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Grade 7/8 Math Circles

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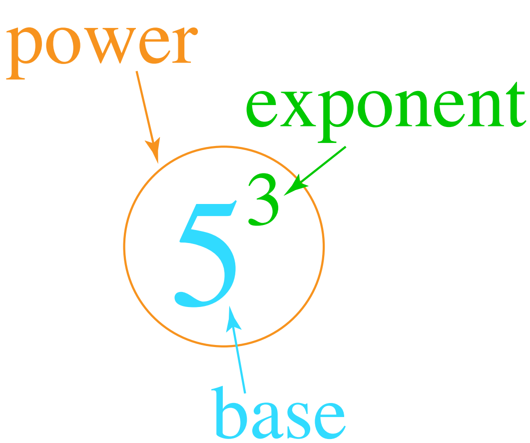
Squares & Radicals

Powers

Multiplication is often introduced as repeated addition e.g. $5 \times 3 = 5 + 5 + 5$. Similarly, **exponentiation** is repeated multiplication e.g. $5^3 = 5 \times 5 \times 5$. This operation makes it easier to write and use recurrent multiplications.

Take a look at the picture below and observe the notation used for exponents.



The large number is called the **base** and the small number at the top is the **exponent**. The exponent indicates how many times to use the base in a multiplication. In the figure to the left with the example 5^3, the 3 signifies that 5 is being multiplied 3 times. It can then be rewritten as $5 \times 5 \times 5$.

When we refer to the base and exponent together, we use the term **power**. In words, we can say that 5^3 is "5 to the exponent 3", "5 to the third power" or simply, "5 cubed". Similarly, when we have a base to the exponent 2 such as 7^2, we can refer to the power as 7 squared.

Example 1: Evaluate the following powers.

a) 6^4 b) 3^5 c) 8^2 d) 9^3

Solution:

a) $6^4 = 1296$ b) $3^5 = 243$ c) $8^2 = 64$ d) $9^3 = 729$

Special Bases & Exponents

The base and exponent of a power can be any real number, including the numbers 0 and 1 and negative numbers as well.

Base to the Exponent 1: Any base, b, to the exponent 1 returns the value of the base.

$$b^1 = b$$

For example, 99^1 simplifies to 99. In other words, we are multiplying 99 by itself just once which is 99. This can be thought of in the same way that 99×1 means to add 99 to itself just once, which is again 99.

Base to the Exponent 0: Any base, b, not equal to zero to the exponent 0 is equivalent to 1.

$$b^0 = 1$$

Using 99 as our base again, we see that $99^0 = 1$. We will show one explanation of this rule later on.

Base 0 & 1: If 1 is our base, for any exponent, x, the power is equal to 1. Similarly, when the base of our power 0, for any exponent, y, not equal to 0, the power is equal to 0.

$$1^x = 1 \qquad\qquad 0^y = 0$$

So, 1^{1597} is equal to 1 and 0^{1900} is 0.

Negative Exponent: When the exponent is negative, $-x$, take the **reciprocal** of the base, b, to the corresponding positive exponent. The **reciprocal** of a number is easily found by calculating 1 divided by that number.

$$b^{-x} = (1 \div b)^x = \left(\frac{1}{b}\right)^x$$

For example, to evaluate 7^{-2}, first take the reciprocal of 7 which is $1 \div 7 = \frac{1}{7}$. Then, evaluate $\left(\frac{1}{7}\right)^2$. To take the power of a rational base, take the power of the numerator and denominator. So, $\left(\frac{1}{7}\right)^2 = \frac{1^2}{7^2}$ or $\frac{1}{49}$.

Negative Base: Let $(-b)$ be a negative base, x be an even exponent, and y be an odd exponent. A negative base to an even exponent will result in a positive value while a negative base to an odd exponent will result in a negative value.

$$(-b)^x = b^x \qquad\qquad (-b)^y = (-1) \times b^y = -(b^y)$$

Consider the power $(-5)^4$. Since the exponent is even, we can directly evaluate $(-5)^4$ as 5^4 which computes to 625. Now, look at $(-6)^3$. The exponent is odd so the result will be negative. Thus, $(-6)^3 = -(6^3)$. As 6^3 equals 216, we get that $(-6)^3 = -216$.

Expand: Watch this video to explore the rule of powers with negative bases: https://youtu.be/Z2S7N2kuBBY.

Example 2: Evaluate the following powers.

a) 101^0

b) 8^{-3}

c) 0^{27}

d) $(-16)^4$

e) $(7+e)^1$

f) $\left(\frac{1}{19}\right)^{-2}$

g) 1^π

h) $(-3)^{-5}$

Solution:

a) $101^0 = 1$

b) $8^{-3} = \left(\frac{1}{8}\right)^3 = \frac{1}{512}$

c) $0^{27} = 0$

d) $(-16)^4 = 16^4 = 65,536$

e) $(7+e)^1 = 7+e$

f) $\left(\frac{1}{19}\right)^{-2} = 19^2 = 361$

g) $1^\pi = 1$

h) $(-3)^{-5} = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$

Squares

As you are probably familiar, a **square** is a shape with 4 equal sides and 4 equal right angles. To calculate the area of a square, we multiply the length, l, and width, w, of the square. So,

$$A_{square} = l \times w$$

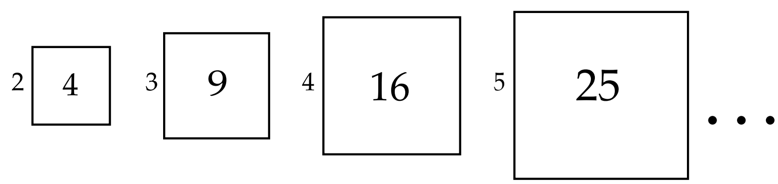
However, the length and width are equivalent. If we denote the side length of a square as s, then

$$A_{square} = s \times s$$

Using our knowledge of exponents, we can simplify this formula to $A_{square} = s^2$.

A **perfect square** is a number that formed by squaring a whole number. In other words, when the side length of a square is a whole number, the area of the square is called a perfect square.

In the figure to the right, take a look at the squares and note the given side length and area. Since the side lengths are all whole numbers, the value of the area formed by each square would be called a perfect square.



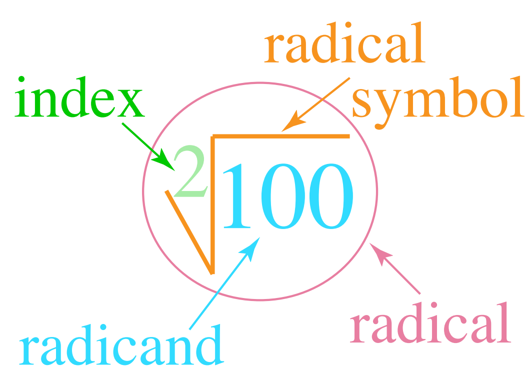
Square Roots

The *inverse*, or opposite, operation of exponents is roots. A **square root** is when we are given a square and must find the *positive number* that is multiplied by itself, or squared, to get the original number. We can describe it as being given the area of a square and computing the corresponding side length. If given a square, n, we use the notation, $\sqrt{n}$, to signify that we are calculating the square root of n.

As an example, let's look at the perfect square 169. Then, $\sqrt{169}$ will correspond to the positive number that when multiplied by itself, or squared, equals 13. Observe that $13^2 = 169$ so, the square root of 169 is 13.

Radicals

We can generalize the above operation to solve for the n^{th} root of any number such that n is a positive integer. That is to say, for a number, x, we can ask what number multiplied by itself n times results in x. This can be denoted as $\sqrt[n]{x}$. For example, the cube root of 27, or $\sqrt[3]{27}$ is 3 since $3^3 = 27$.



The term $\sqrt[n]{x}$ is referred to as a radical. Similar to powers, radicals have different parts to their notation. Underneath the radical symbol, $\sqrt{}$, is the **radicand**; the value we are taking the root of. To the left of the $\sqrt{}$ symbol is the **index** which indicates how many times the root has been multiplied by itself to get the radicand. In the given example, the index is 2, representing that the root has been multiplied 2 times to get the radicand 100. When the index is 2, it does not need to be indicated. Also, if an index is omitted, it is assumed to be 2.

Example 3: Evaluate the following radicals.

a) $\sqrt[4]{81}$ b) $\sqrt[5]{32}$ c) $\sqrt{144}$ d) $\sqrt[3]{125}$

Solution:

a) $\sqrt[4]{81} = 3$ b) $\sqrt[5]{32} = 2$ c) $\sqrt{144} = 12$ d) $\sqrt[3]{125} = 5$

Most of the radicals that you will see are square roots. As such, the rest of this lesson will deal with these exclusively.

Operations with Radicals

Radicals of the form $a\sqrt{x}$ and $b\sqrt{x}$, where a, b can be any number and x is a positive integer, can be combined through addition and subtraction. That is, two radicals with the same radicand can be added or subtracted.

(*Note:* The radical $a\sqrt{x}$ is $a \times \sqrt{x}$. When the integer a is equal to 1, it is not indicated and the radical is left as $\sqrt{x}$. For example, $\sqrt{2}$.)

When adding two radicals, we add the two numbers a and b. Similarly, when subtracting two radicals, we subtract the two numbers a and b. So,

$$a\sqrt{x} + b\sqrt{x} = (a+b)\sqrt{x}$$
$$a\sqrt{x} - b\sqrt{x} = (a-b)\sqrt{x}$$

When multiplying two radicals, $a\sqrt{x}$ and $b\sqrt{y}$, the product is $(a \times b)\sqrt{x \times y}$. Note that if both radicals have the same radicand i.e.$\sqrt{x} \times \sqrt{x}$, the product is the radicand. For example, $\sqrt{2} \times \sqrt{2}$ returns the value 2. So, a square root *squared*, $(\sqrt{x})^2$, returns the number x.

Example 4: Express the following expressions as a single radical.

a) $8\sqrt{3} - \sqrt{3} + 7\sqrt{3}$

b) $\sqrt{26} \times \sqrt{26} \times \sqrt{15}$

Solution:

a) $8\sqrt{3} - \sqrt{3} + 7\sqrt{3} = (8 - 1 + 7)\sqrt{3} = 14\sqrt{3}$

b) $\sqrt{26} \times \sqrt{26} \times \sqrt{15} = 26 \times \sqrt{15} = 26\sqrt{15}$

Simplifying Radicals

Use the following steps to simplify radicals of the form $\sqrt{x}$, where x is a positive integer:

1. Find two factors of the radicand, x, such that at least one of the two factors is a perfect square greater than 1.
2. Simplify the square root of the perfect square to its corresponding value.
3. Repeat the above two steps if the radicand still has a factor that is a perfect square greater than 1.

We consider the radical to be in simplified form if the radical is simplified into an integer, or an integer multiplied by a radical whose radicand has no factor greater than 1 that is a perfect square.

Example 5: Simplify.

a) $\sqrt{14} \times \sqrt{8}$

b) $\sqrt{1024}$

c) $\sqrt{735}$

Solution:

a) $\sqrt{14} \times \sqrt{8} = \sqrt{14 \times 8} = \sqrt{112} = \sqrt{4} \times \sqrt{28} = 2 \times \sqrt{4} \times \sqrt{7} = 2 \times 2 \times \sqrt{7} = 4\sqrt{7}$

b) $\sqrt{1024} = \sqrt{64} \times \sqrt{16} = 8 \times 4 = 32$

c) $\sqrt{735} = \sqrt{49} \times \sqrt{15} = 7 \times \sqrt{15} = 7\sqrt{15}$

Not all radicals can be expressed as a whole number. For example, $\sqrt{2}$ cannot be expressed as a whole number and is approximately equal to 1.4132135624. Thus, it is often better represented as simply $\sqrt{2}$.