The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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From the archives of the CEMC

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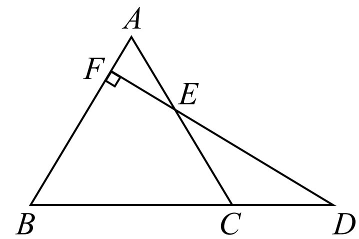
In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *1997 Cayley Contest, Question 22*
   In the diagram, $\triangle ABC$ is equilateral, $BC = 2CD$, $AF = 6$, and DEF is perpendicular to AB. What is the area of quadrilateral $FBCE$?

   (A) $144\sqrt{3}$ **(B)** $138\sqrt{3}$ **(C)** $126\sqrt{3}$
   (D) $108\sqrt{3}$ **(E)** $66\sqrt{3}$

2. *2012 Canadian Senior Mathematics Contest, Question B2*
   Consider the equation $x^2 - 2y^2 = 1$, which we label ①. There are many pairs of positive integers (x, y) that satisfy equation ①.

   (a) Determine a pair of positive integers (x, y) with $x \leq 5$ that satisfies equation ①.

   (b) Determine a pair of positive integers (u, v) such that
   $$(3 + 2\sqrt{2})^2 = u + v\sqrt{2}$$
   and show that (u, v) satisfies equation ①.

   (c) Suppose that (a, b) is a pair of positive integers that satisfies equation ①. Suppose also that (c, d) is a pair of positive integers such that
   $$(a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}$$
   Show that (c, d) satisfies equation ①.

   (d) Determine a pair of positive integers (x, y) with $y > 100$ that satisfies equation ①.

3. *2003 Fryer Contest, Question 2*
   In a game, Xavier and Yolanda take turns calling out whole numbers. The first number called must be a whole number between and including 1 and 9. Each number called after the first must be a whole number which is 1 to 10 greater than the previous number called.

   (a) The first time the game is played, the person who calls the number 15 is the winner. Explain why Xavier has a winning strategy if he goes first and calls 4.

   (b) The second time the game is played, the person who calls the number 50 is the winner. If Xavier goes first, how does he guarantee that he will win?

4. *2013 Euclid Contest, Question 7b*
   Determine all linear functions $f(x) = ax + b$ such that if $g(x) = f^{-1}(x)$ for all values of x, then $f(x) - g(x) = 44$ for all values of x. (Note: f^{-1} is the inverse function of f.)

5. *2016 Canadian Intermediate Mathematics Contest, Question A4*
   Dina has a calculating machine, labelled f, that takes one number as input and calculates an output. The machine f calculates its output by multiplying its input by 2 and then subtracting 3. For example, if Dina inputs 2.16 into f, the output is 1.32. If Dina inputs a number x into f, she gets a first output which she then inputs back into f to obtain a second output, which is -35. What is the value of x?