



CEMC Math Circles - Grade 11/12

February 24 - March 2, 2021

It's True Because it Isn't False!

Background

What do you think of the following multiple choice question?

Exactly one of the following statements is true. Which one is it?

- (a) 6 is a prime number.
- (b) $x = 2$ is a solution to the equation $x + 3 = 4$.
- (c) The point $(1, 2)$ lies on the line with equation $y = x + 10$.
- (d) The CEMC was founded in 1995 with origins dating back to the 1960s.
- (e) $2^0 = 0$

The correct answer is (d) and you probably got it right without knowing anything about the history of the CEMC. How did you do that? You may have used a useful trick for answering a multiple choice question: elimination!

Answer (a) is false because $6 = 2 \times 3$ so 6 is not prime.

Answer (b) is false because when $x = 2$, the value of $x + 3$ is $2 + 3 = 5$ and $5 \neq 4$.

Answer (c) is false because when $x = 1$, the value of $x + 10$ is $1 + 10 = 11$ and $11 \neq 2$.

Answer (e) is false because $2^0 = 1$ and $1 \neq 0$.

Now that we have eliminated four of the five answers, we know that (d) must be the true statement. Elimination works here because we are told that exactly one of the statements is true. Without that information, we cannot confidently answer the question (unless we are CEMC history buffs!).

Example 1: Consider the statement “There is no greatest positive integer.”

How can we eliminate the possibility that this statement is false?

We know that the statement is either true or false, but let's say that we do not know which is the case. Let's suppose that the statement is false, and see where this leads.

Suppose that there is a greatest positive integer.

Let's call this greatest positive integer k .

Now, we proceed as we normally would in a mathematical argument, using sound logic and facts that we know to be true.

Consider the number $k + 1$.

We know that the number $k + 1$ must be a positive integer and must satisfy $k + 1 > k$.

Here we see that something is wrong. We have shown that if the given statement is false, then we can deduce that the following are both true about the number k :

- k is the greatest positive integer, and
- $k + 1$ is a positive integer that is greater than k .

But these cannot both be true of k . We claim that this means that the given statement could not possibly be false. Can you see why we can conclude this?



Explanation of a Proof Method (Proof by Contradiction)

In mathematics, we deal with sentences that have a definite state of being either true or false. We call these sentences *statements*. Since a statement is either true or false, we can use elimination to argue that a statement must be true by eliminating the possibility that it is false.

How can we eliminate the possibility that a statement is false? We can suppose that it is false and show that this assumption leads us to a *contradiction*. A contradiction is a combination of ideas that are opposed to one another, and hence cannot be simultaneously true. (For example, if we deduce that a number x must satisfy $x > 1$ and $x < -1$, then we have reached a contradiction.)

If we reach a contradiction in our argument, then we can be sure that there is at least one flaw in our argument. If the only possible error in our argument was our initial assumption (“the statement is false”) then this is the only thing that could have caused us to reach a contradiction. If we are sure that the assumption “the statement is false” is wrong, then the only remaining option is that the statement is actually true!

We call this logical method a *proof by contradiction*.

Example 2: Here is an example of a proof by contradiction.

<p><i>Statement:</i> The sum of a rational number and an irrational number is an irrational number.</p> <p><i>Proof:</i> Suppose, for a contradiction, that there is a rational number r and an irrational number α for which the sum $r + \alpha$ is rational. Since r is rational, we can write $r = \frac{a}{b}$ for integers a, b ($b \neq 0$). Since $r + \alpha$ is rational, we can write $r + \alpha = \frac{c}{d}$ for integers c, d ($d \neq 0$). This means we have $\frac{a}{b} + \alpha = \frac{c}{d}$. It follows that $\alpha = \frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd}$. This means α is rational. We have reached a contradiction. Therefore, it cannot be the case that there is a rational number and an irrational number whose sum is rational. We conclude that it must be the case that the sum of a rational number and an irrational number is an irrational number.</p>	<p>A rational number is a real number that can be expressed as a fraction of two integers (with non-zero denominator). An irrational number is a real number that cannot be expressed in this way.</p> <p>We suppose that the given statement is false.</p> <p>We proceed using logic and mathematical facts we know to be true.</p> <p>We have deduced the following:</p> <ul style="list-style-type: none"> • α is irrational • α is rational <p>But of course, they cannot both be true!</p> <p>We have eliminated the possibility that our statement is false.</p>
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Example 3: Try the following problem and watch a video of the proof: <https://youtu.be/E5IPLzQsSRA>.

The ten integers from 1 to 10 are placed randomly around a circle. Prove that there must be three integers in adjacent positions around the circle that have a sum which is greater than or equal to 17.

Practice: Your turn! Prove each of the following statements using a proof by contradiction approach.

1. There do not exist integers x and y such that $10x - 25y = 6$.
Start by supposing that there do exist integers x and y that satisfy $10x - 25y = 6$. Then think about the factors of the integer $10x - 25y$.
2. If x and y are positive real numbers, then $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$.
Start by supposing that there are positive real numbers x and y for which $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.
3. Extra Challenge: If the parabola $y = ax^2 + bx + c$ (with a, b, c non-zero real numbers) touches or crosses the x -axis, then a, b, c cannot form a geometric sequence, in that order.