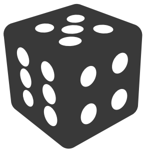
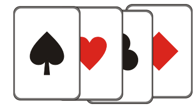


CEMC Math Circles - Grade 11/12
November 11 - 17, 2020
Mathematical Games - Solutions



Here are the solutions to this weeks problems. There is a description of each game, followed by a discussion about the winning strategy.

Warm-up Activity: 21 Flags (described here)

The rules of **21 Flags** are the following:

- There are 21 flags.
- Two players take turns removing either 1, 2, or 3 flags.
- The player who takes the last flag wins.

Solution. In this game, you always want to go first! This is because if you go first, no matter what your opponent does, you can arrange that the number of flags remaining is a multiple of 4. For instance, if there are 4 flags left and it's your opponents turn, *no matter what they do*, you can win! Similarly, if there are 8 flags left and it's your opponents turn, you can make sure that on their next turn, there are 4 flags left (and so on). Since there are 21 flags to start, if you go first you will always be able to win.

This game illustrates a concept we'll see a few more times: the idea of *preserving* some kind of condition. Most often (though not in this game), this takes the form of *parity*, i.e. whether something is even or odd. □

A Rook on a Chessboard:

A rook is placed on the bottom right square of an 8×8 chessboard. On each players turn, they move it any number of spaces to the left, or any number of spaces up (never to the right or down). The player who moves the rook to the top left square wins.

Solution. In this game, you always want to go second. No matter what your opponent does, move back to a diagonal square. Since you can't move the rook to a new diagonal square without first moving off, you will eventually win! □

The Calendar Game:

Players take turns writing down dates. The first player must begin by writing down January 1. After this, the next player takes the previous date and may increase either the month or the day arbitrarily, but *not both*. For example, the second player could choose January 12, or May 1, but not Feburary 2. The player who writes down December 31 wins.

Solution. This game is secretly the same as **The Calendar Game**! Working backwards, notice that if you write down November 30, you will win. Indeed, no matter what your opponent chooses, you will be able to write down December 31. Similarly, if you write down October 29, no matter what your opponent chooses, you will be able to write down either November 30, or December 31. Continuing in this fashion, the same logic applies to September 28, August 27, and so on. In particular, if you go first and write down January 20, you will be able to win!

To see that this is the same as **The Calendar Game**, write the dates and months down on a grid: the winning positions are just the diagonal squares! □

The Left Handed Queen:

A queen is placed near the bottom right square of an 8×8 chessboard. On each players turn, they can move it any number of spaces to the left, diagonally up and to the left, or up. The player who moves the queen to the top left square wins.

Solution. Working backwards, we can label the chessboard with winning and losing positions. To start, we can label all squares where we can get to the top left corner in a single move; these are winning positions. Next, if a square *only* has moves to winning positions, it must be a losing position! Similarly, any square with a move to a losing position is a winning position, and so on. Continuing in this way, we can fill in the whole chessboard. In particular, the starting position given is a winning position: if you go first, you can always win.

If we add labels to the chessboard starting at zero, then the (top) losing positions have coordinates:

$$\{(1,2),(3,5),(4,7),(6,10),(8,13),(9,15),(11,18),(12,21),\ldots\}$$

Can you figure out what this sequence is? □

Easy Nim:

The game begins with two piles of stones. One has 5 and the other has 7. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone loses.

Solution. **Easy Nim** is much easier than **Nim**. This is just another example of a diagonal game! The winning strategy here is to go first, and make the number of stones in both piles equal. Once you get to two piles of two stones, *no matter what your opponent does*, you will be able to force them to take the last stone. □

Nim: (can be played online here)

The game begins with five piles of stones. There are $1, 2, 3, 4$, and 5 stones in each pile (to save space, we might write this as $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$). On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone loses.

Solution. This game is by far the hardest on this list! If there are only two piles (**Easy Nim**) then this is just a diagonal game like **A Rook on a Chessboard**. In other words, $m \oplus n$ is a winning position exactly when $m \neq n$ (if this is the case, your move should be to make them equal). However, if there are three or more piles, you might have to draw out all the possible moves to understand the winning and losing positions. For instance, $1 \oplus 2 \oplus 3$ is a losing position, and $1 \oplus 2 \oplus 3 \oplus 4$ is a winning position. However, there's a very nice (but slightly complicated) solution that works for *any* number of piles and stones.

One thing we'll need is the idea of a *binary number*. It's basically the same thing as a decimal number, but we're only allowed to use ones and zeros. Instead of expanding the digits of a number by powers of ten, we use powers of two (we usually include a subscript of 2 to indicate that this is a binary number). For example:

$$101101_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 32 + 8 + 4 + 1 = 45$$

To work backwards, write your decimal number as a sum of single powers of two (think about why you can always do this!). For example:

$$61 = 32 + 16 + 8 + 4 + 1 = 111101_2$$

What does this have to do with **Nim**? Let's write the number of stones in each pile in binary. For example, think about the piles $13 \oplus 4 \oplus 7$ and $11 \oplus 3 \oplus 9 \oplus 1$. In fact, these are examples of a winning and losing position. If we add up the number of ones in each column, we get the following:

$$\begin{array}{rr} 13: & 1101 \\ 4: & 100 \\ 7: & \underline{111} \\ & 1312 \end{array} \qquad\qquad \begin{array}{rr} 11: & 1011 \\ 3: & 11 \\ 9: & 1001 \\ 1: & \underline{1} \\ & 2024 \end{array}$$

Notice that in the second case ($11 \oplus 3 \oplus 9 \oplus 1$) every column had an even number of ones in it. In the first case ($13 \oplus 4 \oplus 7$), there are some columns with an odd number of ones in them. In fact, this is how we can distinguish winning and losing positions! If every column has an even number of ones in it, then it is a losing position, and if there is a column with an odd number of ones in it, then it is a winning position. We just have to figure out why this works.

To see that we can always go from a winning position to a losing position, think about our example above. Consider the leftmost column that has an odd number of ones in it, and pick one of the piles that has a one in it. You can change the numbers in this row so that all columns will have an even number of ones in them! For instance, we can change the 13 pile from 1101 to 0011 (or in other words, from 13 to 3).

$$\begin{array}{rr} 3: & 0011 \\ 4: & 100 \\ 7: & \underline{111} \\ & 0222 \end{array} \qquad \begin{array}{rr} 7: & 0111 \\ 3: & 11 \\ 9: & 1001 \\ 1: & \underline{1} \\ & 1024 \end{array}$$

To see that any move from a losing position ends in a winning position, think about what happens if you take any number of stones away from some pile. In particular, you must change *at least one* digit in some row, but no other rows! In other words, there will be a column with an odd number of ones in it. For example, think about what happens if we change the pile of 11 to a pile of 7 (or any other number). If you try this strategy for **Easy Nim**, you will see that this is the same diagonal strategy as before.

Once again, parity was important! Can you use this to check whether $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ is a winning or losing position? □

Disclaimer: There are no solutions provided for the challenge problems. If you're interested in one of them or you think you have a solution, get in touch!