The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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May 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *1985 Euclid Contest, Question A6*

   L and M are fixed points, 5 cm apart. The set of all points P in the plane, for which the triangle LMP has an area of 20 cm^2, is

   (A) a circle, diameter LM

   (B) a pair of lines perpendicular to LM

   (C) one line perpendicular to LM

   (D) a pair of lines parallel to LM

   (E) one line parallel to LM

   Solution

   If the area of triangle LMP is 20 cm^2 and the base is 5 cm, then if the height is h cm, we have $\frac{1}{2}(5)h = 20$ or $h = 8$.

   Thus, the set of all points P is the set of points when the perpendicular distance to LM is 8 cm.

   The set of points 8 cm from LM is a pair of lines parallel to LM.

   ANSWER: (D)

2. *1968 Ontario Senior Mathematics Problems Contest, Question 2*

   (a) Given that the equation

   $$x^3 + ax^2 + bx + c = 0$$

   has roots r_1, r_2, r_3, develop formulae for $r_1 + r_2 + r_3, r_1r_2 + r_1r_3 + r_2r_3$, and $r_1r_2r_3$.

   (b) For the equation

   $$x^3 + 2x^2 + 7x - 19 = 0$$

   compute the values of $r_1^2 + r_2^2 + r_3^2$ and $r_1^3 + r_2^3 + r_3^3$.

   Solution

   (a) Since r_1, r_2, r_3 are roots, then $(x - r_1)(x - r_2)(x - r_3) = x^3 + ax^2 + bx + c$.

   Expanding, we obtain

   $$(x^2 - (r_1 + r_2)x + r_1r_2)(x - r_3) = x^3 + ax^2 + bx + c$$
   $$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = x^3 + ax^2 + bx + c$$

Comparing coefficients in this identity, we find

$$\begin{aligned} r_1 + r_2 + r_3 &= -a \\ r_1r_2 + r_1r_3 + r_2r_3 &= b \\ r_1r_2r_3 &= -c \end{aligned}$$

(b) For the given equation,

$$\begin{aligned} r_1 + r_2 + r_3 &= -2 \\ r_1r_2 + r_1r_3 + r_2r_3 &= 7 \\ r_1r_2r_3 &= 19 \end{aligned}$$

Since

$$(r_1 + r_2 + r_3)^2 = r_1^2 + r_2^2 + r_3^2 + 2r_1r_2 + 2r_1r_3 + 2r_2r_3$$

then

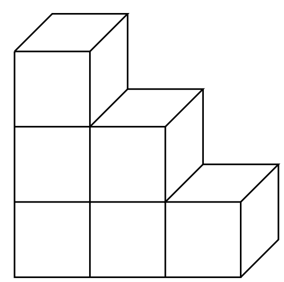
$$r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_1r_3 + r_2r_3) = 4 - 14 = -10$$

Also, $r_1^3 + 2r_1^2 + 7r_1 - 19 = 0$ and $r_2^3 + 2r_2^2 + 7r_2 - 19 = 0$ and $r_3^3 + 2r_3^2 + 7r_3 - 19 = 0$. Thus,

$$\begin{aligned} r_1^3 + r_2^3 + r_3^3 &= -2(r_1^2 + r_2^2 + r_3^2) - 7(r_1 + r_2 + r_3) + 3(19) \\ &= -2(-10) - 7(-2) + 57 \\ &= 91 \end{aligned}$$

3. *1981 Gauss Contest, Question 22*

A base row of blocks is formed and rows of blocks are added so that each new row has one fewer block that the row below it. If the base has nine blocks and the final row has one block, the total number of blocks used is



(A) 6 **(B)** 36 **(C)** 40
(D) 45 **(E)** 81

Solution

From the given information, the total number of blocks is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.

ANSWER: (D)

4. *1986 Descartes Contest, Question 9*

In $\triangle ABC$, $ab^2 \cos A = bc^2 \cos B = ca^2 \cos C$. Prove that the triangle is equilateral.

Solution

By the cosine law, $a^2 = b^2 + c^2 - 2bc \cos A$ and so $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Similarly, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Therefore,

$$\frac{ab^2(b^2 + c^2 - a^2)}{2bc} = \frac{bc^2(c^2 + a^2 - b^2)}{2ca} = \frac{ca^2(a^2 + b^2 - c^2)}{2ab}$$

Multiplying by $2abc$ gives

$$a^2b^2(b^2 + c^2 - a^2) = b^2c^2(c^2 + a^2 - b^2) = c^2a^2(a^2 + b^2 - c^2)$$

Dividing by $a^2b^2c^2$ gives

$$\frac{b^2+c^2-a^2}{c^2} = \frac{c^2+a^2-b^2}{a^2} = \frac{a^2+b^2-c^2}{b^2}$$

Now if $r = \frac{p}{q} = \frac{s}{t}$, then $\frac{p+s}{q+t} = \frac{qr+rt}{q+t} = r$.
Therefore, each of these fractions is equal to

$$\frac{(b^2+c^2-a^2)+(c^2+a^2-b^2)+(a^2+b^2-c^2)}{c^2+a^2+b^2}$$

But this fraction equals 1 after simplification.
Thus, $b^2+c^2-a^2=c^2$ and $b^2=a^2$ or $b=a$.
Similarly, $a=c$, so $a=b=c$, and the triangle is equilateral.

5. *1977 Gauss Contest, Question 10*
The maximum number of points of intersection of 4 distinct straight lines is

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** none of these

Solution
The first line drawn will have 0 intersections.
The second line can have at most 1 intersection.
The third line can have at most 2 intersections.
The last line can have at most 3 intersections.
Thus at most there will be $0+1+2+3=6$ intersections.

ANSWER: (C)