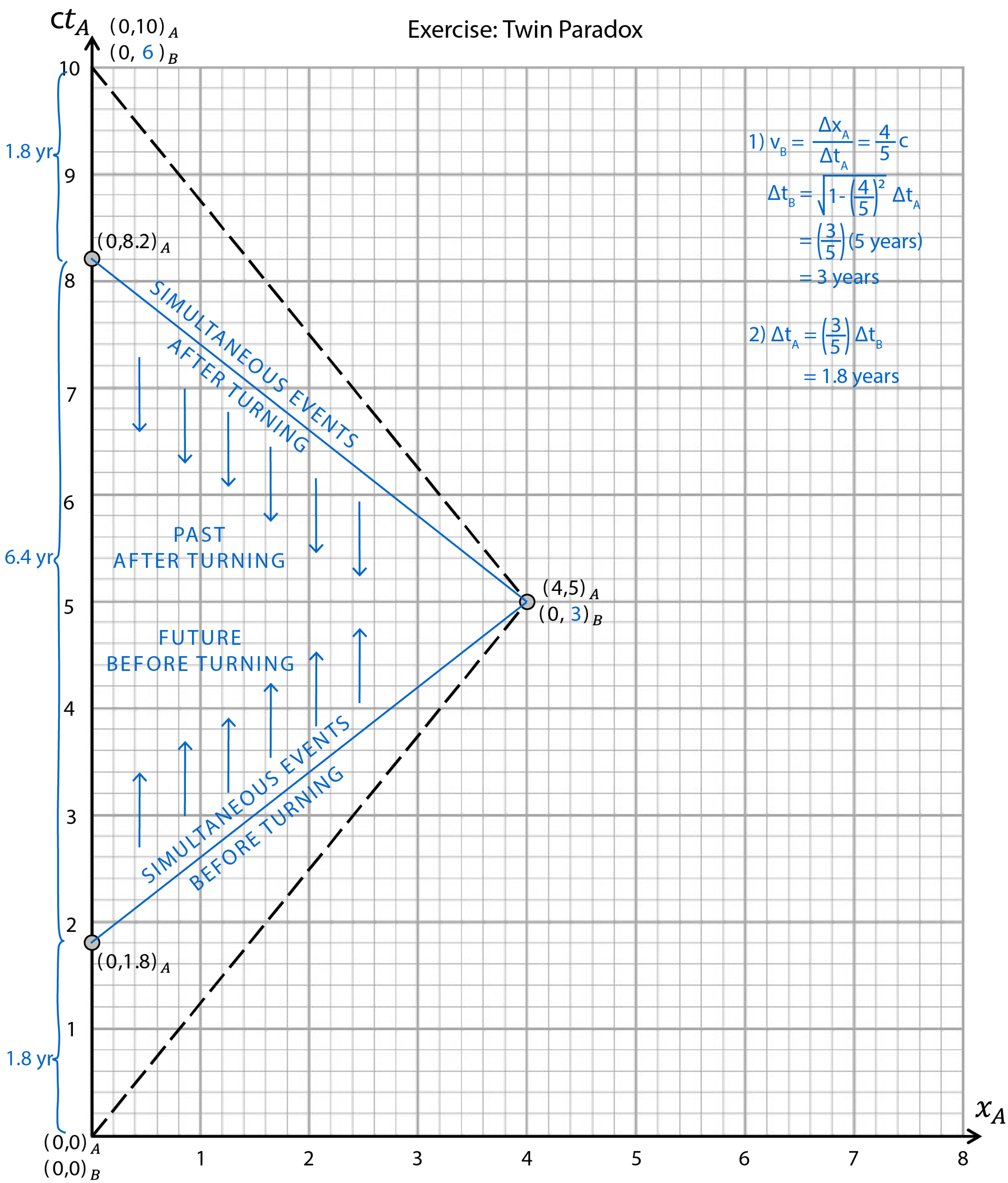
However, he also believed (before turning around) that $(0, 1.8)_A$ and $(4, 5)_A$ were simulataneous.

This means that in the process of turning around, everything on Alice's clock from $t_A = 1.8$ to $t_A = 8.2$ (a total time of 6.4 years) goes from being in Bob's future to his past.

In other words, Bob skips 6.4 years of Alice's time. This is how it is possible for Alice to determine Bob's trip would take 6 years but for Bob to still reckon that he should only "see" a total of 3.6 years elapse on Alice's clock since she is moving relative to him.

I put the word "see" in quotations above because Bob would actually be seeing light coming from Alice and he would observe all 10 years of Alice's time on Earth but the rate at which he sees events occuring increases (by a factor of 9) as soon as he turns around. These light signals all begin in Bob's past but what he determines to be in "his past" changes.

Exercise: Twin Paradox



1) What is Bob's velocity? In his frame, how long does each leg of the trip take?
2) For Bob's outward trip, how much time does he think elapses on Alice's clock?
3) Draw Bob's spatial axis when he arrives at the star but before turning around?
4) Draw Bob's spatial axis just after he turns around?