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Grade 7/8 Math Circles

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Triangles

Introduction

Triangles are polygons with three edges and three vertices. They are one of the basic shapes in geometry. Trigonometry is the study of triangles. The word trigonometry is derived from the Greek words *trignon* meaning "triangle" and *metron* meaning "measure". It is used in many STEM (Science, Technology, Engineering, Mathematics) disciplines and in everyday applications such as, computer animation and architecture.

Triangles have two main properties: they have three sides and three angles AND the sum of the three angles equals 180°.

A triangle can be categorized by the number of equal sides or angles. They can be further categorized by the type of angle inside the triangle.

Type	# of Equal Sides	# of Equal Angles
Equilateral	3	3
Isosceles	at least 2	at least 2
Scalene	0	0

Table 1: Types of Triangles

Observe that we can have a triangle that can be described in two ways. For example, an equilateral triangle is also an acute triangle or we can have an isosceles right triangle.

Type	Definition
Acute	All angles are less than 90°
Obtuse	Has an angle greater than 90°
Right	Has a right angle (90° angle)

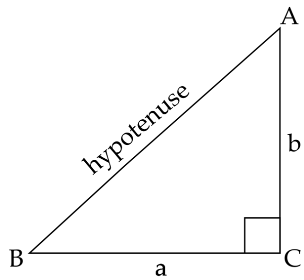
Table 2: Triangle Definitions by Angle

For this lesson, we will focus on the most interesting triangle, the right-angled triangle.

Pythagorean Theorem

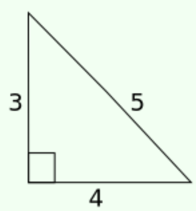
As defined in the previous section, a right-angled triangle, or simply a right triangle, is a triangle where one of the three angles is a right angle. That is, one angle is equal to 90°. We use a little box to show the right angle.

The side opposite to the right angle has a special name and is called the **hypotenuse** of the triangle. It is the longest side of a right triangle.



Pythagoras was a Greek mathematician who observed that the square of the hypotenuse is equal to sum of the squares of the other two sides. We summarize this as $a^2 + b^2 = c^2$ for a right triangle ABC where c is the length of the hypotenuse. This is called Pythagoras' Theorem, or the Pythagorean Theorem.

Example 1: Consider the triangle below.



$$\begin{aligned} 3^2 + 4^2 &= 9 + 16 \\ &= 25 \\ &= 5^2 \end{aligned}$$

Observe that if we substitute the two legs into the Pythagorean Theorem, the sum of their squares is equal to the hypotenuse squared. So, this right triangle upholds the Pythagorean Theorem.

Using this relationship between the three sides of a right triangle, we can solve for a missing side using the other two sides.

Example 2: While bored in geography class, Hans drew a right triangle with a base of 9 units and height of 12 units. Before he could measure the hypotenuse, the bell rang. How long is the hypotenuse of Hans' triangle?
Solution: Since we know the two sides of the triangle, we can substitute a and b in the formula for the Pythagorean Theorem and solve for c, the hypotenuse.

$$\begin{aligned} (9)^2 + (12)^2 &= c^2 \\ 81 + 144 &= c^2 \\ 225 &= c^2 \\ \sqrt{225} &= \sqrt{c^2} \\ c &= 15, \text{since } c > 0 \end{aligned}$$

Therefore, Hans' triangle has a hypotenuse of length 15 units.

Observe that while $\sqrt{c^2}$ has two possible answers, $-c$ and c, we only consider positive c. This is because a triangle must have a side length greater than zero (i.e. must be positive).

Additionally, the two sides that we are given do not necessarily have to be the two legs of the right-angled triangle. We can solve for the remaining side given the hypotenuse and one of the two sides.

Example 3: Rio looked at Hans' triangle and realized that the hypotenuse is actually 15 units long and the height is 6 units long. What is the length of the remaining side?
Solution: This time we know the hypotenuse, c, and one of the sides, let's call it a. After substituting a and c in the formula for the Pythagorean Theorem, we can arrange the formula and solve for the remaining side, b.

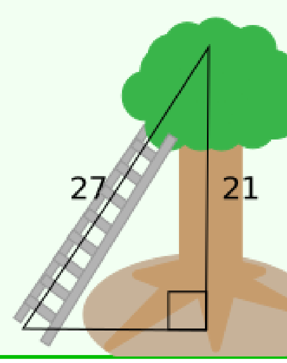
$$\begin{aligned}(6)^2 + b^2 &= (15)^2 \\ 36 + b^2 &= 225 \\ b^2 &= 225 - 36 \\ b^2 &= 189 \\ \sqrt{b} &= \sqrt{189} \\ b &= \sqrt{189}\end{aligned}$$

Therefore, the remaining side length is $\sqrt{189}$ units.

(*__Note:__ The number* 189 *is not a perfect square so,* $\sqrt{189}$ *is better represented as a radical. However, it can be simplified to* $3\sqrt{21}$.)

The Pythagorean Theorem is applicable to real-life scenarios such as, calculating the distance between two points or the height of an object.

Example 4: A cat is stuck on top of a tree that is 21 m tall. The firefighters use a 27 m tall ladder to save the cat. How far is the base of the ladder from the base of the tree?
Solution: If we draw a picture of the problem, we can note that a right triangle is formed using the tree, ground, and ladder. Observe that the length of the hypotenuse is 27 m or the length of the ladder and we know one of the two sides is 21 m since that is the height of the tree.



Using the Pythagorean Theorem, we can solve for the distance on the ground between the tree and ladder as it is the remaining side of the right triangle. So,

$$\begin{aligned} 21^2 + b^2 &= 27^2 \\ 441 + b^2 &= 729 \\ b^2 &= 729 - 441 \\ b^2 &= 288 \\ b &= \sqrt{288} \\ b &\approx 16.971 \end{aligned}$$

Therefore, the base of the ladder is approximately 16.971 m from the base of the tree.

There are multiple proofs of the Pythagorean Theorem. In fact, while this theorem is named after Pythagoras, it was discovered by other Indian, Greek, Chinese, and Babylonian mathematicians way before it was popularized by Pythagoras.

The following activity guides you through one particular proof of Pythaogras' Theorem.

Try it Yourself: Prove the Pythagorean Theorem

To do this activity, you will need two pieces of paper, a pencil or pen, colouring utensils, and scissors. Use this GeoGebra app to help visualize the proof: `https://www.geogebra.org/m/efyy6fza`.

Part 1: Square of the Hypotenuse

1. Draw a large square in the center of one page (this is square $WXYZ$ in the GeoGebra app).
2. Colour in the square.
3. On a second piece of paper, draw a right triangle. Let's call it Triangle A. The two legs of Triangle A can be labeled a and b, and the hypotenuse will be c.
4. Colour Triangle A a different colour than the square.
5. Cut out the triangle and line up the two legs with the sides of the square, aligning the right angle of the triangle with the top right corner of the square.

6. Draw three right triangles to align with the other three corners of the square. Let's call them Triangle B, Triangle C, and Triangle D. These will be congruent to Triangle A so, each triangle will have two legs of side length a and b, and hypotenuse of length c.

7. Colour these triangles the same colour. Choose a colour that is different than the one used for the square and Triangle A.

8. Cut out the triangles and line up the two legs of each triangle with the sides of the square, such that, the right angle of each triangle is aligned with the corner of the square. Place Triangle B to be on the bottom left corner, Triangle C on the top right corner, and Triangle D on the bottom right corner.

Observe that the four triangles make a square in the center which has the same side length as the hypotenuse of the triangles, specifically, c. If the hypotenuse of Triangle A is c, then the area of this square is c^2.

Part 2: Square of the Base and Height

1. Move Triangle A and align its hypotenuse with the hypotenuse of Triangle D.

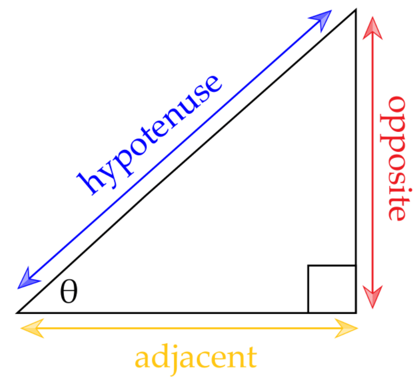
2. Slide Triangle B up to the top left corner of the square.

3. Slide Triangle C left, aligning its hypotenuse with the hypotenuse of Triangle B.

Observe that the four triangles now form two rectangles. The space not covered by these two rectangles forms two squares: one that has the same side length as the height of Triangle A and one that has the same side length as the base of Triangle A. So, if the height of Triangle A is a, and the base is b, then the area of the two squares are a^2 and b^2 respectively.

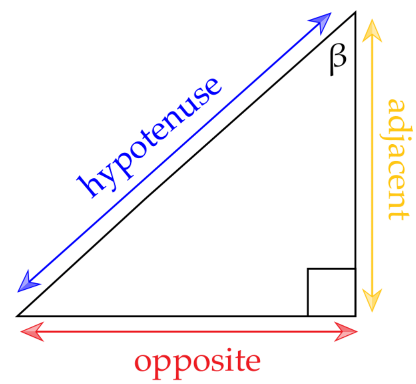
Throughout this proof, we are still working with the same square $WXYZ$ and the same four triangles. So, the area of the square originally formed by the four squares is equal to the areas of the two squares formed by rearranging the triangles. Therefore, the area of the square formed using the hypotenuse of Triangle A, c^2 is the same as the area of the other two squares, $a^2 + b^2$. So, we get $a^2 + b^2 = c^2$ or the Pythagorean Theorem.

Sine, Cosine and Tangent



Consider our original right-angled triangle with the labeled hypotenuse. If we label one of the two angles other than the right angle as θ, then the other two sides of our triangle have specific names.

The side next to, or adjacent, to the angle θ is called the **adjacent** side. Then, the side opposite to the angle θ is called the **opposite** side. If we look at a triangle with the remaining angle labeled as β, then the adjacent and opposite sides change. In either case, the hypotenuse does not change.



In trigonometry, there are three primary ratios between the three sides of a right-angled triangle. They are called **sine**, **cosine**, and **tangent** which are often abbreviated to sin, cos and tan.

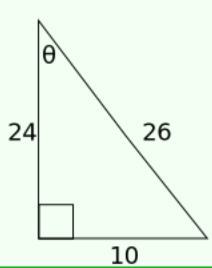
For an angle θ, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

Example 5: Let $\theta = 15°$. Use the sin, cos, and tan, buttons on a calculator to solve all three primary ratios of the corresponding right-angled triangle. So, the trigonometric ratios are

$$\sin(15°) = 0.258819... \approx 0.26$$
$$\cos(15°) = 0.965925... \approx 0.97$$
$$\tan(15°) = 0.267949... \approx 0.27$$

If the side lengths of the triangle from the above example are scaled to be larger or smaller, the ratios of the angle θ will remain the same. That is, the sine, cosine, and tangent values will be equal to the ones calculated above. However, even without the angle θ, we can compute the ratios of the triangle using its side lengths.

Example 6: Consider the right-angled triangle below. Find all three primary trigonometric ratios of the angle θ.



Solution: In relation to our angle θ, the three sides of the triangle can be labeled as such - opposite $= 10$ adjacent $= 24$ hypotenuse $= 26$ Since sine is the ratio between the opposite side and the hypotenuse, $\sin\theta = \frac{10}{26} = \frac{5}{13}$. Since cosine is the ratio between the adjacent side and the hypotenuse, $\cos\theta = \frac{24}{26} = \frac{12}{13}$. Since tangent is the ratio between the opposite and adjacent sides, $\tan\theta = \frac{10}{24} = \frac{5}{12}$.