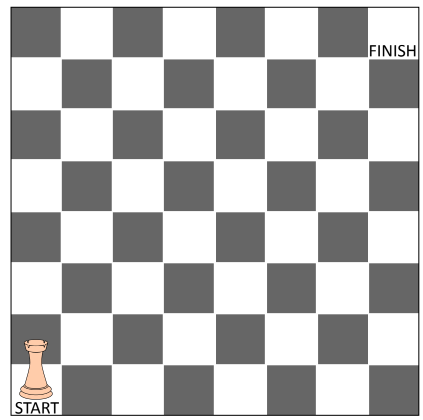
CEMC at Home

Grade 11/12 - Monday, March 23, 2020

Rook to the Top

Do you have a chessboard at home? Get it out, grab another person and let's play a game! If you don't know how to play chess, don't worry!



You Will Need:

- Two players
- A chessboard or checkerboard

  If you can't find a board, then you can draw an 8×8 grid on a piece of paper.
- A rook (as shown)

  If you can't find a rook, then you can use a coin or any small object in place of the rook.

How to Play:

1. Place the rook in the bottom left corner of the board.
2. The two players will alternate turns moving the rook. Decide which player will go first.
3. On your turn, you can move the rook as many squares as you want either to the right or up.

   You must move the rook at least one square and you cannot move the rook both right and up on the same turn. And of course you cannot run the rook off the board!
4. The player to place the rook in the top right corner of the board wins the game!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy* for this game? Does the winning strategy depend on whether you move the rook first or second?

> * A *strategy* is a pre-determined set of rules that a player will use to play the game. The strategy dictates what the player will do for every possible situation in the game. It's a *winning strategy*, if the strategy allows the player to always win, regardless of what the other player does.

Variation:

- Cover up the bottom 3 rows of the chessboard and start with the rook in the new bottom left corner. Play the game with the same rules. Does this change the winning strategy?

More Info:

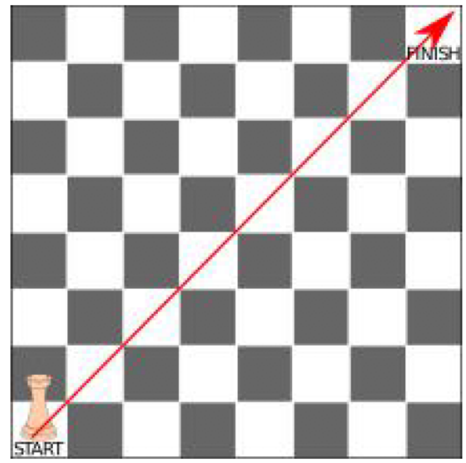
Check out the CEMC at Home webpage on Monday, March 30 for a discussion of a strategy for this game. We encourage you to discuss your ideas online using any forum you are comfortable with.

We sometimes put games on our math contests! Check out Question 2 on the 2003 Hypatia Contest for another game where we are looking for a winning strategy.

CEMC at Home

Grade 11/12 - Monday, March 23, 2020
Rook to the Top - Solution

We are going to call the diagonal line of white squares indicated in diagram, the *main diagonal*. Playing this game, you probably realized that the main diagonal is important to the strategy of this game.

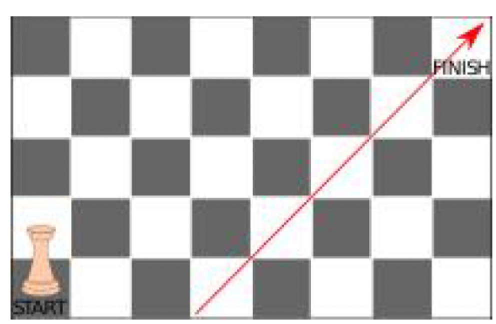


The rook begins on the main diagonal. The first player moves the rook and no matter what move they make, they will have to move the rook off of the main diagonal. If the first player moves the rook n squares to the right, then the second player can move the rook n squares up and the rook will be back on the main diagonal. If the first player moves the rook n squares up, then the second player can move the rook n squares to the right and the rook will be back on the main diagonal. In such a way the second player can guarantee that the rook will be on the main diagonal after their turn and the rook will be closer to the top right square (and maybe even at this square)!

Since the rook is back on the main diagonal, the first player must again move the rook off of the main diagonal and the second player can again put it back on to the main diagonal. Repeating this process, the second player will always be able to place the rook on the main diagonal closer to the top right square. Since there are a finite number of squares on the chessboard, the second player will eventually place the rook in the square at the top right corner.

Thus, we can see that the second player has a winning strategy for this game.

Variation:
In the variation of this game, we have a board with only five rows. We refer to the diagonal shown as the main diagonal. In this variation, the rook does not start on the main diagonal. If the first player moves the rook three spaces to the right, the rook will then be on the main diagonal. After this first move, the second player has no choice but to move the rook off of the main diagonal, leaving the first player the opportunity to place it back on the diagonal. Then the strategy continues as described for the first version. Therefore, the first player has the winning strategy in this variation of the game.



Extension: Consider a chessboard with any number of rows and any number of columns. For what size of chessboard will the first player have a winning strategy? For what size of chessboard will the second player have a winning strategy?

CEMC at Home

Grade 11/12 - Tuesday, March 24, 2020
Divisors and Primes

There are lots of problems that involve divisors of integers: counting divisors, looking for particular divisors, identifying common divisors, and more. For the following problems it might be helpful to review what a prime number is and how to find the prime factorization of an integer. Let's practice:

1. Find the prime factorization of 72 600.

   To help towards a solution, think about the following questions:

   - What are prime numbers?
   - Is 2 a divisor of 72 600? Is 3 a divisor of 72 600?
   - For each prime divisor p of 72 600, how many copies of p can we factor out of 72 600?

2. For how many integers n is $72\left(\frac{3}{2}\right)^n$ equal to an integer?

   To help towards a solution, think about the following questions:

   - Try some values of n. What if $n = 1$? What if $n = 10$? What if $n = -4$?
   - How big can n be? How small can n be?
   - Could prime factorizations help us here?

3. Determine the number of positive divisors of the integer 14!.

   Note: The *factorial* of a positive integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$.

4. For a positive integer n, $f(n)$ is defined as the largest power of 3 that is a divisor of n.

   What is $f\left(\dfrac{100!}{50!20!}\right)$?

More Info:

Check the CEMC at Home webpage on Wednesday, March 25 for a solution to Divisors and Primes. We encourage you to discuss your ideas online using any forum you are comfortable with.

These problems were taken from the CEMC's free online course *Problem Solving and Mathematical Discovery*. Check it out here: `https://courseware.cemc.uwaterloo.ca/40`

CEMC at Home

Grade 11/12 - Tuesday, March 24, 2020
Divisors and Primes - Solution

1. Find the prime factorization of 72 600.

Solution:

First, we factor 72 600 into two factors, $72\,600 = 726 \times 100$. Next, we factor each of these factors into a product of two factors, $726 = 2 \times 363$ and $100 = 10 \times 10$. We repeat this process until all the factors are prime numbers, (some of these prime factors will be repeated),

$$\begin{aligned} 72\,600 &= 726 \times 100 \\ &= 2 \times 363 \times 10 \times 10 \\ &= 2 \times 3 \times 121 \times 2^2 \times 5^2 \\ &= 2 \times 3 \times 11^2 \times 2^2 \times 5^2 \\ &= 2^3 \times 3 \times 5^2 \times 11^2. \end{aligned}$$

2. For how many integers n is $72\left(\frac{3}{2}\right)^n$ equal to an integer?

Solution:

Notice that the prime factorization of 72 is $2^3 \times 3^2$, so the expression $72\left(\frac{3}{2}\right)^n$ can be written as

$$72\left(\frac{3}{2}\right)^n = 2^{3-n} \times 3^{2+n}.$$

The expression will be an integer whenever the exponents $3-n$ and $2+n$ are non-negative integers. So, $3-n \geq 0$ and $2+n \geq 0$ imply that $n \leq 3$ and $n \geq -2$. Hence, there are six possible values of n, which are $-2, -1, 0, 1, 2, 3$.

3. Determine the number of positive divisors of the integer 14!.

Solution:

First, we find the prime factorization of 14!, which is the following:

$$14! = 2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13.$$

Any divisor of 14! has a prime factorization of the form $2^p \times 3^q \times 5^r \times 7^s \times 11^t \times 13^u$, where p, q, r, s, t, u are integers and $0 \leq p \leq 11$, $0 \leq q \leq 5$, $0 \leq r \leq 2$, $0 \leq s \leq 2$, $0 \leq t \leq 1$, and $0 \leq u \leq 1$.

Hence, the total number of choices for the exponents p, q, r, s, t, u are $12, 6, 3, 3, 2, 2$ respectively, and therefore, the total number of possible divisors is $12 \times 6 \times 3 \times 3 \times 2 \times 2 = 2592$.

4. For a positive integer n, $f(n)$ is defined as the largest power of 3 that is a divisor of n. What is $f\left(\dfrac{100!}{50!20!}\right)$?

Solution:

First, we count the number of factors of 3 included in 100!. Every multiple of 3 includes at least 1 factor of 3. The product 100! includes 33 multiples of 3 (since $33 \times 3 = 99$). Counting one factor of 3 from each of the multiples of 3 (these are $3, 6, 9, 12, 15, 18, \cdots, 93, 96, 99$), we see that 100! includes at least 33 factors of 3.

However, each multiple of $3^2 = 9$ includes a second factor of 3 (since $9 = 3^2, 18 = 3^2 \times 2$, etc.) which was not counted in the previous 33 factors. The product 100! includes 11 multiples of 9 (since $11 \times 9 = 99$), and thus there are at least 11 additional factors of 3 in 100!.

Similarly, 100! includes 3 multiples of $3^3 = 27$, each of which contribute an additional factor of 3 (these are $27 = 3^3, 54 = 3^3 \times 2$, and $81 = 3^4$).

Finally, there is one multiple of $3^4 = 81$ which contributes one more factor of 3.

Since $3^5 > 100$, then 100! does not include any multiples of 3^5 and so we have counted all possible factors of 3. Thus, 100! includes exactly $33 + 11 + 3 + 1 = 48$ factors of 3, and so $100! = 3^{48} \times t$ for some positive integer t that is not divisible by 3.

Counting in a similar way, the product 50! includes 16 multiples of 3, 5 multiples of 9, and 1 multiple of 27, and thus includes $16+5+1 = 22$ factors of 3. Therefore, $50! = 3^{22} \times r$ for some positive integer r that is not divisible by 3.

Also, 20! includes $6 + 2 = 8$ factors of 3, and thus $20! = 3^8 \times s$ for some positive integer s that is not divisible by 3. Therefore,

$$\frac{100!}{50!20!} = \frac{3^{48} \times t}{(3^{22} \times r)(3^8 \times s)} = \frac{3^{48} \times t}{3^{30} \times rs} = 3^{18} \times \frac{t}{rs}.$$

Since $\dfrac{t}{rs}$ is an integer (as $\dfrac{100!}{50!20!}$ is an integer), and each of r, s and t does not include any factors of 3, then 3 is not a divisor of the integer $\dfrac{t}{rs}$.

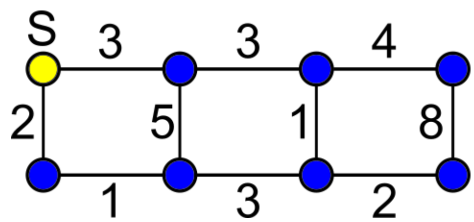
Therefore, the largest power of 3 which divides $\dfrac{100!}{50!20!}$ is 3^{18}, and so $f\left(\dfrac{100!}{50!20!}\right) = 18$.

Note: Solutions to these problems can also be found in the Number Theory section of the CEMC's free online course *Problem Solving and Mathematical Discovery*. Check out these problems and more by visiting `https://courseware.cemc.uwaterloo.ca/40`

CEMC at Home

Grade 11/12 - Wednesday, March 25, 2020
Byber Path

You drive for a delivery service called Byber. You start at location S and you have to drop off a package at each of the seven other locations, shown as circles. The locations are joined by roads shown as lines. You cannot visit any location more than once on your route. You can finish at any location that you wish. The number beside a line is the toll you need to pay for taking the corresponding road.



What is the least total amount you need to pay in order to drop off all seven packages?

You may end up with the correct answer for this problem without being completely convinced that your answer is indeed correct! Think about what it would take to completely justify the validity of your answer. You would need to show that there is a route that will result in exactly your amount, and explain why every other route would cost at least as much as yours.

More Info:

Check the CEMC at Home webpage on Wednesday, April 1 for the correct answer with justification.

This problem almost ended up on the 2019 Beaver Computing Challenge (BCC) which is a problem solving contest with a focus on computational and logical thinking.

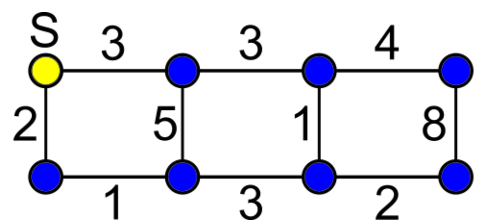
While only officially open to students in Grades 5 to 10, students in Grades 11 and 12 can have fun and learn something by trying the BCC problems. You can find more problems like this on past BCC contests.

CEMC at Home

Grade 11/12 - Wednesday, March 25, 2020
Byber Path - Solution

Problem

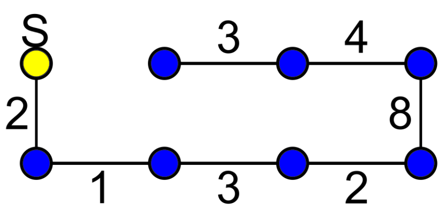
You drive for a delivery service called Byber. You start at location S and you have to drop off a package at each of the seven other locations, shown as circles. The locations are joined by roads shown as lines. You cannot visit any location more than once on your route. You can finish at any location that you wish. The number beside a line is the toll you need to pay for taking the corresponding road.

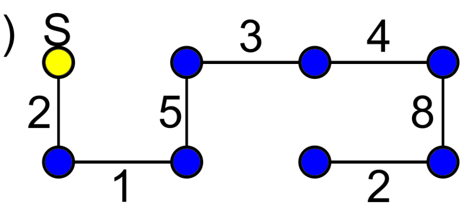


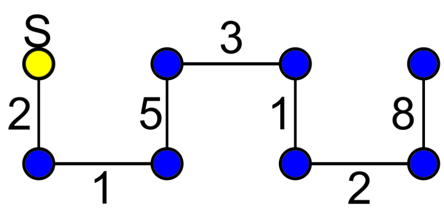
What is the least total amount you need to pay in order to drop off all seven packages?

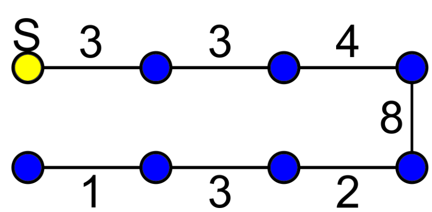
Solution

There are only four possible paths that start at S and visit each of the locations exactly once:

1)


2)


3)


4)


To justify this, notice that we have two choices for where to travel from S: down or right.

- If we travel right first, then the rest of our path is determined. If we move down before reaching the right-most location in the top row, then we would need to backtrack to reach all of the bottom locations. This would mean revisiting at least one location on the way. Therefore, we must travel right until we reach the right-most location in the top row, and then must follow path 4) for the remainder.

- If we travel down first, then we must travel right next, but then there we again have two options: up or right.
  - If we travel right, then the rest of our path is determined. We cannot move up next, as then we cannot reach the right-most locations without backtracking, and so we must move right to get to the right-most bottom location. From here we would have to follow path 1) for the remainder.
  - If we travel up, then we must travel right next. From here we have two choices: down or right.
    - If we travel down, then the rest of our path is determined. We must follow path 3).
    - If we travel right, then the rest of our path is determined. We must follow path 2).

These four possible paths have total paid amounts of:

1) $2 + 1 + 3 + 2 + 8 + 4 + 3 = 23$
2) $2 + 1 + 5 + 3 + 4 + 8 + 2 = 25$
3) $2 + 1 + 5 + 3 + 1 + 2 + 8 = 22$
4) $3 + 3 + 4 + 8 + 2 + 3 + 1 = 24$

(where the sum is shown starting at S and moving through the path).

The least total paid amount of these four paths is 22.

Connections to Computer Science

For each Beaver Computing Challenge problem, we include a short description of its connections to computer science. The italicized keywords emphasize terminology that can be used to search online, if you are interested in learning more.

In this particular problem, the locations and roads can be modelled by a *graph*. Locations are the *vertices* and roads are the *edges* of the graph. To use a computer to solve a problem like this, we need to

- figure out how to represent the graph, and
- discover and implement an *algorithm* to produce the final answer.

In a *programming language*, different *data structures* exist or can be built to represent virtually anything we can image. Amazingly, everything is ultimately modelled by a long *binary* sequence of 0s and 1s.

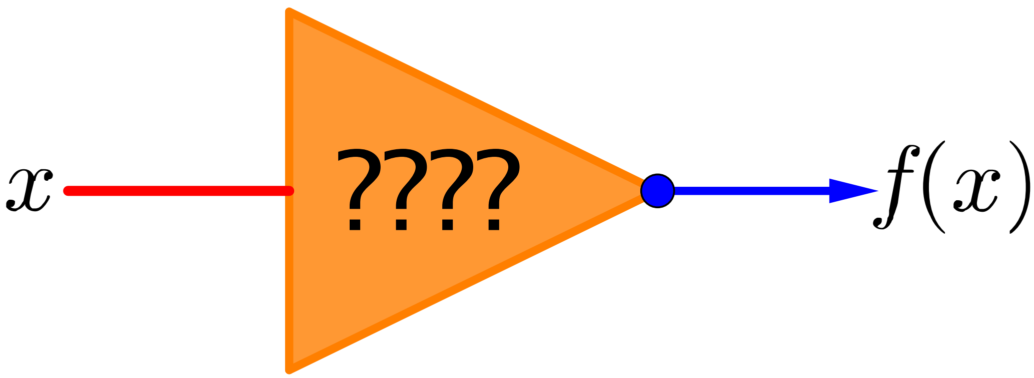
Different computer algorithms are used for finding the best or the worst path through a graph. In this problem, one of the restrictions is to find a path which visits all the vertices exactly once. This is called a *Hamiltonian path*. Problems involving Hamiltonian paths are well-known and considered very difficult. They are closely related to what is known as the famous *Travelling salesman problem*. You can read more about this problem here.

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, March 26, 2020
Functionally Possible

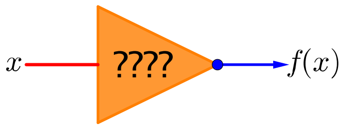
The function $f(x) = x^5 - 3x^4 + ax^3 - x^2 + bx - 2$ has a value of 5 when $x = 3$.

Determine the value of the function when $x = -3$.



More Info:

Check the CEMC at Home webpage on Thursday, April 2 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 2.



Problem of the Week

Problem E and Solution

Functionally Possible

Problem

The function $f(x) = x^5 - 3x^4 + ax^3 - x^2 + bx - 2$ has a value of 5 when $x = 3$.

Determine the value of the function when $x = -3$.

Solution

We know that the function has a value of 5 when $x = 3$. Therefore, $f(3) = 5$.

$$\begin{aligned} f(3) &= 5 \\ (3)^5 - 3(3)^4 + a(3)^3 - (3)^2 + b(3) - 2 &= 5 \\ 243 - 243 + 27a - 9 + 3b - 2 &= 5 \\ 27a + 3b &= 16 \qquad (1) \end{aligned}$$

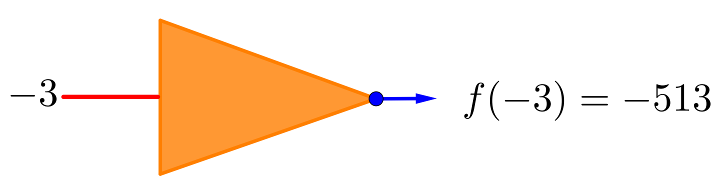
At this point we seem to have used up the given information. Maybe we can learn more by looking at precisely what we are asked to determine.

In this problem, we want the value of the function when $x = -3$. In other words, we want $f(-3)$.

$$\begin{aligned} f(-3) &= (-3)^5 - 3(-3)^4 + a(-3)^3 - (-3)^2 + b(-3) - 2 \\ &= -243 - 243 - 27a - 9 - 3b - 2 \\ &= -27a - 3b - 497 \end{aligned}$$

But from (1) above, $27a + 3b = 16$ so
$f(-3) = -27a - 3b - 497 = -(27a + 3b) - 497 = -16 - 497 = -513$.

Therefore, the value of the function is -513 when $x = -3$.



We are not given enough information to find the precise values of a and b but enough information is given to solve the problem.

CEMC at Home

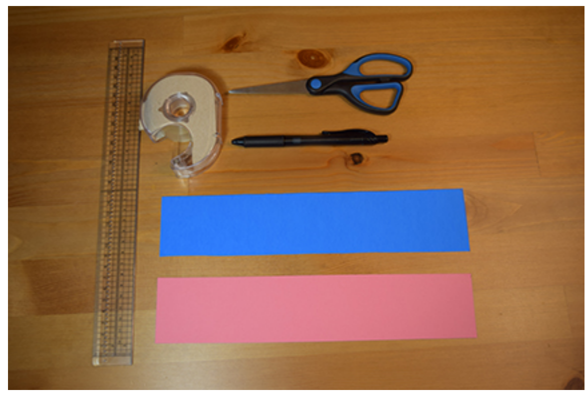
Grade 11/12 - Friday, March 27, 2020
The Möbius Strip

A Möbius strip is a mathematical object that has interesting properties. It is a surface with only one face (or side) and only one edge (or boundary). In this activity we will build a Möbius strip and investigate its curious properties. The purpose of this activity is not to fully understand the mathematics of a Möbius strip, but rather to hopefully surprise and intrigue you!

You will need:

- A pencil
- A ruler
- Scissors
- Tape
- Two rectangular strips of paper of different colours.

  Strips of around 6 cm wide and 30 cm long will work well. We will use one blue strip and one pink strip, but you can use any colours you want.



How to construct a Möbius strip:

1. Use your ruler to draw a line along the blue strip that divides the strip into two equal parts. Do the same on the other side of the strip.
2. Use your ruler to draw two lines along the pink strip that divides the strip into three equal parts. Do the same on the other side of the strip.

3. Grab the blue strip by the two short edges. Twist one end of the strip half of the way around and join the two short edges together. (Make sure this is a "half twist" and not a "full twist".) Line up the short edges and tape them together from end to end.

4. Repeat the same process with the pink strip. You now have two Möbius strips.

Let's explore some properties of our Möbius strips!

1. Take one of the strips you made and answer the following questions:

   (a) How many faces does the Möbius strip have?

   You might need to spend some time thinking about what is meant by a "face" here.

   (b) How many edges does the Möbius strip have?

   You might need to spend some time thinking about what is meant by an "edge" here.

   (c) Does the Möbius strip have an "inside" and an "outside"?

2. Take the blue Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along the line drawn in the middle of the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the blue strip along the middle line. You will need to carefully cut or puncture the strip somewhere along this line in order to start the cut. What happens once you cut along this line? Is it what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

3. Take the pink Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along one of the two lines that we drew down down the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the pink strip along one of the lines. When you cut, you might notice that it doesn't actually matter which of the two lines you chose to cut along. Is the result what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

More Info:

Check the CEMC at Home webpage on Friday, April 3 for further discussion on The Möbius Strip.

CEMC at Home

Grade 11/12 - Friday, March 27, 2020
The Möbius Strip - Solution

1. Take one of the strips you made and answer the following questions:

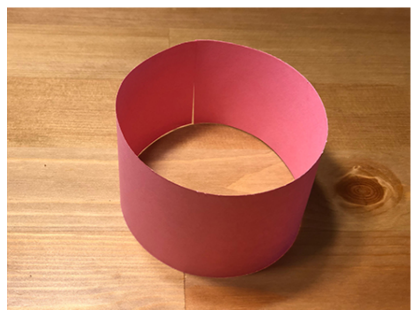
   (a) How many faces does the Möbius strip have?

   (b) How many edges does the Möbius strip have?

   (c) Does the Möbius strip have an "inside" and an "outside"?

   Discussion:

   We will compare our Möbius strip with a cylinder. A cylinder can be made by taking the two short edges of a strip, lining them up, and taping them together from end to end.

   A cylinder has an "outside" face and an "inside" face. An ant walking on one of these faces must cross an edge (or boundary) to get to the other face. The Möbius strip has only one face (or side). An ant can walk along the entire surface of the Möbius strip without crossing an edge (or boundary). In particular, an ant that begins on any part of either line we drew, can follow this line and will end up back where it started. In doing this, it will have travelled the full length of *both* lines drawn on the two sides of the original blue strip.

   Similarly, a cylinder has a "top" edge and "bottom" edge, but a Möbius strip has only one edge (or boundary). Imagine the ant walking along the edge of the Möbius strip. The ant will travel along the entire edge of the strip and will end up back where it started.

   In summary, a cylinder has two faces and two edges, but a Möbius strip only has one face and one edge. Both can be created from a single strip of paper.

2. Take the blue Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along the line drawn in the middle of the strip? How many detached pieces do you think will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the blue strip along the middle line. You will need to carefully cut or puncture the strip somewhere along this line in order to start the cut. What happens once you cut along this line? Is it what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

Discussion:

Following our intuition with the cylinder, we know that if we cut the cylinder along a middle line parallel to its two edges, then we will obtain two smaller detached cylinders. However, if we cut the Möbius strip along the middle line, the result might surprise us in two ways:

- We get one strip instead of two detached pieces.
- The strip we get is not a Möbius strip!

This result is illustrated in Figure 1.

Figure 1

One way to help us understand this is to think about the process of "gluing and cutting" in a different order. We can *begin* by cutting along the middle line and taping the two pieces together leaving a gap to show our cut. *After* doing this, we can do a "half twist" and join the short edges together as in the original instructions. This is illustrated in Figure 2. Notice that the green ends connect to each other and the black ends also connect to each other.

Figure 2

Try to use the analogy of an ant walking on the surface and edges of the resulting strip to convince yourself that this strip is an attached single piece with two faces and two sides. An effective way to do this is to do the construction shown in Figure 2 yourself and follow the path an ant might take with your finger. Interestingly, this resulting strip is what we would get if we followed the original instructions using two "full twists" instead of a "half twist".

3. Take the pink Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along one of the two lines that we drew down down the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the pink strip along one of the lines. When you cut, you might notice that it doesn't actually matter which of the two lines you chose to cut along. Is the result what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

Discussion:

As before, Figure 3 is an illustration of the result and Figure 4 illustrates what happens if we *begin* by cutting and taping before doing a "half twist" and joining the short edges together.

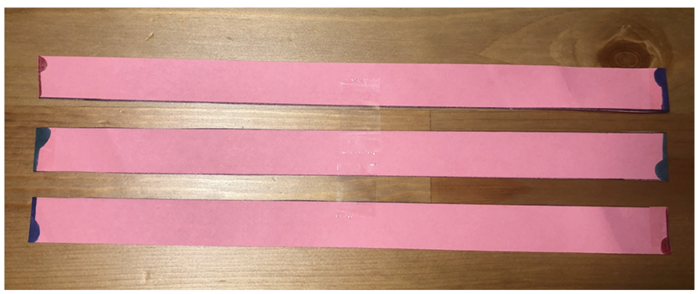


Figure 3

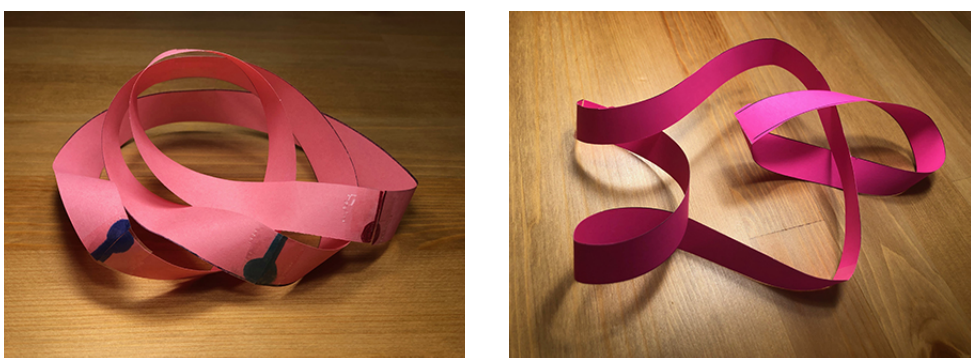


Figure 4

Some interesting and surprising things happen:

- The opposite ends of the upper and lower strips connect to each other making a longer strip which is similar in structure (but narrower) than the one we obtained from the blue strip.
- We also get a shorter strip which is a Möbius strip.
- These two strips are linked together!

CEMC at Home

Grade 11/12 - Monday, March 30, 2020
Check Your Calendar

You Will Need:

- Two players

How to Play:

1. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
2. Player 1 starts the game by saying the date "January 1".
3. Player 2 then says a date later in the year which has either the same month or the same day as the date said by Player 1 ("January 1"). For example, Player 2 could say "January 5" (same month) or "March 1" (same day), but not "February 4" (different month and day).

   Note: If you are playing with the 2020 calendar, then February 29 may be used!
4. The players now alternate saying dates, based on the date said previously, following the same rules as given in #3.
5. The player who says "December 31" wins the game!

An example of a complete game:
Alexis and David are playing the game. They decide that Alexis will be Player 1.

Alexis: January 1
David: January 16
Alexis: April 16
David: July 25
Alexis: David!!! You can't change the month and day!
David: Oh yeah, right. July 16
Alexis: July 31
David: December 31! I win!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy for this game? Does the winning strategy depend on whether you are Player 1 or Player 2?

Is there a connection between this game and the game we played on March 23 (Rook to the Top)?

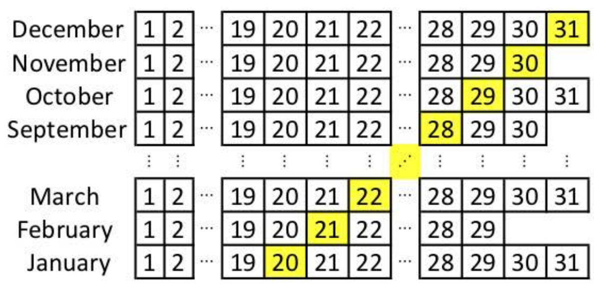
More Info:
Check out the CEMC at Home webpage on Monday, April 6 for a solution to Calendar Game.
We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Monday, March 30, 2020
The Calendar Game - Solution

It turns out that the strategy for winning the Calendar Game is similar to the strategy for winning the game Rook to the Top (that we played on March 23). You might want to refresh your memory by having a look at the strategy for Rook to the Top.

In Rook to the Top, we played on an 8 by 8 grid. In some sense, we can also think of the Calendar Game as being played on a "grid". In this case it will be a 12 by 31 grid with some spaces not open for play.



Notice that the grid has a row for each of the 12 months, and the rows contain either 29, 30 or 31 squares, depending on how many days are in that particular month (in 2020). The diagonal that is highlighted on the grid is the one we will focus on for the winning strategy of this game. We will refer to it as the *winning diagonal*.

The winning diagonal consists of the following dates:

January 20	July 26
February 21	August 27
March 22	September 28
April 23	October 29
May 24	November 30
June 25	December 31

Since Player 1 must say "January 1", we see that the first "move" lands off of the winning diagonal. Player 2 can then "move" onto the winning diagonal by saying "January 20". Now Player 1 must change either the month or the day (but not both) and so any allowed date will represent a move off of the winning diagonal. If they change the month, this corresponds to a vertical move upwards and if they change the day, this corresponds to a horizontal move to the right. Player 2 can now choose the appropriate date from the table above to move back onto the winning diagonal. For example, if Player 1 changes the month and says "May 20", Player 2 can then change the day and say "May 24" (from the table above). If instead Player 1 changes the day and says "January 27", Player 2 can then change the month and say "August 27" (from the table above).

Repeating this process, Player 1 will always have to move off of the winning diagonal, and Player 2 will always be able to return to the winning diagonal, closer to December 31. Since there are a finite number of dates to choose from, Player 2 will eventually say December 31.

Thus, Player 2 has a winning strategy for this game.

CEMC at Home

Grade 11/12 - Tuesday, March 31, 2020
Making a List

Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. Ellie makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.)

1. Suppose that Ellie starts with $a = 5$ and $b = 16$. Can you determine the four numbers that each appear twice in the third list without writing out all 36 numbers in the third list?

2. Suppose that Ellie's third list has the following properties:

   (i) the integer 49 appears in the third list,

   (ii) there is no number in the third list that is a multiple of 64, and

   (iii) there is at least one number in the third list that is larger than 75.

   Determine all possibilities for the pair (a, b).

More Info:

Check out the CEMC at Home webpage on Tuesday, April 7 for the solution to Making a List.

Part of this question appeared on a past Euclid Contest. You can see the original question and the rest of the contest here: 2015 Euclid Contest.

CEMC at Home

Grade 11/12 - Tuesday, March 31, 2020
Making a List - Solution

Problem:

Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. Ellie makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.)

1. Suppose that Ellie starts with $a = 5$ and $b = 16$. Can you determine the four numbers that each appear twice in the third list without writing out all 36 numbers in the third list?

2. Suppose that Ellie's third list has the following properties:

   (i) the integer 49 appears in the third list,

   (ii) there is no number in the third list that is a multiple of 64, and

   (iii) there is at least one number in the third list that is larger than 75.

   Determine all possibilities for the pair (a, b).

Solution to 1.

The first list is 5, 6, 7, 8, 9, 10 and the second list is 16, 17, 18, 19, 20, 21.

The prime numbers that are a factor of at least one number in the first two lists are as follows:

$$2, 3, 5, 7, 17, 19$$

We examine the 36 products in the third list based on their prime factors.

Products with a factor of 17 or 19

Note that 17 is the only number in the first or second list that has a factor of 17. This means that there will be exactly six numbers in the third list that have a factor of 17 (obtained by multiplying each number in the first list by 17) and these six numbers will be all be different. The situation is similar for the six numbers in the third list that have a factor of 19. We can see that all twelve of these numbers only appear once in the third list.

Products with a factor of 7

Note that 7 and 21 are the only numbers in the first or second list that have a factor of 7. Suppose we have $7 \times m = n \times 21$ where $m \neq 21$ comes from the second list and $n \neq 7$ comes from the first list. Since m must have a factor of 3, the only possibility for m is 18, which forces n to be 6, and gives

$$7 \times 18 = 6 \times 21 = 126$$

Products with a factor of 5

Note that 5, 10, and 20 are the only numbers in the first or second list that have a factor of 5. Suppose we have $5 \times m = n \times 20$ where $m \neq 20$ comes from the second list and $n \neq 5$ comes from the first list. Since m must have a factor of 4, the only possibility for m is 16, which would force n to be 4, which is not possible as 4 is not in the first list. Now suppose we have $10 \times m = n \times 20$ where $m \neq 20$ comes from the second list and $n \neq 10$ comes from the first list. Since m must have a factor of 2, the only possibilities for m are 16 and 18, which would force n to be 8 and 9, respectively, giving us

$$10 \times 16 = 8 \times 20 = 160$$

$$10 \times 18 = 9 \times 20 = 180$$

Products whose only prime factors are 2 or 3

All remaining numbers in the list must be formed by multiplying one of 6, 8, or 9 by one of 16, or 18. There is only one final duplication here, formed by

$$8 \times 18 = 9 \times 16 = 144$$

Therefore the four numbers that each appear twice in the third list are as follows:

$$\begin{aligned}126 &= 7 \times 18 = 6 \times 21\\144 &= 8 \times 18 = 9 \times 16\\160 &= 8 \times 20 = 10 \times 16\\180 &= 9 \times 20 = 10 \times 18\end{aligned}$$

Solution to 2.

We will start by considering what condition (i) tells us about the values of a and b as it seems to be the most restrictive of the three conditions.

Condition (i) tells us that 49 must be a product of an integer from the first list and an integer from the second list. Since $49 = 7^2$, 7 is prime, and all integers in the two lists are positive, these integers must be either 1 and 49 or 7 and 7. We will find all possible values of a and b by considering two cases separately:

- Case 1: 49 was obtained in the third list by multiplying 1 and 49
- Case 2: 49 was obtained in the third list by multiplying 7 and 7

Note: It is not possible for 49 to be obtained in both of these ways at once because if a list contains 49 then it cannot also contain 7. However, knowing this will not be important for our solution.

Case 1: 49 was obtained by multiplying 1 and 49.

Since the number 1 is in one of the lists, we must have either $a = 1$ or $b = 1$. The condition of $a < b$ means we must have $a = 1$. This means that the first list must be

$$1, 2, 3, 4, 5, 6$$

and the number 49 must appear somewhere in the second list.

Therefore, the second list is one of the following six lists (with each list appearing horizontally):

$$\begin{array}{cccccc} 44, & 45, & 46, & 47, & 48, & 49 \\ 45, & 46, & 47, & 48, & 49, & 50 \\ 46, & 47, & 48, & 49, & 50, & 51 \\ 47, & 48, & 49, & 50, & 51, & 52 \\ 48, & 49, & 50, & 51, & 52, & 53 \\ 49, & 50, & 51, & 52, & 53, & 54 \end{array}$$

Notice that $4 \times 48 = 192 = 64 \times 3$. Since 4 is in the first list, and no number in the third list can be a multiple of 64, the second list cannot contain the number 48. This leaves just one possibility for the second list (the last one above):

$$49, 50, 51, 52, 53, 54$$

This case leads us to exactly one possibility for the pair (a, b), namely $(1, 49)$.

We can verify that the third list for the pair $(a, b) = (1, 49)$ actually satisfies conditions (ii) and (iii). For (ii), we note that $64 = 2^6$ and that we can get at most two factors of 2 from a number in the first list and at most two factors of 2 from a number in the second list. It follows that any product in the third list will have at most 4 factors of 2, and hence cannot be a multiple of 64. For (iii), we note that $2 \times 49 = 98$ is in the third list and is greater than 75.

Case 2: 49 was obtained by multiplying 7 and 7.

In this case, we know that the number 7 must appear in both the first list and the second list. In order for this to happen we need to have $2 \le a \le 7$ and $2 \le b \le 7$. Since $a < b$, we actually must have $3 \le b \le 7$. (The smallest a can be is 2 and so b must be at least one more than that.)

Since $3 \le b \le 7$, the second list *must* contain the number 8. This means that to satisfy condition (ii), the first list *cannot* contain the number 8. Therefore, we must have $a = 2$, making the first list

$$2, 3, 4, 5, 6, 7$$

Since $7 \times 10 = 70$ and $7 \times 11 = 77$, the third list can only satisfy condition (iii) if the second list contains a number at least as large as 11. This means we cannot have $b = 3$, $b = 4$, or $b = 5$, leaving the only possible values to be $b = 6$ or $b = 7$. These values produce the following second lists:

$$b = 6: \qquad 6, 7, 8, 9, 10, 11$$

$$b = 7: \qquad 7, 8, 9, 10, 11, 12$$

Therefore, this case leads us to two additional possibilities for the pair (a, b), namely $(2, 6)$ and $(2, 7)$.

We can verify that the third list for each of the the pairs $(a, b) = (2, 6)$ and $(a, b) = (2, 7)$ satisfies conditions (ii) and (iii) using a similar argument to the one given in Case 1.

Combining the two cases, we conclude that there are exactly three pairs, (a, b), that satisfy all three conditions. They are as follows:

$$(1, 49), (2, 6), (2, 7)$$

CEMC at Home

Grade 11/12 - Wednesday, April 1, 2020
Collecting Pollen and Wood

Video

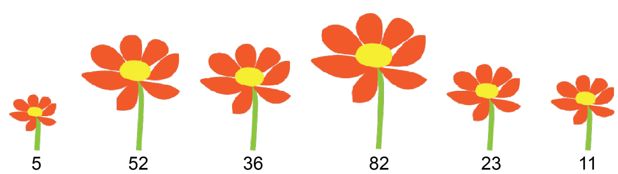
Watch the following presentation on algorithmic paradigms, based on two past problems from the Beaver Computing Challenge:

https://youtu.be/XuR1a_9orJQ

The two problems discussed in the presentation are included below for your reference. Links to the two apps used in the video are also provided should you wish to do some exploration on your own.

Collecting Pollen

Beever the Bee flies to a field of flowers to collect pollen. On each flight Beever visits only one flower and can collect up to 10 mg of pollen. Beever may return to the same flower more than once. The field contains 6 flowers, each containing a different amount of pollen (in mg) as shown.



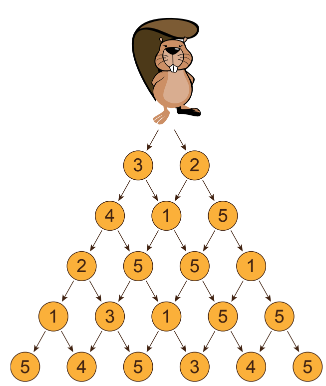
If Beever flies to the field 20 times, what is the maximum total amount of pollen Beever can collect?

Note: This problem was also given as a Grade 4/5/6 exercise last week, but our focus will be different. Our goal is not just to arrive at the answer to the problem, but rather to discuss the algorithm used to arrive at the optimal solution.

App for exploration: https://www.geogebra.org/m/guzzeqn4

Collecting Wood

A beaver collects wood while descending from a mountaintop. Each stop contains a different amount of wood as shown.



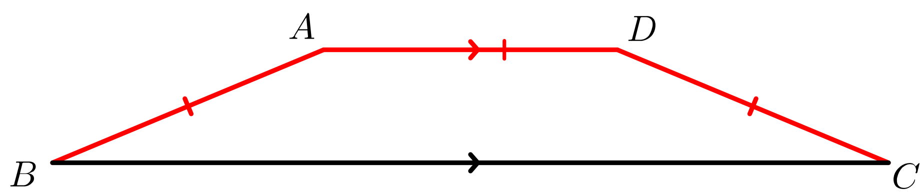
The beaver can only follow the arrows down. What is the maximum total amount of wood the beaver can collect?

App for exploration: https://www.geogebra.org/m/nsmtks3u

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 2, 2020
Three Equal Sides

In trapezoid $ABCD$, the lengths of AB, AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides.



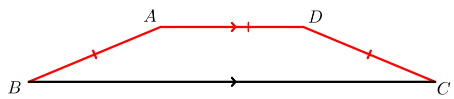
If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

More Info:

Check the CEMC at Home webpage on Thursday, April 9 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 9.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem E and Solution

Three Equal Sides

Problem

In trapezoid $ABCD$, the lengths of AB, AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides. If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

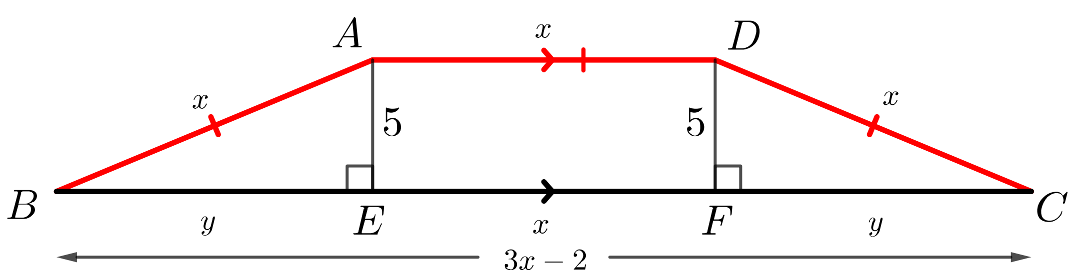
Solution

Let x represent the length of AB. Then $AB = AD = DC = x$. Since the base BC is two less than the sum of the three equal sides, $BC = 3x - 2$.

Construct altitudes from A and D meeting BC at E and F, respectively. Then $AE = DF = 5$, the distance between the two parallel sides.

Let y represent the length of BE. We can show that $BE = FC$ using the Pythagorean Theorem as follows: $BE^2 = AB^2 - AE^2 = x^2 - 5^2 = x^2 - 25$ and $FC^2 = DC^2 - DF^2 = x^2 - 5^2 = x^2 - 25$. Then $FC^2 = x^2 - 25 = BE^2$, so $FC = BE = y$ since $FC > 0$.

Since $\angle AEF = \angle DFE = 90^\circ$ and AD is parallel to EF, it follows that $\angle DAE = \angle ADF = 90^\circ$ and $AEFD$ is a rectangle so $EF = AD = x$. The following diagram contains all of the given and found information.



We can now determine a relationship between x and y.

$$\begin{aligned} BC &= BE + EF + FC \\ 3x - 2 &= y + x + y \\ 2x - 2 &= 2y \\ x - 1 &= y \end{aligned}$$

In right $\triangle ABE$, $AB^2 = BE^2 + AE^2$ gives $x^2 = y^2 + 5^2$. Substituting $y = x - 1$ we get $x^2 = (x-1)^2 + 25$. Solving, $x^2 = x^2 - 2x + 1 + 25$, or $2x = 26$, or $x = 13$.

Since $x = 13$, $3x - 2 = 3(13) - 2 = 37$. Therefore, $AD = x = 13$ and $BC = 3x - 2 = 37$.

Therefore,

$$\begin{aligned} \text{area of trapezoid } ABCD &= AE \times (AD + BC) \div 2 \\ &= 5 \times (13 + 37) \div 2 \\ &= 125 \text{ units}^2 \end{aligned}$$

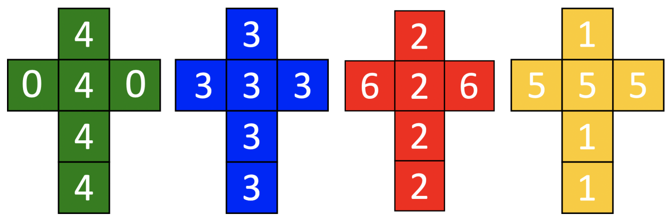
CEMC at Home

Grade 11/12 - Friday, April 3, 2020
A Dicey Situation

In this activity, we will investigate the properties of various non-standard dice.

You Will Need: Two players and four standard six-sided dice.

Description of the Dice: Alter your dice so that the sides of the dice are labelled according to the following diagrams. For example, you could put stickers on each side of the dice.



If you do not have an easy way to alter your dice, you can make use of the following conversion table for each of the dice. This table will allow you to roll a standard die as if it were one of the dice shown above. For example, if you roll a 3 on the standard die representing the green die, then you interpret this as rolling a 4 on the green die (using the second "Green" row in the table).

Die Colour	Number Rolled on Standard Die	Number Rolled on Altered Die
Green	1, 6	0
	2, 3, 4, 5	4
Blue	1, 2, 3, 4, 5, 6	3
Red	1, 6	6
	2, 3, 4, 5	2
Yellow	1, 4, 6	5
	2, 3, 5	1

If all of your dice are identical, you will need a way to keep track of which die is which "colour".

Investigation:

To start a game, each player chooses a die to use for the entire game. One game consists of 20 rounds. In each round, each player rolls their die and the player that rolls the higher number wins the round. The winner of the game is the person who won the most rounds. No round can end in a tie, but the game may end in a tie. Play the game a number of times and alternate the pair of dice used for each game. Keep track of the scores of each game, along with which player had which die for each game.

Follow-up Questions:

1. If you choose your die first, is there a best choice for your die?
2. If you choose your die second, is there a best choice for your die?
   Your answers can depend on what die the other player chooses.
3. Try to justify your answers for 1. and 2. by analyzing the four dice and calculating probabilities.

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for the solution to A Dicey Situation.

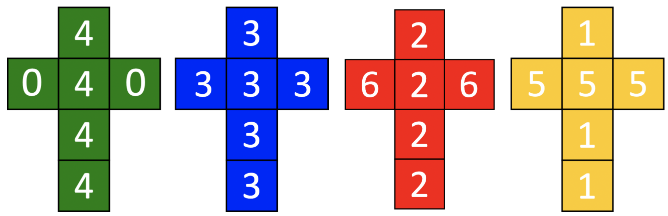
To refresh your knowledge of probabilities, check out this lesson from the CEMC courseware.

CEMC at Home

Grade 11/12 - Friday, April 3, 2020
A Dicey Situation - Solution

Investigation:

To start a game, each player chooses a die to use for the entire game.



One game consists of 20 rounds. In each round, each player rolls their die and the player that rolls the higher number wins the round. The winner of the game is the person who won the most rounds. No round can end in a tie, but the game may end in a tie. Play the game a number of times and alternate the pair of dice used for each game. Keep track of the scores of each game, along with which player had which die for each game.

Follow-up Questions:

1. If you choose your die first, is there a best choice for your die?
2. If you choose your die second, is there a best choice for your die?
   Your answers can depend on what die the other player chooses.
3. Try to justify your answers for 1. and 2. by analyzing the four dice and calculating probabilities.

Solution: You may have answered 1. and 2. using information gathered during your investigation. We will give answers to 1. and 2. that are based upon theoretical probabilities related to the four dice. We start by examining the game that arises if certain pairs of dice are chosen by the two players. Note that the rolls of the two dice are independent.

Green Versus Blue

What happens if the two players pick the green and blue dice (in some order)? Since each of the dice has 6 faces, and each face is equally likely to end up as the top face on a roll, there are $6 \times 6 = 36$ equally likely outcomes when these two dice are rolled. We indicate which die wins in each of the 36 cases the table below:

	3	3	3	3	3	3
0	B	B	B	B	B	B
0	B	B	B	B	B	B
4	G	G	G	G	G	G
4	G	G	G	G	G	G
4	G	G	G	G	G	G
4	G	G	G	G	G	G

Based on this table, we can see that the probability of green winning is

$$\text{P(Green Winning Against Blue)} = \frac{24}{36} = \frac{2}{3}$$

and the probability of blue winning is

$$\text{P(Blue Winning Against Green)} = \frac{12}{36} = \frac{1}{3}$$

Since rolls of the two dice are independent, we can calculate probabilities without listing the possible outcomes. We know that the blue die will always roll a 3, and so we only need to consider the roll of the green die. The probability that the green die rolls a 0 is $\frac{2}{6} = \frac{1}{3}$, and the probability that the green die rolls a 4 is $\frac{4}{6} = \frac{2}{3}$. If the green die rolls a 0, then the blue die wins, and if the green die rolls a 4, then the green die wins. Therefore, we have

$$\text{P(Green Winning Against Blue)} = \frac{2}{3}$$

$$\text{P(Blue Winning Against Green)} = \frac{1}{3}$$

So between green and blue, green is a better choice.

Blue Versus Red

What happens if the two players pick the blue and red dice (in some order)? We know that the blue die will always roll a 3, and so we only need to consider the roll of the red die. If the red die rolls a 2, which happens with probability $\frac{4}{6} = \frac{2}{3}$, then the blue die wins. If the red die rolls a 6, which happens with probability $\frac{2}{6} = \frac{1}{3}$, then the red die wins. Therefore, we have

$$\text{P(Blue Winning Against Red)} = \frac{2}{3}$$

$$\text{P(Red Winning Against Blue)} = \frac{1}{3}$$

So between blue and red, blue is the better choice.

Red Versus Yellow

What happens if the two players pick the red and yellow dice (in some order)? In the table below we indicate which die wins for each of the 36 equally likely rolls.

	1	1	1	5	5	5
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
6	R	R	R	R	R	R
6	R	R	R	R	R	R

From this table, we see that

$$\text{P(Red Winning Against Yellow)} = \frac{24}{36} = \frac{2}{3}$$

$$\text{P(Yellow Winning Against Red)} = \frac{12}{36} = \frac{1}{3}$$

Since the rolls of the two dice are independent, we can calculate probabilities without listing all of the outcomes. There is only one type of roll that will result in a win for the yellow die: 5 rolled on the yellow die and 2 rolled on the red die. The probability that a 5 is rolled on the yellow die is $\frac{3}{6} = \frac{1}{2}$, and the probability that a 2 is rolled on the red die is $\frac{4}{6} = \frac{2}{3}$.
Multiplying these two probabilities we determine that

$$\text{P(Yellow Winning Against Red)} = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$$

and since there are only two possibilities (yellow wins or red wins), the two probabilities must sum to 1, and so we have that

$$\text{P(Red Winning Against Yellow)} = 1 - \frac{1}{3} = \frac{2}{3}.$$

So between red and yellow, red is the better choice.

Yellow Versus Green

What happens if the two players pick the yellow and green dice (in some order)? Similar reasoning to the case above allows us to determine that

$$\text{P(Green Winning Against Yellow)} = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$$
$$\text{P(Yellow Winning Against Green)} = 1 - \frac{1}{3} = \frac{2}{3}$$

So between yellow and green, yellow is the better choice.

At this point there are two more pairs to consider (green and red, and blue and yellow) which are left to you. Let's summarize our findings for the four pairs we considered:

- In a game played with the green and blue dice, the better die is green.
- In a game played with the blue and red dice, the better die is blue.
- In a game played with the red and yellow dice, the better die is red.
- In a game played with the yellow and green dice, the better die is yellow.

One interesting thing that our results show is that the dice form a kind of "cycle", rather than an ordering from "best" to "worst". For every choice of die, there exists another die which is a better choice, and another die that is a worse choice. Now we discuss the follow-up questions:

If you choose your die first, then no matter what die you choose, the other player will be able to choose a die that beats your die with probability $\frac{2}{3}$. If you choose your die second, then the best choice of die is outlined above: if the first player chooses blue, then you should choose green; if the first player chooses red, then you should choose blue; if the first player chooses yellow then you should choose red, and if the first player chooses green then you should choose yellow. Can you convince yourself that these are the best choices? For example, if your opponent chooses the red die, then we know the blue die is a good choice (from bullet 2) and the yellow die is a bad choice (from bullet 3), but what about the green die? Since the blue die will win against the red die with a probability of $\frac{2}{3}$, you need to convince yourself that the green die has worse odds than this against the red die.

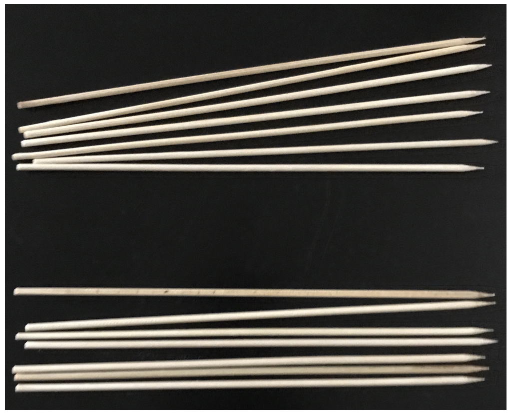
Background of the dice: These dice were invented by Bradley Efron, an American statistician. The dice provide an example of four *non-transitive** dice. You can create different sets of non-transitive dice. For example, you can create a set of three 6-sided non-transitive dice using the values from 1 to 9 on the faces of the dice. Non-transitive dice are an interesting topic about which you might want to do more reading.

*You may have heard the word *transitive* used in math before, and even if you have not, you have likely worked with many comparisons that are transitive: for example, if $x < y$ and $y < z$ then we must have $x < z$. In the case of Efron's dice, we can compare the dice, but these comparisons may not be transitive: for example, if "green is better than blue" and "yellow is better than green", must "yellow be better than blue"?

CEMC at Home

Grade 11/12 - Monday, April 6, 2020

Pick up Sticks



You Will Need:

- Two players
- 14 sticks (these could be chopsticks, popsicle sticks, toothpicks, pencils, etc.)

How to Play:

1. Arrange the 14 sticks into two piles with 7 sticks each.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On your turn, you can pick up as many sticks as you want, as long as all of the sticks you pick up come from the same pile.
4. The player who picks up the last stick wins the game!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy for this game? If so, which player (Player 1 or Player 2) has a winning strategy?

Is there a connection between this game and the game we played on March 23 (Rook to the Top)?

Variations:

A. Change the number of sticks that you have in the two piles. There should still be an equal number of sticks in each pile. How does this variation affect your strategy for the game?

B. Play the game instead starting with two piles with a different number of sticks. How does this variation affect your strategy for the game?

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for a discussion of a strategy for Pick Up Sticks. We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Monday, April 6, 2020
Pick Up Sticks - Strategy

As you might have expected based on our first two games in this game series, Pick Up Sticks is just another variation of the first game Rook to the Top (from March 23).

Please take a minute to remind yourself of the Rook to the Top game and its winning strategy.

How are these games related? To see the connection, we will first describe the game Pick Up Sticks in terms of ordered pairs. Let's label our two piles A and B. The ordered pair (a, b) will refer to the state of the game where we have a sticks remaining in pile A and b sticks remaining in pile B. Therefore, we start with the state corresponding to the ordered pair $(7, 7)$ and the winning state corresponds to the ordered pair $(0, 0)$. There are 64 different possible states during the game and they correspond to the ordered pairs of integers (a, b), where $0 \leq a \leq 7$ and $0 \leq b \leq 7$.

A valid move in the game takes us from an ordered pair (a, b) to another ordered pair (a^*, b^*) where either $a^* < a$ and $b^* = b$, or $a^* = a$ and $b^* < b$. The first case represents removing some number of sticks from pile A (decreasing the first component and leaving the second component unchanged), and the second case represents removing some number of sticks from pile B (decreasing the second component and leaving the first component unchanged).

So how does this help us see the connection to the game Rook to the Top? Remember that Rook to the Top is played on an 8 by 8 grid with 64 squares. We begin by labelling the 64 squares on our grid with the ordered pairs as shown. (Notice that the convention in this table is not the same as the normal labelling of ordered pairs in the plane.)

(0,7)	(0,6)	(0,5)	(0,4)	(0,3)	(0,2)	(0,1)	(0,0)
(1,7)	(1,6)	(1,5)	(1,4)	(1,3)	(1,2)	(1,1)	(1,0)
(2,7)	(2,6)	(2,5)	(2,4)	(2,3)	(2,2)	(2,1)	(2,0)
(3,7)	(3,6)	(3,5)	(3,4)	(3,3)	(3,2)	(3,1)	(3,0)
(4,7)	(4,6)	(4,5)	(4,4)	(4,3)	(4,2)	(4,1)	(4,0)
(5,7)	(5,6)	(5,5)	(5,4)	(5,3)	(5,2)	(5,1)	(5,0)
(6,7)	(6,6)	(6,5)	(6,4)	(6,3)	(6,2)	(6,1)	(6,0)
(7,7)	(7,6)	(7,5)	(7,4)	(7,3)	(7,2)	(7,1)	(7,0)

We know that each ordered pair corresponds to a possible state in a game of Pick Up Sticks. A valid move where the first component does not change and the second component decreases corresponds to a move to the right in the table. A valid move where the second component does not change and the first component decreases corresponds to a move upward in the table.

As with the grid in our solution of Rook to the Top, we highlight an important diagonal on the grid, called the *winning diagonal*. Again, the strategy is to make moves, if possible, to end up on the winning diagonal. The ordered pairs on the winning diagonal correspond to the states where the two piles have an equal number of sticks.

From our knowledge of the winning strategy for Rook to the Top, we know that the second player has a winning strategy and this strategy is to always move the rook back onto the winning diagonal. Similarly, the second player in Pick Up Sticks has a winning strategy, and this strategy is to always "move" the game into a state located on the winning diagonal. This means always picking up sticks in a way that leaves the two piles with an equal number of sticks.

Since we start with two equal piles, the first player has no choice but to make the piles unequal. The second player can then make the piles equal and the total number of sticks in the piles will have decreased. The first player again must make the piles unequal and the second player can again make the piles equal. Repeating this process, the second player will always be able to return to a state on the winning diagonal with fewer sticks in the piles, in total. Since there is a finite number of sticks, the second player will eventually make the piles equal with no sticks left and win the game. Therefore, the second player has a winning strategy for this game.

Variations:

A. *Different number of sticks in the piles, but piles still equal in size.*

   Now that we know the winning strategy for the original version of Pick Up Sticks, we can see that the number of sticks we start with has no effect on the winning strategy, provided that we still start with piles that are equal in size. The second player will still have a winning strategy and it will be the same as described above.

B. *Piles that are not equal in size.*

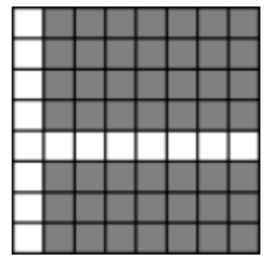
   If we start the game with two piles with different numbers of sticks, then the strategy is similar, but the first player in the game would have the winning strategy. On the first move, the first player can remove sticks from one pile in order to leave the piles with an equal number of sticks. The second player then has no choice but to make the piles unequal, giving the first player the chance to make them equal again (with fewer sticks in total). The argument continues as it did for the strategy in the original game.

CEMC at Home

Grade 11/12 - Tuesday, April 7, 2020
Counting and Probability

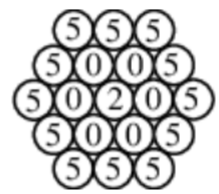
Today's problems are from some of our past Fermat Contests. These problems were compiled using our Problem Set Generator. The Problem Set Generator can be used to create a randomly generated problem set or a problem set focussed on a specific topic and/or a specific level of difficulty. We chose the topic of *counting and probability* to generate this problem set. Try using the Problem Set Generator to create your own problem set!

1. In the diagram, how many 1×1 squares are shaded in the 8×8 grid?
   (A) 53 (B) 51 (C) 47
   (D) 45 (E) 49



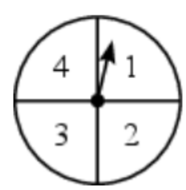
(Source: 2017 Fermat (Grade 11), #2)
Primary Topics: **Geometry and Measurement | Counting and Probability**
Secondary Topics: **Area | Counting**

2. Starting with the 2 in the centre, the number 2005 can be formed by moving from circle to circle only if the two circles are touching. How many different paths can be followed to form 2005?
   (A) 36 (B) 24 (C) 12
   (D) 18 (E) 6



(Source: 2005 Fermat (Grade 11), #12)
Primary Topics: **Counting and Probability**
Secondary Topics: **Counting | Digits**

3. On each spin of the spinner shown, the arrow is equally likely to stop on any one of the four numbers. Deanna spins the arrow on the spinner twice. She multiplies together the two numbers on which the arrow stops. Which product is most likely to occur?
   (A) 2 (B) 4 (C) 6
   (D) 8 (E) 12



(Source: 2014 Fermat (Grade 11), #15)
Primary Topics: **Counting and Probability | Number Sense**
Secondary Topics: **Probability | Counting**

4. Amina and Bert alternate turns tossing a fair coin. Amina goes first and each player takes three turns. The first player to toss a tail wins. If neither Amina nor Bert tosses a tail, then neither wins. What is the probability that Amina wins?
   (A) $\frac{21}{32}$ (B) $\frac{5}{8}$ (C) $\frac{3}{7}$ (D) $\frac{11}{16}$ (E) $\frac{9}{16}$

(Source: 2015 Fermat (Grade 11), #21)
Primary Topics: **Counting and Probability**
Secondary Topics: **Probability**

More Info: Visit our webpage to find past Fermat Contests and their solutions.

CEMC at Home

Grade 11/12 - Wednesday, April 8, 2020
Silly Square Roots

The connections between mathematics and computer science run deep. Here we explore these connections through the following problem which was Question 5a on the 2019 Euclid Contest. No programming experience is needed to follow along but there is also a challenge for you if you do have some programming experience.

Problem 1

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{50}$.

Solution to Problem 1 Using Mathematics

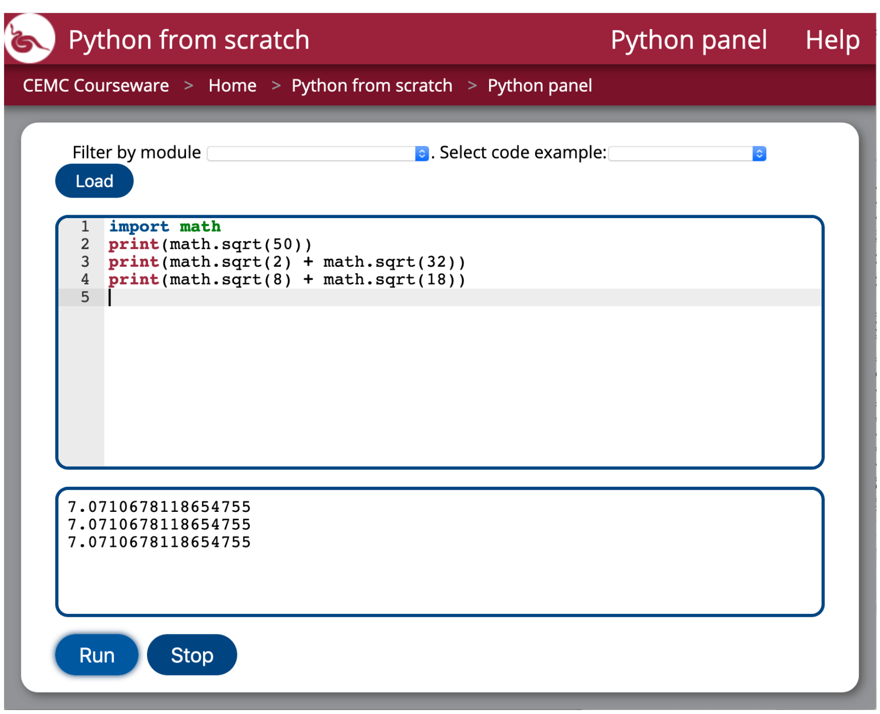
On a math contest or test, we would normally use known mathematical facts, such as the properties of square roots, to find the two pairs. Attempt the problem above this way and then check your answer. Note that your solution can rely on the fact that we were told there are exactly two pairs satisfying the equation.

Solution to Problem 1 Using Computer Programming

Now visit our Python from scratch panel and enter the following code in the upper code box.

```
import math
print(math.sqrt(50))
print(math.sqrt(2) + math.sqrt(32))
print(math.sqrt(8) + math.sqrt(18))
```

Hit the Run button to see the result of this program. Your screen should look something like this:



Congratulations if you just entered a computer program and ran it for the first time!

The key thing to observe is that this program uses a library function called `math.sqrt` to display the three values $\sqrt{50}$, $\sqrt{2}+\sqrt{32}$ and $\sqrt{8}+\sqrt{18}$. They all appear to be equal suggesting that the two pairs are $(2, 32)$ and $(8, 18)$. But where did these numbers come from?

In order to discover these two pairs, we could generate possible pairs (a, b) and check if $\sqrt{a}+\sqrt{b} = \sqrt{50}$ for each pair. Think about how you would generate possible pairs systematically and check whether the equality holds. The next program will simulate one way to do this. Replace your previous program with the following program and run it to see what it displays.

```
import math
for a in range(1, 50):
    for b in range(a + 1, 50):
        if (math.sqrt(a) + math.sqrt(b)) == math.sqrt(50):
            print((a,b))
```

Try to get a rough understanding of why this produces us the right answer. One advantage of programs written in Python is that they tend to be readable by beginners. However, if you are new to programming or new to the language Python, here are a few notes to help explain some of the details:

- The first line gives the rest of the program access to a library of mathematical functions such as `math.sqrt`. If we do not include this line, Python will produce an error message.
- The second line tells us that a *variable* `a` will take on the integer values starting at 1 and ending at $50 - 1 = 49$. (Python does not include the second number in the range.) The block of three lines indented below this will be executed once for each of these values of `a`. Together, this is typically called a *loop*.
- The third line is similar to the second line except the values of `b` begin at the value `a + 1`. The result is that we have one loop nested inside another. For each value of `a`, the variable `b` will take on its own range of values.
- To see the how the values of the variables `a` and `b` change, you can `print` (or display) them at a strategic point in your program. Here is an example using 5 in place of 50:

```
for a in range(1, 5):
    for b in range(a + 1, 5):
        print((a,b))
```

```
(1, 2)
(1, 3)
(1, 4)
(2, 3)
(2, 4)
(3, 4)
```

- The fourth line executes a conditional test involving variables `a` and `b`. The `==` operator attempts to determine if the two expressions are equal. If they are, the indented line below is executed. Otherwise, it is skipped.

Now ask yourself why 50 is used as an upper bound in `range(1,50)` and `range(a + 1, 50)`. Can this be improved? Why is 1 used as a lower bound in `range(1, 50)` but not in `range(a + 1, 50)`? We will discuss these details when solutions to the next two problems are presented.

Problem 2

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{75}$.

Attempt to Solve Problem 2

The two pairs that solve this problem are $(3, 48)$ and $(12, 27)$. However, our programming solution to Problem 1 where each 50 is replaced by 75, does not work! You should try it yourself to see the result. Why is nothing displayed? We get a clue by doing this:

```python
import math
print(math.sqrt(75))
print(math.sqrt(3) + math.sqrt(48))
print(math.sqrt(12) + math.sqrt(27))
```

```
8.660254037844387
8.660254037844386
8.660254037844386
```

The fundamental problem is that the `math.sqrt` function does not compute exact values. We can see that the displayed results above are extremely close to each other but the first one, the purported value of $\sqrt{75}$, is just a bit larger.

This is not a flaw with Python! It is impossible to store let alone calculate irrational numbers like $\sqrt{75}$, π and e exactly. An infinite amount of memory would be needed to do this. We already know this from mathematics where we understand that an irrational number is one for which its decimal expansion does not terminate or end with a repeating sequence.

Can you find a way to use a combination of mathematical insight and Python to solve this second problem?

Problem 3

Determine *all* pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a} + \sqrt{b} = \sqrt{147}$.

More Info:

Check out the CEMC at Home webpage on Wednesday, April 15 for a solution to Silly Square Roots.

Our webpage Computer Science and Learning to Program is the best place to find the CEMC's computer science resources. Two resources through which you can explore Python further are:

Python from scratch
A gentle introduction to programming designed with the beginner in mind.

CS Circles
Interactive lessons teaching the basics of writing computer programs in Python. This is also an introduction but moves at a bit of a faster pace. The CS Circles console is an alternative tool that can be used to enter and run all the code explored in this resource.

CEMC at Home

Grade 11/12 - Wednesday, April 8, 2020
Silly Square Roots - Solution

Here we expand on our use of writing Python programs to solve Question 5a on the 2019 Euclid Contest and two variations of this problem.

Problem 1

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{50}$.

Discussion of Programming Solution

We saw that the following program gives the two correct answers.

```
import math
for a in range(1, 50):
    for b in range(a + 1, 50):
        if (math.sqrt(a) + math.sqrt(b)) == math.sqrt(50):
            print((a,b))
```

Here are some observations about the mathematics used to inform and write the code:

- The smallest value to consider for a is 1 because it must be a positive integer.
- The smallest value to consider for b is $a + 1$ because we want $a < b$ and both a and b must be integers.
- If $a \geq 50$, then $\sqrt{a} + \sqrt{b} \geq \sqrt{50} + \sqrt{b} \geq \sqrt{50}$. This means we don't need to consider values of a that are greater than or equal to 50. The same applies for b.
- Since we want $a < b$, we have $2\sqrt{a} = \sqrt{a} + \sqrt{a} < \sqrt{a} + \sqrt{b}$. So if $\sqrt{a} + \sqrt{b} = \sqrt{50}$, we get $2\sqrt{a} < \sqrt{50}$. Squaring both sides and rearranging gives $a < \frac{50}{4} < 13$. This means that we could have replaced `range(1,50)` with `range (1,13)`, and hence checked fewer pairs.

Problem 2

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{75}$.

Programming Solution

We noticed that changing 50 to 75 in the programming solution to Problem 1, does not produce the correct answer for Problem 2. In this sense, our solution to Problem 1 was "lucky". One approach that does give us the right answer is the following:

```
import math
for a in range(1,75):
    for b in range(a+1,75):
        if abs(math.sqrt(a) + math.sqrt(b) - math.sqrt(75)) < 0.0001:
            print((a,b))
```

Since the `math.sqrt` function gives close approximate values, we can test if $\sqrt{a}+\sqrt{b}$ is approximately equal to $\sqrt{75}$. To do this, we look at the positive difference of these two values. This is called the absolute value and is written $|\sqrt{a}+\sqrt{b}-\sqrt{75}|$ in mathematics and computed using

```
abs((math.sqrt(a) + math.sqrt(b)) - math.sqrt(75))
```

in Python. It turns out that being within 0.0001 is "close enough". That is, if we use Python to search for pairs (a, b) satisfying $|\sqrt{a}+\sqrt{b}-\sqrt{75}| < 0.0001$ as outlined earlier, then we happen to find exactly two pairs: $(3, 48)$ and $(12, 27)$. We were told that there are exactly two solutions to the equation, and so we can be sure that our program has found all of the solutions. On the other hand, if we had used Python to find pairs (a, b) satisfying $|\sqrt{a}+\sqrt{b}-\sqrt{75}| < 0.001$, then we would have instead found three pairs: the two solutions as well as one extraneous pair, $(8, 34)$. If we changed the bound to 0.01 or 0.1, then the true solutions would be hidden among an even larger number of extraneous pairs. You can explore the number of pairs in these cases on your own.
How would you use the results of these searches to help determine the complete set of solutions if you were not told in advance that there were exactly two solutions?

Problem 3

Determine all pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b}=\sqrt{147}$.

Programming Solution

Here we do not know how many pairs of positive integers to look for. A solution to this problem uses a nice combination of mathematics and computer science.

Consider positive integers a and b and suppose that

$$\sqrt{a}+\sqrt{b}=\sqrt{147}.$$

By squaring both sides we get

$$a+2\sqrt{ab}+b=147.$$

Rearranging gives

$$2\sqrt{ab}=147-a-b.$$

Squaring both sides again yields

$$4ab=(147-a-b)^2.$$

This is an equation that does not involve square roots. But we do have to be a bit careful because squaring equations can introduce solutions. Now, we are only considering cases where a and b are positive, so the first time we squared both sides of the equation above did not introduce solutions. It is also true that the solutions we are looking for must satisfy $a+b<147$ (try to show this!) which means the second time we squared both sides of the equation also did not introduce solutions. Now we can test values for a and b that satisfy this equation only using integer operations, which give exact values in Python:

```
import math
for a in range(1,147):
    for b in range(a+1,147):
        if 4*a*b == (147-a-b)*(147-a-b) and a + b < 147:
            print((a,b))
```

Running this program gives us the answers of $(3, 108)$, $(12, 75)$ and $(27, 48)$.

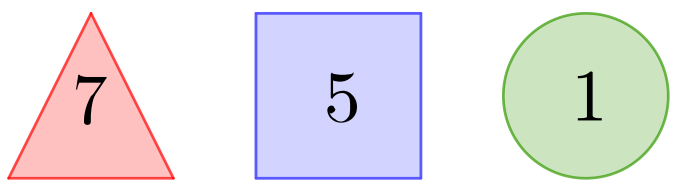
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 9, 2020
What's Your Unlucky Number?

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky.

Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $20 \times 12 \times 8 = 1920$ different combinations of tokens that can be created by selecting one token from each bag. Note that the order of selection does not matter. Also note that selecting the 7 red token, the 5 blue token and 1 green token is different than selecting the 5 red token, 7 blue token and the 1 green token.



Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?

More Info:

Check the CEMC at Home webpage on Thursday, April 16 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 16.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week
Problem E and Solution
What's Your Unlucky Number?

Problem

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky. Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $20 \times 12 \times 8 = 1920$ different combinations of tokens that can be created by selecting one token from each bag. Note that the order of selection does not matter. Also note that selecting the 7 red token, the 5 blue token and 1 green token is different than selecting the 5 red token, 7 blue token and the 1 green token.

Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?

Solution

There are 20 different numbers which can be selected from the green bag, 12 different numbers which can be selected from the red bag, and 8 different numbers which can be selected from the blue bag. So there are $20 \times 12 \times 8 = 1920$ different combinations of numbers which can be selected from the three bags.

Let (g, r, b) represent the outcome of a selection where g is the number on the token selected from the green bag, r is the number on the token selected from the red bag and b is the number on the token selected from the blue bag. Also, $1 \leq g \leq 20$, $1 \leq r \leq 12$, and $1 \leq b \leq 8$, for integers g, r, b.

Numbers that are divisible by 13 are $13, 26, 39, 52, \cdots$. The maximum sum that can be reached any selection is $8 + 12 + 20 = 40$. To count the number of possibilities for sums which are divisible by 13, we will consider three cases: a sum of 13, a sum of 26 and a sum of 39. Within the first two cases, we will look at sub-cases based on the possible outcome for the 8 possible selections from the blue bag.

1. **The sum of the numbers on the 3 tokens is 13.**
   - **1 is on the token selected from the blue bag**
     The sum of the numbers on the other two tokens is 12. Selecting the 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 11, 1), (2, 10, 1), (3, 9, 1), \cdots, (11, 1, 1)$, 11 possibilities in total.
   - **2 is on the token selected from the blue bag**
     The sum of the numbers on the other two tokens is 11. Selecting the 11 or 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 10, 2), (2, 9, 2), (3, 8, 2), \cdots, (10, 1, 2)$, 10 possibilities in total.

- **3 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 10. Using reasoning similar to the preceding two cases, the possibilities for (g, r, b) are $(1, 9, 3), (2, 8, 3), (3, 7, 3), \cdots, (9, 1, 3)$, 9 possibilities in total.
- **4 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 9. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 8, 4), (2, 7, 4), (3, 6, 4), \cdots, (8, 1, 4)$, 8 possibilities in total.
- **5 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 8. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 7, 5), (2, 6, 5), (3, 5, 5), \cdots, (7, 1, 5)$, 7 possibilities in total.
- **6 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 7. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 6, 6), (2, 5, 6), (3, 4, 6), \cdots, (6, 1, 6)$, 6 possibilities in total.
- **7 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 6. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 5, 7), (2, 4, 7), (3, 3, 7), (4, 2, 7), (5, 1, 7)$, 5 possibilities in total.
- **8 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 5. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 4, 8), (2, 3, 8), (3, 2, 8), (4, 1, 8)$, 4 possibilities in total.

Summing the results from the 8 cases, there are $11 + 10 + 9 + \cdots + 5 + 4 = 60$ combinations so that the sum of numbers on the 3 tokens is 13.

2. **The sum of the numbers on the 3 tokens is 26.**

- **1 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 25. The largest number possible from the green bag is 20 so the smallest possible number from the red bag would be 5. The largest number possible from the red bag is 12 so the smallest possible number from the green bag would be 13. The numbers from the green bag go from 20 to 13 while the numbers from the red bag go from 5 to 12. The possibilities for (g, r, b) are $(20, 5, 1), (19, 6, 1), (18, 7, 1), \cdots, (13, 12, 1)$, 8 possibilities in total.
- **2 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 24. Using similar reasoning to the first case in this section, the possibilities for (g, r, b) are $(20, 4, 2), (19, 5, 2), (18, 6, 2), \cdots, (12, 12, 2)$, 9 possibilities in total.
- **3 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 23. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 3, 3), (19, 4, 3), (18, 5, 3), \cdots, (11, 12, 3)$, 10 possibilities in total.

- **4 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 22. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are
  $(20, 2, 4), (19, 3, 4), (18, 4, 4), \cdots, (10, 12, 4)$, 11 possibilities in total.
- **5 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 21. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are
  $(20, 1, 5), (19, 2, 5), (18, 3, 5), \cdots, (9, 12, 5)$, 12 possibilities in total.
- **6 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 20. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 8. The highest number possible from the green bag is 19 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are
  $(19, 1, 6), (18, 2, 6), (17, 3, 6), \cdots, (8, 12, 6)$, 12 possibilities in total.
- **7 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 19. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 7. The highest number possible from the green bag is 18 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are
  $(18, 1, 7), (17, 2, 7), (16, 3, 7), \cdots, (7, 12, 7)$, 12 possibilities in total.
- **8 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 18. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 6. The highest number possible from the green bag is 17 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are
  $(17, 1, 8), (16, 2, 8), (15, 3, 8), \cdots, (6, 12, 8)$, 12 possibilities in total.

Summing the results from the cases, there are $8 + 9 + 10 + 11 + 4(12) = 86$ combinations so that the sum of the numbers on the 3 tokens is 26.

3. **The sum of the numbers on the 3 tokens is 39.**

   The maximum sum that can be obtained is $8 + 12 + 20 = 40$. A sum of 39 can only be achieved by keeping two of the three tokens at their maximum and reducing the third token to 1 less than its maximum. The possibilities for (g, r, b) are $(20, 12, 7)$, $(20, 11, 8)$ and $(19, 12, 8)$, 3 possibilities in total.

The total number of combinations in which the sum of the numbers on the three tokens is divisible by 13 is $60 + 86 + 3 = 149$.

Therefore, the probability of selecting three tokens with a sum which is divisible by 13 is $\frac{149}{1920}$.

There is less than an 8% chance that the three tokens selected by Sue will sum to a multiple of her unlucky number.

CEMC at Home

Grade 11/12 - Tuesday, April 14, 2020
One Equation and Two Unknowns

In this problem, we will look at finding solutions to the equation $x^2 - y^2 = n$ where x, y and n are all positive integers.

1. The equation $x^2 - y^2 = 75$ is satisfied by three pairs (x, y) of positive integers. Can you find these three pairs?

   Two of the pairs are not too hard to find by writing out a list of perfect squares or by trial and error. You'll need to be a lot more patient to find the third pair using these methods; you might try using a spreadsheet or writing a program to do this instead.

2. Determine the pair (x, y) of positive integers for which $x + y = 175$ and $x - y = 3$.

   Using our knowledge of differences of squares, $x^2 - y^2 = (x + y)(x - y)$. This means that if $x + y = 175$ and $x - y = 3$, then $x^2 - y^2 = (x + y)(x - y) = 175 \cdot 3 = 525$.

3. There are six pairs (x, y) of positive integers for which $x^2 - y^2 = 525$. Determine all such pairs.

   Note that one way to factor 525 is $525 = 175 \cdot 3$. In question 2. we found one solution to the given equation corresponding to this factor pair of 525. Can you use other factor pairs of 525 to find other solutions the given equation?

Extensions

A. Are there pairs (x, y) of positive integers for which $x^2 - y^2 = 210$. Why or why not?

B. Determine the number of pairs (x, y) of positive integers for which $x^2 - y^2 = 7!$.

Remember that 7! is the product of the positive integers from 1 to 7, inclusive; that is,

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

More Info:

Check out the CEMC at Home webpage on Tuesday, April 21 for a solution to One Equation and Two Unknowns.

CEMC at Home

Grade 11/12 - Tuesday, April 14, 2020
One Equation and Two Unknowns - Solution

1. We start by writing out the squares of the first 15 positive integers:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225$$

From this list, we might see that
$$100 - 25 = 75$$
and
$$196 - 121 = 75$$
From these, we see that $10^2 - 5^2 = 75$ and $14^2 - 11^2 = 75$, which tell us that $(x, y) = (10, 5)$ and $(x, y) = (14, 11)$ are solutions to the equation $x^2 - y^2 = 75$.
The third solution is $(x, y) = (38, 37)$, since $38^2 - 37^2 = 1444 - 1369 = 75$.
The solutions to 2. and 3. below will demonstrate one way to find this last solution.
Alternatively, we can rearrange the given equation to obtain $x^2 = y^2 + 75$.
Since $x > 0$, then $x = \sqrt{y^2 + 75}$. We could then write a computer program or use a spreadsheet to calculate the values of $\sqrt{y^2 + 75}$ starting from $y = 1, 2, 3, \ldots$ until we get a third value of $\sqrt{y^2 + 75}$ that is an integer.

2. If $x + y = 175$ and $x - y = 3$, then $(x + y) + (x - y) = 175 + 3$ or $2x = 178$, which gives $x = 89$.
If $x + y = 175$ and $x = 89$, then $y = 175 - x = 175 - 89 = 86$.
Therefore, $(x, y) = (89, 86)$.

3. First, we write 525 as a product of prime numbers:

$$525 = 105 \cdot 5 = 21 \cdot 5 \cdot 5 = 7 \cdot 3 \cdot 5 \cdot 5 = 3 \cdot 5^2 \cdot 7$$

This means the positive divisors of 525 are

$$1, 3, 5, 7, 15, 21, 25, 35, 75, 105, 175, 525$$

(We can find these by combining the prime factors of 525 in various ways.)
Since 525 has 12 positive divisors, then there are 6 ways of factoring 525 as the product of two positive integers:

$$525 = 525 \cdot 1 = 175 \cdot 3 = 105 \cdot 5 = 75 \cdot 7 = 35 \cdot 15 = 25 \cdot 21$$

Since $x^2 - y^2 = (x + y)(x - y)$, we can use the factorizations above to determine values for $x + y$ and $x - y$ and from them, find x and y as in 2.
For example, suppose that $x + y = 525$ and $x - y = 1$.
If these equations are true, then $x^2 - y^2 = (x + y)(x - y) = 525 \cdot 1 = 525$.
If $x + y = 525$ and $x - y = 1$, then $(x + y) + (x - y) = 525 + 1$ which gives $2x = 526$ and so $x = 263$.
Using $x + y = 525$ and $x = 263$, we obtain $y = 525 - x = 525 - 263 = 262$.

We can verify that $(x, y) = (263, 262)$ is a solution to the equation $x^2 - y^2 = 525$.
All 6 cases can be seen in the following table:

$x+y$	$x-y$	$2x = (x+y)+(x-y)$	x	$y = (x+y) - x$
525	1	526	263	262
175	3	178	89	86
105	5	110	55	50
75	7	82	41	34
35	15	50	25	10
25	21	46	23	2

We can verify that each of the pairs in this table is a solution to the equation $x^2 - y^2 = 525$.

Can you explain why there cannot be any other positive integer solutions?

Extensions:

A. There are no such pairs. If you make a table like the one in the solution to 3. using each factorization of 210, you will see that x and y are never integers (in fact, each of x and y is always halfway between two integers).
It turns out that when x and y are integers, then $x + y$ and $x - y$ have to be both even or both odd. (Try thinking about the various combinations of x and y being even and odd.)
This means that $(x + y)(x - y)$ is either odd (if both factors are odd) or a multiple of 4 (if both factors are even). Since 210 is neither odd nor a multiple of 4, then there cannot be any integer solutions.

B. We note that 7! is a multiple of 4. One way to solve this problem is to count the number of ways of factoring 7! as the product of two positive even integers. Using the thinking from A. and 3., this will allow us to find all of the positive integer solutions to $x^2 - y^2 = 7!$. Can you think through the steps to convince yourself that this is true?

CEMC at Home

Grade 11/12 - Wednesday, April 15, 2020

Escape Room

In the April 8 resource Silly Square Roots, we explored connections between mathematics and computer science. We continue that here, but instead of using computer science to investigate a past math contest problem, we will use mathematics to investigate Problem J5/S2 from the 2020 Canadian Computing Competition (CCC). No computing experience is needed.

Escape Room

An *escape room* is an M-by-N grid with each position (cell) containing a positive integer. The rows are numbered $1, 2, \ldots, M$ and the columns are numbered $1, 2, \ldots, N$. We use (r, c) to refer to the cell in row r and column c.

You start in the cell at $(1, 1)$ and can escape when you reach the cell at (M, N).

If you are in a cell containing the value x, then you can jump to any cell (a, b) with $1 \leq a \leq M$ and $1 \leq b \leq N$ that satisfies $a \times b = x$. For example, from a cell containing a 6, there are up to four cells you can jump to: $(2, 3)$, $(3, 2)$, $(1, 6)$, or $(6, 1)$. If the room is a 5-by-6 grid, there is no row 6 and therefore, only the first three jumps would be possible.

Consider the following example of a 3-by-4 escape room.

	Column 1	Column 2	Column 3	Column 4
Row 1	3	10	8	14
Row 2	1	11	12	12
Row 3	6	2	3	9

Starting in the cell at $(1, 1)$ which contains a 3, one possibility is to jump to the cell at $(1, 3)$. This cell contains an 8 so from it, you could jump to the cell at $(2, 4)$. This jump brings you to a cell containing 12 from which you can jump to $(3, 4)$ and escape.

Questions

1. Give a different sequence of jumps that allows you to escape the room in the example above.
2. Place positive integers in the empty cells of the room below so that it is possible to escape the room visiting each of the six cells along the way. Cells can be visited more than once.

	Column 1	Column 2	Column 3
Row 1	3		
Row 2	1		5

3. In this question, we will consider the 2-by-2 escape rooms for which each entry in the grid is a positive integer less than or equal to 4.
   (a) How many escape rooms of this form are there in total?
   (b) Describe all escape rooms of this form from which it is actually possible to escape.
   (c) One escape room of this form is chosen at random. What is the probability that it is possible to escape from this room?

More Info:

Check out the CEMC at Home webpage on Wednesday, April 22 for a solution to Escape Room.

If you have some programming experience (not much is needed!), you can try programming solutions to past problems from the CCC and test them the using CCC Online Grader.

CEMC at Home

Grade 11/12 - Wednesday April 15 2020
Escape Room - Solutions

Solution to Question 1

Another way to reach the cell at $(3, 4)$ and escape is to start in the cell at $(1, 1)$ which contains a 3, jump to the cell at $(3, 1)$ which contains a 6, jump to the cell at $(2, 3)$ which contains a 12, and jump to the cell at $(3, 4)$.

Solution to Question 2

It is possible to escape the room below by visiting each cell at least once.

	Column 1	Column 2	Column 3
Row 1	3	6	4
Row 2	1	2	5

One way to do this is to visit cells in the following order:

$$(1, 1), (1, 3), (2, 2), (2, 1), (1, 1), (1, 3), (2, 2), (1, 2), (2, 3)$$

Solution to Question 3

(a) Each 2-by-2 grid has exactly 4 entries. Since each entry in the grid must be a positive integer less than or equal to 4, it must be one of 1, 2, 3, or 4. Since there are 4 choices for each of the 4 entries in the grid, there are $4^4 = 256$ possible rooms.

(b) Consider *any* 2-by-2 room. If the cell at $(1, 1)$ contains a 4, then it is possible to immediately jump to $(2, 2)$ and escape. The only other way to escape is to jump to $(2, 2)$ from $(1, 2)$ or $(2, 1)$. This means that at least one of these cells must contain a 4 and $(1, 1)$ must contain a 2.

In summary, it is possible to escape from a 2-by-2 room exactly if it matches at least one of the three possibilities given below where a, b, c, d, e, f and g are positive integers.

	Column 1	Column 2
Row 1	4	a
Row 2	b	c

	Column 1	Column 2
Row 1	2	4
Row 2	d	e

	Column 1	Column 2
Row 1	2	f
Row 2	4	g

This describes *all* 2-by-2 rooms from which it is possible to escape, not just the ones that have entries that are at most 4. To describe only the rooms relevant to this question, we just restrict the parameters a, b, c, d, e, f, and g above to be positive integers less than or equal to 4.

(c) We are considering 2-by-2 rooms such that each integer in the grid is less than or equal to 4. We want to determine the probability $\frac{a}{b}$, where b is the total number of rooms of this form and a is the number of rooms of this form from which it is possible to escape.

From part (a), we have $b = 256$.

From part (b), since the parameters a, b, c, d, e, f, and g are each 1, 2, 3, or 4, there are

- $4 \times 4 \times 4 = 64$ ways to choose a, b and c,
- $4 \times 4 = 16$ ways to choose d and e, and
- $4 \times 4 = 16$ ways to choose d and e.

This gives a total of $64 + 16 + 16 = 96$ ways to pick the values for the parameters, but since there is some overlap between the bottom two sets of rooms from part (b), shown below, we have over counted the number of rooms.

	Column 1	Column 2
Row 1	2	4
Row 2	d	e

	Column 1	Column 2
Row 1	2	f
Row 2	4	g

More precisely, we have counted rooms with $d = f = 4$ twice. To account for this overlap, we must subtract 4 because there are exactly 4 rooms that appear in both sets above.

Therefore, the probability is $\frac{96-4}{256} = \frac{23}{64}$.

More Info:

If you have some programming experience (not much is needed!), you can try programming solutions to past problems from the Canadian Computing Competition and test them the using CCC Online Grader.

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 16, 2020
CUBES

Certain numbers have interesting properties. For example, $1^3 + 5^3 + 3^3 = 153$. That is, the sum of the cubes of the individual digits of the positive integer 153 is the number itself. This may lead you to ask a question like, "Are there other such numbers?" (Yes there are, but that is not our concern today.)

The number 512 stands alone as a three-digit positive integer with three different digits such that the cube of the sum of the digits equals the number itself. That is, $(5 + 1 + 2)^3 = 512$. This is the only three-digit positive integer with three distinct digits that has this property.

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number.

That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that

$$(C+U+B+E+S)^3 = CUBES$$

More Info:

Check the CEMC at Home webpage on Thursday, April 23 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 23.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week

$(C+U+B+E+S)^3 = CUBES$

Problem E and Solution

CUBES

Problem

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number. That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that $(C + U + B + E + S)^3 = CUBES$.

Solution

A straightforward approach to solving this problem is to determine the smallest possible number and the largest possible number. Then, work at finding the numbers in that range that satisfies the given property.

The smallest five-digit number with distinct digits is 10 234. Since $\sqrt[3]{10\,234} \doteq 21.7$, the smallest number to consider is $22^3 = 10\,648$. The largest sum of five distinct digits is $9 + 8 + 7 + 6 + 5 = 35$, so the largest number to consider is $35^3 = 42\,875$. The possibilities, if any exist, are from 22^3 to 35^3. We need to examine these cubes to find the solution.

Number	Number^3	Sum of the Digits	Has the Property?
22	10 648	19	no, $22 \neq 19$
23	12 167		no, digits not distinct
24	13 824	18	no, $24 \neq 18$
25	15 625		no, digits not distinct
26	17 576		no, digits not distinct
27	19 683	27	Yes, $(1 + 9 + 6 + 8 + 3)^3 = 19683$
28	21 952		no, digits not distinct
29	24 389	26	no, $29 \neq 26$
30	27 000		no, digits not distinct
31	29 791		no, digits not distinct
32	32 768	26	no, $32 \neq 26$
33	35 937		no, digits not distinct
34	39 304		no, digits not distinct
35	42 875	26	no, $35 \neq 26$

Since we have examined all possibilities, we can conclude that 19 683 is the only five-digit positive integer with distinct digits such that the cube of the sum of the digits of the number equals the original number.

CEMC at Home

Grade 11/12 - Friday, April 17, 2020

Order from Randomness

Today we will explore a famous geometric figure known as the *Sierpinski triangle*. The Sierpinski triangle is an example of an object called a *fractal* and is formed by the repeated application (or iteration) of a certain process.

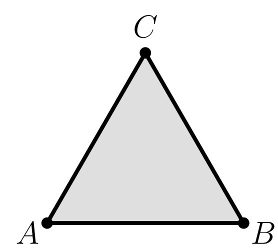
You will need:

- A blue pen and a black pen (or two different coloured pens)
- A ruler
- A six-sided die

Set up:

- On a piece of paper, draw an equilateral triangle with sides of length 16 cm. Label the vertices A, B, and C. *What tools might you use to make sure you draw an accurate triangle?*
- At various times, you will need to randomly choose one of the three vertices of the triangle. To do so, use a standard six-sided die and the following conversion table.

Number Rolled	Vertex
1 or 2	A
3 or 4	B
5 or 6	C



Instructions for Part I:

1. Start at the point A.

   We will call the point we are working with the "current point".

2. Roll the die to choose a second point according to the table above.

   If you roll a 1 or a 2 here, roll the die again until you get a number which is not 1 or 2.

3. Use your ruler to locate the midpoint between the current point and the point corresponding to your roll. Use your blue pen to draw a dot at this midpoint.

4. Repeat Steps 2 and 3 now starting with the midpoint most recently drawn (instead of A). Roll the die to choose a second point (A, B, or C). Draw the midpoint between the current point and the point corresponding to your roll. This new point may not be on the perimeter of $\triangle ABC$.

   Notice that from now on you will not need to re-roll the die if you get a 1 or a 2.

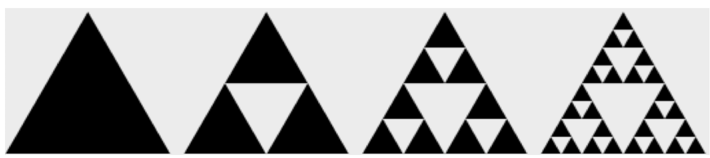
5. Repeat this process until you have drawn at least 10 new points.

   It is important to carefully draw the midpoints in Step 3 since accuracy will be important for our follow-up questions.

Look at the blue dots you drew in and on $\triangle ABC$. You might think that they are drawn not following any particular pattern but let's investigate this further!

Instructions for Part II:

1. Starting with a new copy of $\triangle ABC$, locate the midpoints of the sides of the triangle. Using your ruler and the black pen draw a line between each pair of midpoints. (There are 3 pairs.)
2. You should now have four smaller triangles: one in each corner of the original triangle, which we will call the *corner triangles*, and one *upside-down triangle* in the middle.
3. Perform Step 1 on each of the 3 corner triangles. After this, you should have twelve new even smaller triangles: 9 smaller corner triangles and 3 smaller upside-down triangles.
4. Perform Step 1 on each of the twelve corner triangles. After this, you should have 36 new smaller triangles: 27 smaller corner triangles and 9 smaller upside-down triangles.
5. With your black pen shade all of the 27 smaller corner triangles.

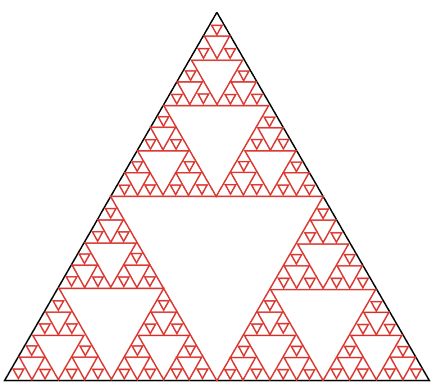


What do you notice when you compare the figure you created in Part I with the figure you created in Part II? If you were precise enough when placing the blue dots for the midpoints in Part I, then you should notice that all your dots ended up in or on the shaded regions in the figure from Part II, and that none of the dots are in the non-shaded regions!

Can you explain why all of the blue dots ended up in the shaded regions?

Follow-up Questions:

Imagine that at each iteration we remove (or cut out) the upside-down triangles from the figure.



1. After the first iteration, what is the total area of the three triangles remaining after the removal of the upside-down triangle? (Recall that our initial triangle has side lengths of 16 cm.)
2. After the second iteration, what is the total area of the nine triangles remaining after the removal of the upside-down triangles?
3. After the nth iteration, what is the total area of the triangles remaining after the removal of the upside-down triangles?

Try to visualize the figure we would obtain if we repeated this process a very large number of times. Imagine what would happen if we could repeat this process infinitely many times. The figure obtained is known as the *Sierpinski triangle*. It is impossible to draw the Sierpinski triangle, but with a bit of imagination, it is possible to visualize it and explore some of its interesting properties. Mathematicians have ways of formalizing various properties of this figure, including its area. What do you think the area of the Sierpinski triangle should be? You can explore this question on your own.

More Info:

Check the CEMC at Home webpage on Friday, April 24 for further discussion on this activity.

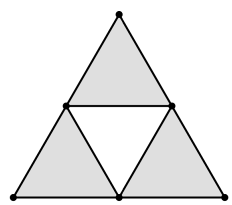
To learn more about fractals, check out this Math circles lesson.

CEMC at Home

Grade 11/12 - Friday, April 17, 2020
Order from Randomness - Solutions

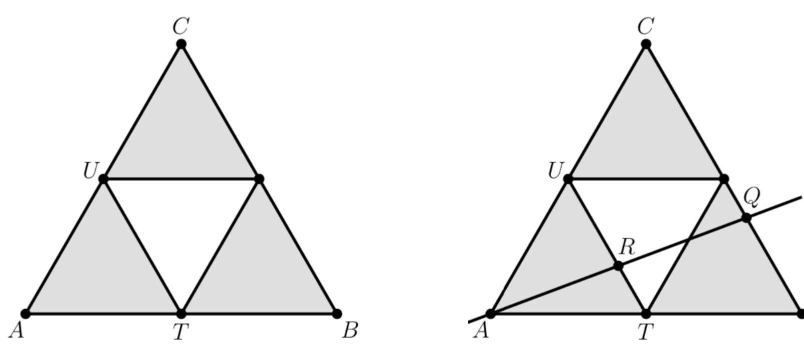
Question: Why did all of the blue dots drawn in Part I of the activity end up in the shaded regions of the figure drawn in Part II?

Discussion: Consider the figure obtained after performing steps 1 and 2 from Part II of the activity. We count the boundaries of the triangular regions as being part of the shaded regions, which we have coloured light grey instead of black for this explanation.



Assume that P is the most recently drawn blue dot. Assume also that after we draw P, we roll a 1 on the die. This means that we have chosen vertex A for the next iteration, and thus we will place a new blue dot at point P', the midpoint of line segment AP. Our goal is to show that P' must land in one of the shaded regions (rather than in the upside-down triangle). In particular, we will show that in this case P' must lie in the bottom leftmost shaded region.

Label the points U and T as shown below and draw the line passing through the points A and P.



You should think about how the picture would change if we rolled something other than a 1 (or 2). Also note that P does not appear in the diagram and we do not know exactly what this line would look like. If P is on the boundary of $\triangle ABC$, then the argument is a bit different. Here we have an illustration of what a "typical line" might look like and we work from here.

This line must intersect UT and CB somewhere; we call these points R and Q, respectively.

We know that P, P', and R all lie on line segment AQ.

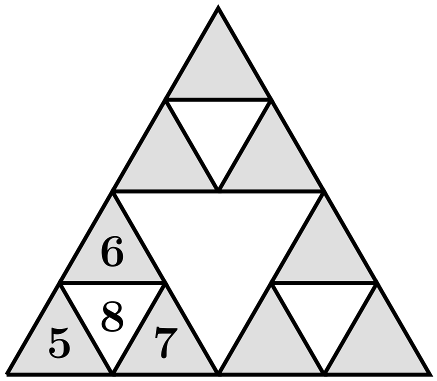
We know that $AP' = \frac{1}{2}AP$, by construction, and we can prove that $AR = \frac{1}{2}AQ$.

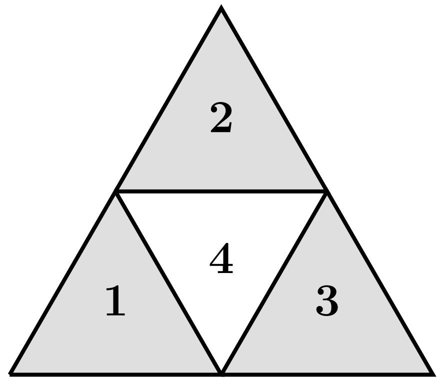
(See the end of the next page for a proof of this fact.)

Since $AP \leq AQ$, we must have $\frac{1}{2}AP \leq \frac{1}{2}AQ$ which means $AP' \leq AR$. This means that point P' must lie in the shaded region $\triangle AUT$.

Think about which region P' would end up in if we used vertex B or C instead of A.

We now know that P' must lie in the shaded region labelled with a 1 in the image below on the right. Now consider the figure obtained after performing Step 3 from Part II, shown below on the left. We will try to argue that P' must actually be in one of the smaller shaded regions, labelled 5, 6, and 7, rather than in the unshaded region labelled 8. This gives us "one more step" in our argument.





Using similar reasoning as before, we can show the following:

i. If P is in region 1, then P' must be in region 5.

ii. If P is in region 2, then P' must be in region 6.

iii. If P is in region 3, then P' must be in region 7.

iv If P is in region 4, then P' must be in region 8.

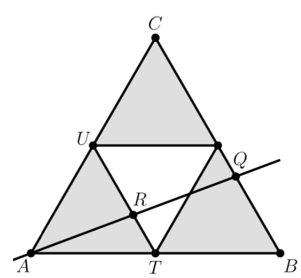
In particular, this shows that if P started off in one of the three larger shaded regions, then P' must end up in one of the three smaller shaded regions. This argument can be repeated to show that P' must lie in one of the even smaller shaded regions in the next figure in the pattern, and so on.

We have now outlined the main building blocks of an argument. A justification of the fact in question involves arguing that the first blue point we draw lies in a shaded region in *every figure* in the pattern. From this we can deduce from our work above that this is also true of every subsequent blue point drawn. This argument can be formalized using something called the principle of mathematical induction.

Proof that $AR = \frac{1}{2}AQ$.

Consider $\triangle AQB$ and $\triangle ART$. Since the segment TU is parallel to the segment BC, $\angle ART = \angle AQB$. As well, $\angle RAT = \angle QAB$ since they are the same angle, which means $\triangle ART \sim \triangle AQB$. Using this similarity and the fact that T is the midpoint of AB, we have the ratios:

$$\frac{AR}{AQ} = \frac{AT}{AB} = \frac{1}{2}.$$



This means R is the midpoint of the segment AQ. Put another way, Q is twice as far away from A as R is. If we let V be the point on ℓ that is twice as far from A as P is, then V will lie farther from A than Q, and hence, V cannot be inside the triangle. Remember, we are trying to show that none of the blue dots are in the upside down triangle. Let us return to this problem.

Follow-up Questions:

Imagine that at each iteration we remove (or cut out) the upside-down triangles from the figure.

1. After the first iteration, what is the total area of the three triangles remaining after the removal of the upside-down triangle? (Recall that our initial triangle has side lengths of 16 cm.)

   Solution:

   Since the three remaining equilateral triangles have the same side length of 8 cm, we will find the area of one of them and multiply it by 3.

   Using the Pythagorean theorem or trigonometry, you can check that an equilateral triangle of side length 8 cm has height of $4\sqrt{3}$ cm. Therefore, the area of one of the remaining triangles is

   $$\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(8 \times 4\sqrt{3}) = 16\sqrt{3}\text{ cm}^2.$$

   Therefore, the total area of the remaining three triangles is $48\sqrt{3}$ cm^2.

2. After the second iteration, what is the total area of the nine triangles remaining after the removal of the upside-down triangles?

   Solution:

   The nine remaining equilateral triangles have the same side length of 4 cm. Similar to the previous solution, we will find the area of one of them and multiply it by 9.

   An equilateral triangle of side length 4 cm has a height of $2\sqrt{3}$ cm, so its area is

   $$\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(4 \times 2\sqrt{3}) = 4\sqrt{3}\text{ cm}^2.$$

   Therefore, the total area of the nine remaining triangles is $36\sqrt{3}$ cm^2.

3. After the nth iteration, what is the total area of the triangles remaining after the removal of the upside-down triangles?

   Solution:

   In each iteration, every remaining triangle is divided into four smaller equilateral triangles of equal area and the middle triangle is removed. This means in each iteration, each remaining triangle has $\frac{1}{4}$ of its area removed. It follows that the total area of the remaining triangles after each iteration is $\frac{3}{4}$ the total area before that iteration.

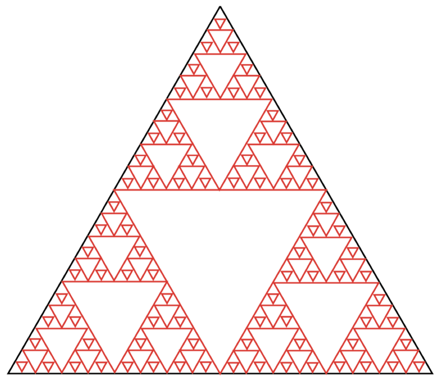
   This agrees with the previous solutions. Indeed, after the first iteration the area is $48\sqrt{3}$ and after the second iteration the area is

   $$36\sqrt{3} = \frac{3}{4}(48\sqrt{3})$$

   After the third iteration, the area will be $\left(\frac{3}{4}\right)^2 48\sqrt{3}$, after the fourth iteration it will be $\left(\frac{3}{4}\right)^3 48\sqrt{3}$, and so on. Continuing in this way, for a positive integer n, the area after the nth iteration we will be

   $$\left(\frac{3}{4}\right)^{n-1} 48\sqrt{3}$$

The Sierpinski Triangle



Since the Sierpinski triangle is obtained by continuing these iterations forever, the area of the Sierpinski triangle should be less than the area remaining after any finite number of iterations. An area should be nonnegative, so if we are to assign an area to this strange object, it should be some nonnegative number A with the property that

$$A < \left(\frac{3}{4}\right)^{n-1} 48\sqrt{3}$$

for *every* positive integer n. However, since $0 < \frac{3}{4} < 1$, the quantity $\left(\frac{3}{4}\right)^{n-1} 48\sqrt{3}$ will eventually be smaller than any positive number. Therefore, the only number that A could be is 0.

You might have come across this kind of idea before if you have studied *limits*.

CEMC at Home

Grade 11/12 - Monday, April 20, 2020
Pick up Sticks - Part 2

You Will Need:

- Two players
- 15 sticks (these could be chopsticks, popsicle sticks, toothpicks, pencils, etc.)

How to Play:

1. Arrange the 15 sticks into three piles: one with 3 sticks, one with 5 sticks, and one with 7 sticks.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On your turn, you can pick up as many sticks as you want, as long as all of the sticks you pick up come from the same pile.
4. The player who picks up the last stick wins the game!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy for this game? If so, which player (Player 1 or Player 2) has a winning strategy?

Can you use your winning strategy from the first game of Pick Up Sticks (from April 6) to develop a winning strategy for this game?

Variations: Change the number of sticks that you have in each of the three piles. (You will need to find more sticks, if you want to start with more than 15 sticks in total.) Does the number of sticks in each pile affect your winning strategy?

More Info:

Check out the CEMC at Home webpage on Monday, April 27 for a discussion of a strategy for this game. We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Monday, April 20, 2020
Pick Up Sticks - Part 2 - Solution

The approach we will use to figure out the winning strategy for this game will be different than the approach used in the similar game of Pick Up Sticks with two piles. The winning strategy will be built around an understanding of what we will call *winning positions*.

A position of the game will be represented by an ordered triple of non-negative integers. The ordered triple (a, b, c) will refer to the position where one pile has a sticks, one pile has b sticks and one pile has c sticks. We will always represent a position with an ordered triple (a, b, c) satisfying $a \leq b \leq c$. This can be done because the order of the piles does not matter. For example, if the three piles at some point have 1, 3, and 5 sticks, then the current position of the game is represented by $(1, 3, 5)$. If 3 sticks are now removed from the pile of 5, then the three piles have 1, 3, and 2 sticks but the new position is represented by $(1, 2, 3)$ rather than $(1, 3, 2)$.

A *winning position* is a position with the property that if we make a move to bring the game to that position, we have a winning strategy from that point forward. In other words, a winning position is a position from which our partner cannot win (unless we make a mistake later). If we move the game to a winning position, our partner cannot make a move to change it to a winning position. Furthermore, no matter what move our partner makes, there will be a move available on our next turn that changes the game back to a winning position.

By the definition of the game, we win if we pick up the last stick. Therefore $(0, 0, 0)$ is a winning position. We are going to develop a table of winning positions for our game. Since the game starts in the position $(3, 5, 7)$, we will not consider any positions (a, b, c) where $a > 3$, $b > 5$, or $c > 7$. Here are two observations.

Observation 1: From our work with Pick Up Sticks with two piles, we know that if a position has exactly two piles and these piles are equal, then it is a winning position. Also, a position having exactly two unequal piles is not a winning position. Therefore, the position $(0, k, k)$ is a winning position and the position $(0, k, \ell)$ with $k < \ell$ is not a winning position.

Observation 2: Suppose we have two position P and Q and that P is a winning position. If two of the piles in Q are the same as two of the piles of P *and* the other pile of Q has more sticks than the other pile of P, then Q is *not* a winning position. This is because there is a move from Q to the winning position P. For example, $(3, 3, 5)$ is not a winning position because $(0, 3, 3)$ is a winning position (see previous observation), the two positions $(0, 3, 3)$ and $(3, 3, 5)$ have two pile sizes in common, and the other pile in $(3, 3, 5)$ is larger than the other pile in $(0, 3, 3)$.

This video uses these two observations to develop a table of winning positions for our game.

Here are the winning positions: $(0, 0, 0)$, $(0, k, k)$, $(1, 2, 3)$, $(1, 4, 5)$, $(2, 4, 6)$, $(2, 5, 7)$, $(3, 4, 7)$, and $(3, 5, 6)$.

Since our starting position is $(3, 5, 7)$ we can see that Player 1 has a winning strategy. Player 1 can remove one stick from any of the three piles to move the game to a winning position.

If you change the starting position of the game, you will change which player has a winning strategy. In general, if the starting position is not a winning position, then Player 1 has a winning strategy, and if the starting position is a winning position, then Player 2 has a winning strategy. The reasoning from the video can be extended to identify the winning positions in a game starting with different numbers of sticks.

CEMC at Home

Grade 11/12 - Tuesday, April 21, 2020
Six Numbers

Problem:

There is a list of 6 unknown numbers p, q, r, s, t, u and they are ordered $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. Adding together the two numbers in each pair we get 15 sums. These 15 sums, in increasing order, are as follows

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$$

Determine the value of $r + s$.

Discussion: Try solving this problem! We will present different approaches for a solution next week, but for now let's look at the problem more closely and discuss some approaches that do not work.

Let's first understand what is being asked. The 15 sums mentioned are obtained by choosing two different numbers from the list p, q, r, s, t, u, and adding them together. For example, the values of the expressions $q + t$ and $u + r$ are among the 15 numbers listed in the question. One of the 15 numbers is $r + s$ and we need to determine which one.

One approach that someone might take is to make the following assumption: Since r and s are the middle two numbers in the list p, q, r, s, t, u, the sum $r + s$ should be the middle number in the list $25, 30, \ldots, 103, 117$, that is $r + s = 63$. This approach does not work. The problem is that the sum $r + s$ might not actually be equal to the middle number in the list of sums. For example, if we had the list of 6 numbers $1, 2, 4, 7, 100, 110$, you can check that the sum $r + s = 4 + 7 = 11$ is not the 8^{th} number in the list of the 15 sums, but is rather the 6^{th} number. The full list of sums is as follows:

$$3, 5, 6, 8, 9, \boxed{11}, 101, 102, 104, 107, 111, 112, 114, 117, 210$$

Expanding on the idea in this first (incorrect) approach, you could decide to write down the expressions for all 15 sums, involving the 6 unknowns, and start assigning values to particular expressions based on what we *do* know about their relative sizes. For example, since p and q represent the smallest two numbers, $p + q$ must represent the smallest sum and so we must have that

$$p + q = 25$$

You can argue that $p + r$ must represent the next smallest sum, so it must be the case that

$$p + r = 30$$

Unfortunately, using only this line of reasoning you will get stuck quite quickly. For example, you cannot be immediately sure of which expression represents the next smallest sum, 38. Can you see why? You can use similar reasoning to assign the largest values in the list of sums, but then will run into a similar problem there as well.

Think about other ways you might approach this problem.

More Info:

Check the CEMC at Home webpage on Tuesday, April 28 for two different solutions to Six Numbers.

CEMC at Home

Grade 11/12 - Tuesday, April 21, 2020
Six Numbers - Solution

Problem:

There is a list of 6 unknown numbers p, q, r, s, t, u and they are ordered $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. Adding together the two numbers in each pair we get 15 sums. These 15 sums, in increasing order, are as follows

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$$

Determine the value of $r + s$.

We will show two different solutions to this problem. For both solutions, we note that besides the original list of 6 numbers p, q, r, s, t, u, there are two additional lists of interest to examine here:

List 1: $25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$
List 2: $p + q, p + r, \ldots, t + u$

What is the relationship between List 1 and List 2? They contain the exact same 15 numbers, just possibly in a different order. In fact, we have not even specified an order for List 2. You can do this if you wish by explicitly writing down the 15 sums.

Solution 1

First, note that since $p < q < r < s < t < u$, the expression $p + q$ must represent the smallest of the 15 sums. Also, $p + r$ must represent the second smallest sum (think about why). This means $p+q = 25$ and $p+r = 30$ and so $(p+r)-(p+q) = 30-25 = 5$. However, $(p+r)-(p+q) = r-q$, so it must be the case that $r - q = 5$. Using this, observe that $(s + r) - (s + q) = r - q = 5$, and similarly $(t + r) - (t + q) = r - q = 5$, and finally $(u + r) - (u + q) = r - q = 5$. That is, each of the following 4 pairs of sums from List 2 has a difference of 5 when the first number in the pair is subtracted from the second: $(p + q, p + r), (s + q, s + r), (t + q, t + r), (u + q, u + r)$.

List 1 contains exactly 4 pairs of numbers with a difference of 5. They are $(25, 30)$,$(49, 54)$, $(63, 68)$, and $(90, 95)$. Therefore, these 4 pairs must be the 4 pairs $(p + q, p + r)$, $(s + q, s + r)$, $(t + q, t + r)$, and $(u+q, u+r)$ in some order. Focusing on second coordinates, this means the numbers 30, 54, 68, and 95 are the sums $p + r$, $s + r$, $t + r$, and $u + r$ in some order. Since $p < s < t < u$, we must have $p+r < s+r < t+r < u+r$, which means $s+r$ is the second smallest of 30, 54, 68, and 95. Thus, $r + s = s + r = 54$.

See the next page for Solution 2.

Solution 2

We start by stating the key to this approach: The sum of all of the numbers in List 1 is equal to the sum of all of the numbers in List 2. This is because addition is a commutative operation. In other words, the order in which you add numbers does not change the sum.

So let's add up all the values in List 1 and then add up all the values in List 2 and compare these two sums. Adding up the values in List 1 gives

$$25 + 30 + 38 + 41 + 49 + 52 + 54 + 63 + 68 + 76 + 79 + 90 + 95 + 103 + 117 = 980$$

What about the sum of the values in List 2? If we add up the 15 expressions, how many times does each variable appear in the sum? Let's start with variable p. There are 5 p's because p appears in exactly five of the 15 expressions in the list. In particular, p appears exactly once with each of the 5 other variables. Similarly, there are 5 q's, 5 r's, 5 s's, 5 t's, and 5 u's. Therefore, adding up all expressions in List 2 results in the expression

$$5p + 5q + 5r + 5s + 5t + 5u$$

Since List 1 and List 2 have the same sum, we have the equality

$$5p + 5q + 5r + 5s + 5t + 5u = 980$$

Dividing both sides by 5 gives us that

$$p + q + r + s + t + u = 196$$

which we rewrite in the following helpful way

$$(p + q) + (r + s) + (t + u) = 196$$

Since p and q are the smallest two numbers in our list of 6 integers, $p+q$ must be equal to the smallest number in List 1 and so $p + q = 25$. Similarly, since t and u are the two largest numbers in our list of 6 integers, $t + u$ must be equal to the largest number in List 1 and so $t + u = 117$. Therefore,

$$r + s = 196 - (p + q) - (t + u) = 196 - 25 - 117 = 54$$

How neat is that! Nobody told us to add up all 15 numbers, but once we do it (in the two different ways, using each of Lists 1 and 2), a nice solution reveals itself!

CEMC at Home

Grade 11/12 - Wednesday, April 22, 2020
Comparison Machine

Your mission, should you choose to accept it, is to develop algorithms to complete tasks involving the the relative order of n distinct integers. The problem is that the integers are random and unknown to you! All you know is that they are named $a_1, a_2, \dots, a_n$. Note that we call i the *index* of the integer a_i.

For each task, your approach must work no matter what the order of the integers is.

Some good news: A helpful machine is available. The machine knows the relative order of these integers. To use it, you enter the indices of two integers into the machine and it will tell you which of the two corresponding integers is larger. For example, if $a_4 = 5$, $a_2 = 7$ and you enter 4 and 2 into the machine, it will tell you that a_2 is larger. We name the machine M and in this case we have $M(4, 2) = 2$ and $M(2, 4) = 2$. Either of these application of M tells you that the integer with index 2 is larger than the integer with index 4.

Some bad news: For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.

Some more good news: Your memory is perfect and you can remember (or record) the result every time you use the machine.

Example

Suppose $n = 4$ and you want to determine which of the integers, a_1, a_2, a_3, or a_4, is the largest, while limiting yourself to only 3 uses of the machine. Here is one way to do this:

1. Compute $M(1, 2)$ and record this answer as the index x.
2. Compute $M(3, 4)$ and record this answer as the index y.
3. The largest integer is the integer with index $M(x, y)$.

The Tasks and a Fun Tool

Below are the three different tasks to be completed. Each task outlines how many integers there will be in the list (n), what you are attempting to answer about the list (Task), and how many times you can ask the machine for help (Limit).

n	Task	Limit
7	Determine the largest integer.	6
8	Determine both the largest integer and the smallest integer.	10
8	Determine the second largest integer.	9

Important: We have written a Python computer program that will generate random integers and simulate the helpful machine. It is a lot of fun to use this interactive tool to test if your solutions are correct. See the next page for instructions on how to use the tool.

Using the Tool

The tool works by repeatedly asking you what you want to enter into the machine and then displaying the result. After the number of times you have used the machine reaches the limit, it will ask you for the index of the largest integer in the list. It will then tell you whether or not you are correct.

You do not need to know anything about Python in order to use the tool.

Getting the correct answer for a few lists does not mean that you have a correct algorithm for the task.

Your algorithm has to work for any choice of integers, regardless of their order. The more you test your algorithm, the more evidence you have that it is correct. After testing out your algorithm using the tool, try to explain why your algorithm will work on all possible lists.

Here are instructions for using the tool:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open this free online Python interpreter in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. You will interact with the tool using the right black panel, and you might want to widen this panel.
5. After completing a test, or if you encounter an error, you can hit *run* to begin another test. If you want to start over during a test, you can hit *stop* and then *run*.

More Info:

Check out the CEMC at Home webpage on Wednesday, April 29 for solutions to the three tasks.

Our webpage Computer Science and Learning to Program is the best place to find the CEMC's computer science resources. Two resources through which you can explore Python further are:

Python from scratch
A gentle introduction to programming designed with the beginner in mind.

CS Circles
Interactive lessons teaching the basics of writing computer programs in Python. This is also an introduction but moves at a bit of a faster pace.

CEMC at Home

Grade 11/12 - Wednesday, April 22, 2020
Comparison Machine - Solutions

Summary of the Tasks

Develop algorithms to complete tasks involving the relative order of n distinct integers.

- The integers are random and unknown to you. All you know is that they are named $a_1, a_2, \ldots a_n$.
- For each task, your approach must work no matter what the order of the integers is.
- A helpful machine M is available. The machine knows the relative order of these integers. To use it, you enter the index of two integers into the machine and it will tell you which of the two corresponding integers is larger.
- For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.
- Your memory is perfect and you can remember (or record) the result every time you use the machine.

Details of the Three Tasks

n	Task	Limit
7	Determine the largest integer.	6
8	Determine both the largest integer and the smallest integer.	10
8	Determine the second largest integer.	9

Solution for Task 1

We will keep track of an index named c. The "current maximum" will be at index c. Begin by setting $c = 1$. Then, repeatedly set $c = M(c, i)$ for $i = 2, 3, \ldots, 7$. (Setting $c = M(c, i)$ means we compute the value of $M(c, i)$ and then update c to be this value.) This uses the machine exactly 6 times. We can then declare a_c to be the largest integer because the machine has told us either directly or indirectly that it is larger than each of the other integers.

Can you see why this algorithm works? If the index of the largest integer is 1, then we will see $M(1, i) = 1$ at each stage and the value of c will never change. If the index of the largest integer is not 1, then at some point the value of c will change. In particular, the value of c will change from 1 to i the first time we see a comparison of the form $M(1, i) = i$ for some $i \geq 2$. The value of c will continue to change each time we find an integer that is larger than our "current maximum".

An example using the provided tool is on the next page.

```
Enter i such that 1 <= i <= 7 : 1
Enter j such that 1 <= j <= 7 : 2
M(i,j) = 2
---------------------------------------
You have used the machine, 1 time(s).
---------------------------------------
Enter i such that 1 <= i <= 7 : 2
Enter j such that 1 <= j <= 7 : 3
M(i,j) = 2
---------------------------------------
You have used the machine, 2 time(s).
---------------------------------------
Enter i such that 1 <= i <= 7 : 2
Enter j such that 1 <= j <= 7 : 4
M(i,j) = 4
---------------------------------------
You have used the machine, 3 time(s).
---------------------------------------
Enter i such that 1 <= i <= 7 : 4
Enter j such that 1 <= j <= 7 : 5
M(i,j) = 5
---------------------------------------
You have used the machine, 4 time(s).
---------------------------------------
Enter i such that 1 <= i <= 7 : 5
Enter j such that 1 <= j <= 7 : 6
M(i,j) = 5
---------------------------------------
You have used the machine, 5 time(s).
---------------------------------------
Enter i such that 1 <= i <= 7 : 5
Enter j such that 1 <= j <= 7 : 7
M(i,j) = 7
---------------------------------------
You have used the machine, 6 time(s).
---------------------------------------

Enter the index of the largest integer: 7

Correct!
The integers in order are: [119, 123, 117, 130, 140, 106, 190]
```

The value of c just before each use of the machine and right after the last use of the machine will be

$$1, 2, 2, 4, 5, 5, 7.$$

Solution for Task 2

Begin by computing $M(1,2)$, $M(3,4)$, $M(5,6)$, and $M(7,8)$ and recording the answers as indices a, b, c, d. Name the other indices e, f, g, h. At this point we have used the machine 4 times and know the largest integer must be at index a, b, c or d. This is because the machine has told us that the integers at indices e, f, g and h are each smaller than at least one of the integers at indices a, b, c or d. Similarly, the smallest integer must be at index e, f, g or h.

Now notice that our solution for Task 1 generalizes to give a way to find the largest of n integers using the machine $n-1$ times. Moreover, if we adjust things by using c to keep track of the index of a "current *minimum*", we can also use it to find the smallest of n integers using the machine $n-1$ times.

Using this generalization, we can find the largest integer among those at indices a, b, c and d using the machine 3 more times. We can also use it to find the smallest integer among those at indices e, f, g and h using the machine 3 more times. These two values are the largest and smallest integers overall and we used the machine $4 + 3 + 3 = 10$ times.

Solution for Task 3

Begin by using the machine 4 times as in the solution for Task 2, and defining a, b, c, and d in the same way. Then compute $M(a,b)$ and $M(c,d)$, recording the answers as the indices x and y. Next, compute $M(x,y)$ and record the answer as the index z. You can think of this approach as a standard "tournament" or "bracket" in a competition where only the winners move on at each stage.

The integer a_z must be the largest integer because it is the only integer that has not been declared smaller than some other integer by the machine. Moreover, the second largest integer can only have been declared smaller than a_z. So to find the second largest integer, we need only find the largest integer among those at indices entered into the machine with z. Note that z was entered into the machine 3 times so we can use our generalized solution to Task 1 by using the machine 2 more times to determine the second largest integer. In total, we used the machine $4+2+1+2=9$ times.

Note

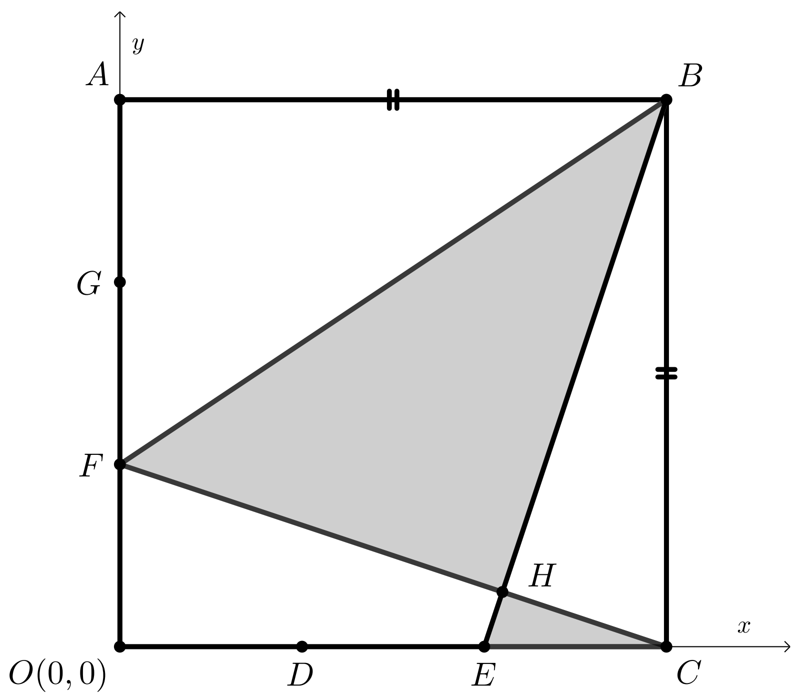
We have only shown that we can complete these tasks within the limits of 6, 10 and 9. Do we need the limits to be this high or can they be lowered? It turns out that they cannot be lowered; they are optimal. That is, for each task, there do not exist correct algorithms that always use the machine strictly fewer times than the given limit. Proving this is very challenging.

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 23, 2020
Maybe One-Third?

In the diagram, square $OABC$ is positioned with O at the origin $(0,0)$, A on the positive y-axis, C on the positive x-axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H.

If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?



More Info:

Check the CEMC at Home webpage on Thursday, April 30 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 30.

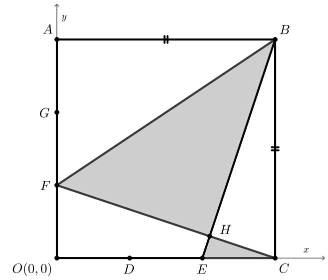
This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week

Problem E and Solution

Maybe One-Third?



Problem

In the diagram, square $OABC$ is positioned with O at the origin $(0,0)$, A on the positive y-axis, C on the positive x-axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H. If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?

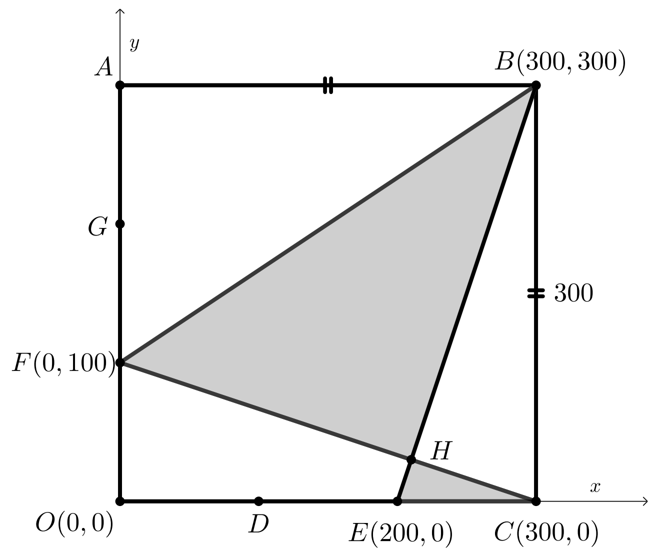
Solution

Since $OF = 100$ and F is on the positive y-axis, the coordinates of F are $(0,100)$.

Since $OD = DE = 100$, it follows that $OE = 200$. Since E is on the positive x-axis, the coordinates of E are $(200,0)$.

Since $OD = DE = EC = 100$, it follows that the side length of the square is $OC = 300$. Since C is on the positive x-axis, the coordinates of C are $(300,0)$.

It then follows that the coordinates of B are $(300,300)$.



The diagram has been updated to reflect the new information.

We will proceed to find the coordinates of H.

Find the equation of the line through $B(300,300)$ and $E(200,0)$.

The slope of $BE = \frac{300-0}{300-200} = 3$. We substitute $x = 200$, $y = 0$ and $m = 3$ into $y = mx + b$ to find b. Then $0 = 3(200) + b$ and $b = -600$ follows. The equation of the line through BE is $y = 3x - 600$. (1)

Find the equation of the line through $C(300,0)$ and $F(0,100)$.

The slope of $CF = \frac{100-0}{0-300} = -\frac{1}{3}$. Since $F(0,100)$ is on the y-axis, the y-intercept is 100. It follows that the equation of the line through C and F is $y = -\frac{1}{3}x + 100$. (2)

Find the coordinates of H, the intersection of the two lines.

At the intersection, the x-coordinates are equal and the y-coordinates are equal. In (1) and (2), since $y = y$, then

$$3x - 600 = -\frac{1}{3}x + 100 \;\Rightarrow\; 9x - 1800 = -x + 300 \;\Rightarrow\; 10x = 2100 \;\Rightarrow\; x = 210$$

Substituting $x = 210$ into (1), $y = 3\,(210) - 600 = 30$. Therefore, the coordinates of H are $(210, 30)$.

At this point we could follow one of two approaches. The first approach would be to find the area of the shaded regions indirectly, by first determining the area of the unshaded regions and then subtracting this from the area of the square. We will leave this approach to the solver.

Our second approach, which is below, is to calculate the areas of the shaded triangles directly.

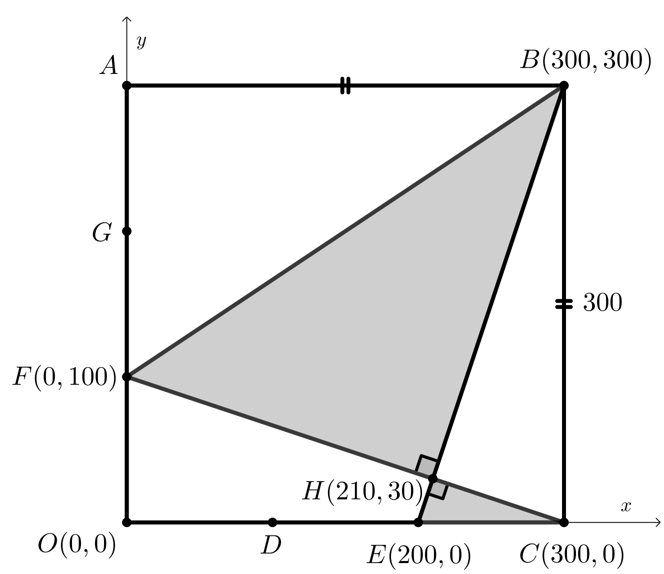
The slope of BE is 3 and the slope of CF is $-\frac{1}{3}$. Since these slopes are negative reciprocals, we know that $BE \perp CF$. It follows that $\triangle BHF$ and $\triangle CHE$ are right-angled triangles.

We will now proceed with finding the side lengths necessary to calculate the area of each shaded triangle.

We first find the area of $\triangle BHF$.

$$\begin{aligned} BH &= \sqrt{(300-210)^2 + (300-30)^2} \\ &= \sqrt{90^2 + 270^2} \\ &= \sqrt{90^2(1+3^2)} \\ &= 90\sqrt{10} \end{aligned}$$

$$\begin{aligned} HF &= \sqrt{(0-210)^2 + (100-30)^2} \\ &= \sqrt{210^2 + 70^2} \\ &= \sqrt{70^2(3^2+1)} \\ &= 70\sqrt{10} \end{aligned}$$

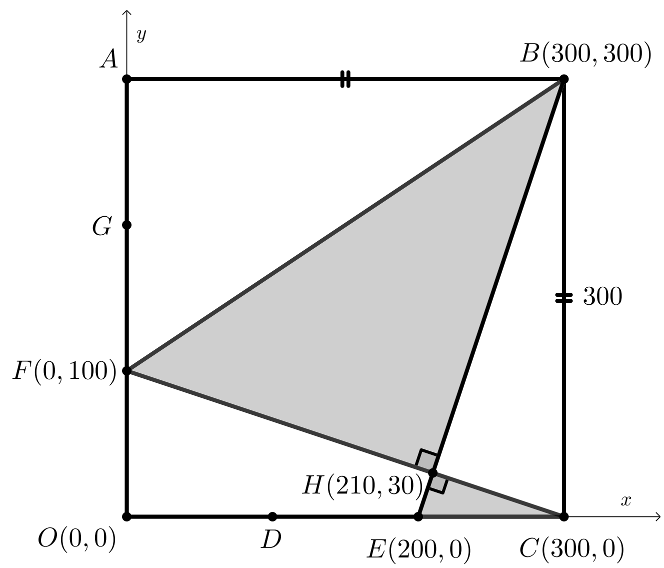


$$\begin{aligned} \text{Area } \triangle BHF &= BH \times HF \div 2 \\ &= 90\sqrt{10} \times 70\sqrt{10} \div 2 \\ &= 31\,500 \end{aligned}$$

Next we find the area of $\triangle CHE$.

$$\begin{aligned} CH &= \sqrt{(300-210)^2+(0-30)^2} \\ &= \sqrt{90^2+30^2} \\ &= \sqrt{30^2(3^2+1)} \\ &= 30\sqrt{10} \end{aligned}$$

$$\begin{aligned} HE &= \sqrt{(210-200)^2+(30-0)^2} \\ &= \sqrt{10^2+30^2} \\ &= \sqrt{10^2(1+3^2)} \\ &= 10\sqrt{10} \end{aligned}$$



$$\begin{aligned} \text{Area } \triangle CHE &= CH \times HE \div 2 \\ &= 30\sqrt{10} \times 10\sqrt{10} \div 2 \\ &= 1500 \end{aligned}$$

We can now calculate the total area shaded, the area of square $OABC$, and the fraction of the area of the square that is shaded.

$$\begin{aligned} \text{Total Area Shaded} &= \text{Area } \triangle BHF + \text{Area } \triangle CHE \\ &= 31\,500 + 1500 \\ &= 33\,000 \end{aligned}$$

$$\begin{aligned} \text{Area } OABC &= OA \times OC \\ &= 300 \times 300 \\ &= 90\,000 \end{aligned}$$

$$\begin{aligned} \text{Fraction of Total Area Shaded} &= \frac{\text{Area } \triangle BHF}{\text{Area } OABC} \\ &= \frac{33\,000}{90\,000} \\ &= \frac{11}{30} \end{aligned}$$

Therefore, $\frac{11}{30}$ of the total area of the square is shaded. This, in fact, is more than one-third.

CEMC at Home

Grade 11/12 - Friday, April 24, 2020
TSP

The Travelling Salesperson Problem (also known as the Travelling Salesman Problem or TSP) is a famous problem in mathematics and computer science. It is widely known because it has applications to many problems that affect our everyday lives and has instances that have remained unsolved for years! Here is one way to state the problem:

> A salesperson wants to visit n cities. What is the shortest route that the salesperson can take in order to visit each of the n cities exactly once and return to their starting point?

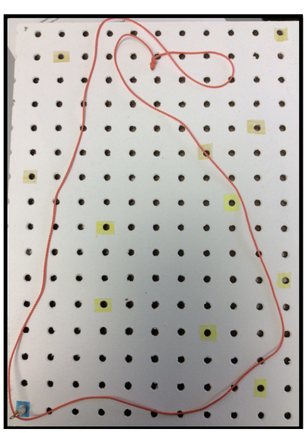
For each pair of cities, the distance between the cities is known. There are many different ways to measure this distance for the purposes of this problem. For example, this distance could be the length of a winding road joining the two cities. Alternatively, it could be the length of the straight line segment joining the two cities. This second option is the measurement of distance that we will use in this activity.

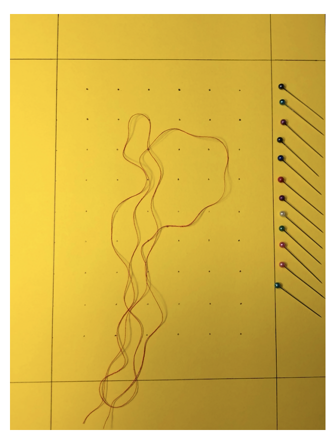
You Will Need:

- Two or more players
- A rectangular pegboard

  We use a board which is 24 cm wide and 30 cm long. See below for alternatives to the pegboard.
- 12 pegs
- A string of length 1 m
- A ruler or measuring tape

If you don't have a pegboard and pegs at home, you can make something similar for yourself. Some good options might include cardboard with pins, foam with toothpicks, or wood with screws. Be as creative as you like.



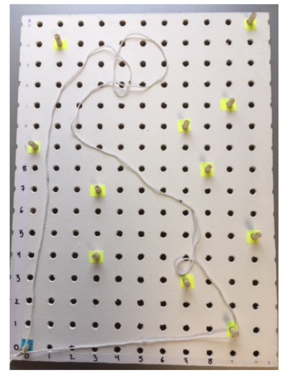


Set Up

We will play a game using the pegboard, the pegs and the string. Here we explain how to setup for the game.

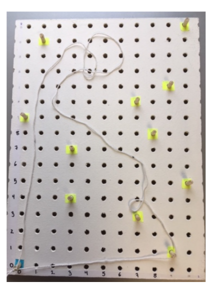
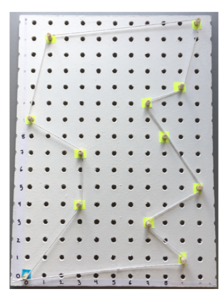
- Attach one of the ends of the string to one of the pegs. You can glue the string to the peg or tie the string around the peg.
- Place the peg with the string attached in the bottom left corner of the board.

During the game, the players will place the remaining 11 pegs in various places on the board. A possible arrangement is shown in the figure.



How To Play:

1. Start with the pegboard as outlined in the set up.
2. To start the game, the players first need to place the remaining 11 pegs into the board.
   The pegs can be placed anywhere on the board. You can decide whether to have one player place all of the pegs, or have the players take turns placing the pegs until all pegs are in place.
3. Players alternate turns working with the board. Decide which player will go first.
4. The first player does the following:
   - Arrange the string so that it touches each peg exactly once and returns to the bottom left peg. Make sure the string is pulled tight. *An example of this is shown below.*

   - Place a finger to mark the place on the string where the string meets itself at the bottom left peg.
   - Unravel the string, making sure to keep the correct mark on the string. Measure the length of the string between the attached peg and the place marked by the finger. Record this measurement as the score for the first player's first turn.
5. Now all other players take a turn doing what the first player did. All players are trying to find a route for the string that touches each peg exactly once and returns to the bottom left peg, *using the smallest possible length of string.*
6. The game ends after each player has had three turns with the string. The winner of the game is the player who achieved the *smallest* score (measurement) on a single turn.

When you are finished a game, you can pull out all but the bottom left peg and try again! New placements of the pegs will lead to whole new games.

Revisiting the Travelling Salesperson Problem (TSP)

The possible routes players can make with the string in this game model the possible routes in particular instances of the TSP with 12 cities. Each time you set up the peg board, with new distances between pairs of pegs, you are setting up a new version of the TSP, with new distances between the pairs of cities.

It is important to note that winning the game does not mean that you have solved the instance of the TSP associated with this particular board. To win the game, you just need to have the shortest route out of all the routes found in the game. But there may be a shorter route that no player found during the game! Think about how you might convince yourself that you have actually found the *shortest route* of *all possible routes*. In general this is hard to do, and gets even harder the more pegs you add to your game (or cities you add to your problem).

Think about the following questions:

- How many different possible routes are there to choose from in each round of the game using 12 pegs?
- If you change the game to include more pegs, say n pegs in total, how many different possible routes are there to choose from in each round of this game?

The TSP has been studied for many decades, yet there is no known efficient algorithm to solve this problem in general. We encourage you to look into this problem more on your own.

More Info:

The TSP can be modelled using *graphs*. Check out this resource about the TSP to learn more!

CEMC at Home

Grade 11/12 - Monday, April 27, 2020
Pick Up Sticks - Part 3

In Pick Up Sticks Part 1 and Part 2 we looked at a game where we removed sticks from piles starting with two piles of sticks and three piles of sticks, respectively.

Refresh your memory on the rules for these games and the winning strategies for these games.

Pick Up Sticks is actually an ancient game called Nim that is said to have originated in China. In Pick Up Sticks Part 2 we were playing Nim with piles of 3, 5 and 7 sticks. In our discussion of the winning strategy for this game we came up with a table of winning positions. If a player could remove sticks so that the piles matched a winning position, then the player had a winning strategy for the game. Here is the table of winning positions:

Winning Positions
$(0, 0, 0)$
$(0, k, k)$
$(1, 2, 3)$
$(1, 4, 5)$
$(2, 4, 6)$
$(2, 5, 7)$
$(3, 4, 7)$
$(3, 5, 6)$

In this activity, we will explore a different way to describe the winning positions in this game. The hope is that we can generalize this new description in order to study the winning positions of a more general version of Nim: a game with n piles, each with an arbitrary number of sticks.

A different view of the winning positions

Can you see a pattern in the table of winning positions shown above? Is there a simple condition that only the winning positions satisfy? At first glance, you may look at the table and think that for a position to be a winning position, the sum of the numbers of sticks in the first two piles must be equal to the number of sticks in the third pile. But we can see that $(3, 5, 6)$ is a winning position and $3 + 5 \neq 6$. We also note that $(1, 3, 4)$ satisfies $1 + 3 = 4$, but $(1, 3, 4)$ is not a winning position.

We need to dive deeper, and we will find the answer in a surprising place! The first step in our analysis is to write the numbers representing the pile sizes in *binary*. (If you have not seen binary numbers before, you can check out this Math Circles lesson before continuing.) The table of winning positions with the pile sizes given in binary is given below.

Winning Positions with Pile Sizes in Binary
$(0, 0, 0)$
$(0, k, k)$
$(1, 10, 11)$
$(1, 100, 101)$
$(10, 100, 110)$
$(10, 101, 111)$
$(11, 100, 111)$
$(11, 101, 110)$

Notice that we have not transformed the winning position $(0, k, k)$ as we do not know the value of k.

Next, we introduce something which is called the *digital sum*. To calculate the digital sum of two binary numbers, we add the digits in the same position (or place value), but follow the rule that $1 + 1 = 0$. For example, the following shows that the digital sum of 100101 and 110011 is 10110:

$$\begin{array}{r} 1\ 0\ 0\ 1\ 0\ 1 \\ \oplus\ 1\ 1\ 0\ 0\ 1\ 1 \\ \hline 0\ 1\ 0\ 1\ 1\ 0 \end{array}$$

We will call the digital binary sum the *Nim-sum*. We will use the notation $x \oplus y$ to refer to the Nim-sum of x and y as shown above. Thus we write $100101 \oplus 110011 = 10110$.

(As a side note to those of you who study computer science: The digital sum calculated in binary is also known as *exclusive or (XOR)*.)

Investigation 1: Nim with piles of size 3, 5, and 7

1. For each winning position (a, b, c) given in our table of winning positions with pile sizes in binary, calculate the Nim-sum, $a \oplus b \oplus c$.

   For example, consider the winning position $(10, 100, 110)$. *We calculate* $10 \oplus 100 = 110$ *and then add* 110 *to the result and get* $(10 \oplus 100) \oplus 110 = 110 \oplus 110 = 0$ *as shown below:*

$$\begin{array}{rccc} & & 1 & 0 \\ \oplus & 1 & 0 & 0 \\ \hline & 1 & 1 & 0 \end{array} \qquad \begin{array}{rccc} & 1 & 1 & 0 \\ \oplus & 1 & 1 & 0 \\ \hline & 0 & 0 & 0 \end{array}$$

2. We know that the positions $(1, 2, 4)$, $(2, 4, 4)$ and $(3, 4, 5)$ are losing positions for this game. For each of these positions, calculate the Nim-sum of their three pile sizes.

3. Based on your work in questions 1 and 2, can you come up with a guess at a condition that can be checked in order to determine whether or not a particular position is a winning position for Nim starting with piles of 3, 5 and 7 sticks?

4. Can you prove that your condition is correct? Your explanation should include these two arguments:

   (a) Show that if position P satisfies your condition, and position Q can be obtained from P in a single move, then Q does not satisfy your condition.

   (b) Show that if position P does not satisfy your condition, then there exists a move that will take position P to some position Q that does satisfy your condition.

Investigation 2: More general Nim games

5. Do you think your condition from Investigation 1 can also be used to determine the winning positions in a game of Nim with 3 piles, each with *any number of sticks*?

6. Do you think your condition from Investigation 1 can also be used to to determine the winning positions in a general game of Nim staring with n *piles, each with any number of sticks*?

See the next page for some properties of the digital binary sum that you might find useful.

More Info:

Check out the CEMC at Home webpage on Monday, May 4 for a solution to Pick Up Sticks - Part 3. We encourage you to discuss your ideas online using any forum you are comfortable with.

Properties of the Nim-sum

In your proofs you might want to use some of the properties of the Nim-sum given below. For extra fun, you can try to prove these properties yourself. You can also investigate the words “commutative”, “associative”, “identity”, and “inverse”.

- $a \oplus b = b \oplus a$ (Nim-sum is commutative)
- $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (Nim-sum is associative)
- $0 \oplus a = a$ (0 is the identity associated with Nim-sum)
- $a \oplus a = 0$ (the inverse of a is itself under Nim-sum)

CEMC at Home

Grade 11/12 - Monday, April 27, 2020
Pick up Sticks - Part 3 - Solution

1. The Nim-sum of the three pile sizes of each of these winning positions is 0. For example, for the position $(3, 5, 6)$, or $(11, 101, 110)$ in binary, Nim-sum of the pile sizes to be:

$$\begin{array}{r} 1\,1 \\ 1\,0\,1 \\ \oplus\ 1\,1\,0 \\ \hline 0\,0\,0 \end{array}$$

2. For the losing position $(1, 2, 4)$, or $(1, 10, 100)$ in binary, we calculate the Nim-sum of the piles sizes to be:

$$\begin{array}{r} 1 \\ 1\,0 \\ \oplus\ 1\,0\,0 \\ \hline 1\,1\,1 \end{array}$$

   For the losing position $(2, 4, 4)$, or $(10, 100, 100)$ in binary, we calculate the Nim-sum of the piles sizes to be:

$$\begin{array}{r} 1\,0 \\ 1\,0\,0 \\ \oplus\ 1\,0\,0 \\ \hline 0\,1\,0 \end{array}$$

   For the losing position $(3, 4, 5)$, or $(11, 100, 101)$ in binary, we calculate the Nim-sum of the pile sizes to be:

$$\begin{array}{r} 1\,1 \\ 1\,0\,0 \\ \oplus\ 1\,0\,1 \\ \hline 0\,1\,0 \end{array}$$

3. Based on our answers to 1. and 2. we may hypothesize that a position is a winning position if the Nim-sum of its pile sizes is 0 and it is a losing position if the Nim-sum of its pile sizes is not 0. Mathematicians might also express this condition as follows: a position is a winning position if and only if the Nim-sum of its pile sizes is 0.

4. Let's try to justify that our hypothesis from 3. is correct. We will argue the following:

   (a) Show that if position P satisfies your condition, and position Q can be obtained from P in a single move, then Q does not satisfy your condition.

   (b) Show that if position P does not satisfy your condition, then there exists a move that will take position P to some position Q that does satisfy your condition.

Suppose one of the players is faced with a game position P (other than $(0, 0, 0)$) having a Nim-sum equal to 0. Below we show an illustration of the Nim-sum calculation, where each letter represents either a 0 or a 1.

$$\begin{array}{rccc} & A & B & C \\ & D & E & F \\ \oplus & G & H & I \\ \hline & 0 & 0 & 0 \end{array}$$

Note that every positive integer less than 8 (and hence every pile size in this particular game) can be represented using a binary number with at most three digits. Remember that leading 0s are often omitted, for example if $A = 0$ then we may write the first number as simply BC.

We will show that if some number of sticks are removed from the "top pile" on this player's turn, then the resulting position Q cannot have Nim-sum 0. (Since the order of the sum doesn't matter, we can put whatever pile we want as the "top pile" and all we need to do is to make sure that this pile has at least one stick to proceed.)

Since the Nim-sum of position P is 0, this means there must be an even number of 1s in each column in the Nim-sum calculation. Let's set a_1 to be the number of sticks in the top pile before this player's turn and a_2 to be the number of sticks in the top pile after this player's turn. This means $a_2 < a_1$, and so the binary representations of a_1 and a_2 must differ in at least one place (digit).

Now we consider the Nim-sum of position P (on the left) and the new position Q (on the right) after this player's turn. Note that we are assuming a_1 is ABC and a_2 is $A^*B^*C^*$.

$$\begin{array}{rccc} & A & B & C \\ & D & E & F \\ \oplus & G & H & I \\ \hline & 0 & 0 & 0 \end{array} \qquad \begin{array}{rccc} & A^* & B^* & C^* \\ & D & E & F \\ \oplus & G & H & I \\ \hline & & & \end{array}$$

We know that we must have $A \neq A^*$, $B \neq B^*$, or $C \neq C^*$. Let's suppose we have $A \neq A^*$. Since the first column of the Nim-sum on the left must have an even number of 1s, we must have that the Nim-sum on the right must have an odd number of 1s. This happens because changing the digit from A to A^* will either add exactly one 1 to this column, or remove exactly one 1 from this column. In this case we see that the Nim-sum of Q (on the right) must have at least one digit of 1, and hence cannot be 0. The argument is similar if we have $B \neq B^*$, $C \neq C^*$ (or any combination of the three). Therefore, Q cannot have Nim-sum 0.

Now suppose one of the players is faced with a game position P having a Nim-sum *not* equal to 0. We will show that there is a move this player can make in order to bring the game to a position Q that *has* Nim-sum 0. We know that at least one column in the Nim-sum of position P totals to 1. Find the leftmost column in this Nim-sum that totals to 1, and pick any pile whose pile number has a 1 in this column. In the illustration below, we assume this pile has a_1 (or ABC) sticks.

$$\begin{array}{rccc} & A & B & C \\ & D & E & F \\ \oplus & G & H & I \\ \hline & & & \end{array} \qquad \begin{array}{rccc} & A^* & B^* & C^* \\ & D & E & F \\ \oplus & G & H & I \\ \hline & & & \end{array}$$

The player will change this "top pile" to a_2 (or $A^*B^*C^*$) sticks to get to the next position, Q. How should the player choose a_2 in order to achieve a Nim-sum of 0 on the right?

If the leftmost column in the Nim-sum of position P (on the left) has an even number of 1s, then we set $A^* = A$. If the leftmost column in the Nim-sum of position P has an odd number of 1s, then we set $A^* \neq A$, that is, if $A = 1$ then we set $A^* = 0$ and if $A = 0$ then we set $A^* = 1$. We choose the other digits of a_2 in a similar way. There is certainly a number a_2 with these binary digits, and choosing $A^*B^*C^*$ in this way will ensure a Nim-sum of 0 (on the right). But, we need to know that this a_2 is *less than* a_1 so that changing the pile from a_1 to a_2 is actually a legal move.

Can you see why this a_2 must be less than a_1? The idea is to argue that the numbers a_1 and a_2 must differ in at least one place (or digit), and the leftmost place (or digit) where they differ must have a 1 in a_1 and a 0 in a_2. Can you see why this is true? This will ensure that no matter what other digits may differ to the right in these representations, a_1 must be larger than a_2.

The General Game of Nim

It turns out that for any game of Nim (with n piles, each with an arbitrary number of sticks) a position is a winning position if and only if the Nim-sum of the n pile sizes is equal to 0.

We encourage you to think about this yourself, or read on if you are interested in a formal proof of this result.

First, consider the following result.

The Nim-sum Theorem
Suppose that the pile sizes before a move are $a_1, a_2, \ldots, a_n$ and that the pile sizes after the move are $b_1, b_2, \ldots, b_n$.
Let s be the Nim-sum of the pile sizes before the move, that is

$$s = a_1 \oplus a_2 \oplus \cdots \oplus a_n.$$

Let t be the Nim-sum of the pile sizes after the move, that is

$$t = b_1 \oplus b_2 \oplus \cdots \oplus b_n.$$

If the move changed the size of pile k, then $t = s \oplus (a_k \oplus b_k)$.

Proof: We begin by noting that $a_i = b_i$ for all $i \neq k$.

We will prove the statement using the properties of Nim-sum which were given in Question 5.

$$\begin{aligned}
t &= 0 \oplus t && \text{(0 is the identity)}\\
&= (s \oplus s) \oplus t && (s \oplus s = 0)\\
&= s \oplus (s \oplus t) && \text{(associativity)}\\
&= s \oplus [(a_1 \oplus a_2 \oplus \cdots \oplus a_n) \oplus (b_1 \oplus b_2 \oplus \cdots \oplus b_n)] &&\\
&= s \oplus (a_1 \oplus b_1) \oplus (a_2 \oplus b_2) \oplus \cdots \oplus (a_n \oplus b_n) && \text{(associativity and commutativity)}\\
&= s \oplus (a_1 \oplus a_1) \oplus \cdots \oplus (a_k \oplus b_k) \oplus \cdots \oplus (a_n \oplus a_n) && (a_i = b_i \text{ for } i \neq k)\\
&= s \oplus 0 \oplus \cdots \oplus (a_k \oplus b_k) \oplus \cdots \oplus 0 && (a_i \oplus a_i = 0)\\
&= s \oplus (a_k \oplus b_k) && \text{(0 is the identity)}
\end{aligned}$$

which is our desired result.

Now, using the Nim-sum Theorem we will prove the two necessary results. In each of these results, s and t are defined as in the Nim-sum Theorem.

Result 1: If $s = 0$, then $t \neq 0$.

Proof: Assume $s = 0$. Then, using the Nim-sum Theorem we get that $t = 0 \oplus (a_k \oplus b_k) = a_k \oplus b_k$.

Is it possible that $a_k \oplus b_k = 0$? Using the properties of Nim-sum we would obtain:

$$\begin{aligned} a_k \oplus b_k &= 0 \\ (a_k \oplus b_k) \oplus b_k &= 0 \oplus b_k \\ a_k \oplus (b_k \oplus b_k) &= b_k && \text{(associativity and 0 is the identity)} \\ a_k \oplus 0 &= b_k && \text{(the inverse of } b_k \text{ is itself)} \\ a_k &= b_k && \text{(0 is the idenity)} \end{aligned}$$

However, we know that $a_k \neq b_k$ because we changed the pile size of pile k. So it must be that $a_k \oplus b_k \neq 0$ and therefore, $t \neq 0$.

Result 2: If $s \neq 0$, then there exists a move for which $t = 0$.

Proof: Let 2^j be the largest power of 2 such that $2^j \leq s$. This power of 2 corresponds to the leftmost non-zero digit in the binary representation of s. This means that there must be at least one pile i, for which the digit corresponding to 2^j in the binary representation of a_i is a 1.

We will remove sticks from pile i so that the resulting pile size b_i has the property that $b_i = a_i \oplus s$. For this to be a legal move, we must be able to choose b_i so that $b_i < a_i$.

Why can this be done? We leave this part of the proof for you to think about on your own. Recall how we made this argument in the case of the three piles.

After we do this move we get

$$\begin{aligned} t &= s \oplus (a_i \oplus b_i) && \text{(using the Nim-sum Theorem)} \\ &= s \oplus (a_i \oplus (a_i \oplus s)) && \text{(substituting } b_i = a_i \oplus s\text{)} \\ &= s \oplus ((a_i \oplus a_i) \oplus s) && \text{(associativity)} \\ &= s \oplus (0 \oplus s) \\ &= s \oplus s && \text{(0 is the identiy)} \\ &= 0 \end{aligned}$$

as required.

The General Winning Strategy

Since you know how to calculate the winning positions in any game of Nim, you can form a winning strategy for this game. However, changing the pile sizes into binary and computing the Nim-sum of these pile sizes is a lot of mental math to do if there are many piles and/or large piles in the game. One trick that makes things easier is to think of the piles sizes as the sum of powers of 2 that would be used to determine their binary representation. Then “cancel out” any pairs of powers of 2 that match. If they all cancel out, then you have a winning position. If you have any odd number of any power, then you have a losing position.

CEMC at Home

Grade 11/12 - Tuesday, April 28, 2020
Think Before You Solve

There is often more than one possible approach to solving a particular math problem, and some approaches may take more time and effort than others. Before jumping into the algebra or arithmetic involved in a problem, it can be helpful to take a few minutes to think about some initial steps you might take that could greatly simplify your work overall. Is it helpful to simplify any expressions before substituting values for the variables? Is it helpful to either factor or expand any expressions before trying to solve the given equation(s)? Think about this while solving the problems given below.

---

1. Let $x = \frac{1}{3}$, $y = \frac{1}{7}$, and $z = \frac{1}{11}$. Determine the exact value of

$$\frac{xy + xz + yz}{xyz}$$

2. Determine the x-intercepts and y-intercepts of the graph with equation

$$y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4).$$

3. Determine all pairs (x, y) of real numbers that satisfy the following system of equations:

$$x^2 + x + y - y^2 = 20$$
$$(x+y)^2(x-y) = 75$$

You may find that this problem is more challenging than the first two. Think about how you might transform this system into a simpler system, perhaps by changing your view of what the "variables" are.

More Info:

Check the CEMC at Home webpage on Tuesday, May 5 for a solution to Think Before You Solve.

Problem 2 is from a past Euclid Contest, and problem 3 is from a past Canadian Team Mathematics Contest (CTMC). You can find past CEMC contests here.

CEMC at Home

Grade 11/12 - Tuesday, April 28, 2020
Think Before You Solve - Solution

1. Let $x = \frac{1}{3}$, $y = \frac{1}{7}$, and $z = \frac{1}{11}$. Determine the exact value of

$$\frac{xy + xz + yz}{xyz}$$

Solution: It is tempting to immediately substitute the values for x, y, and z into the given expression, however it is helpful to first observe that this expression can be simplified. The arithmetic involved in this question is made easier if the expression is first simplified as shown below:

$$\begin{aligned}
\frac{xy + xz + yz}{xyz} &= \frac{xy}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} \\
&= \frac{1}{z} + \frac{1}{y} + \frac{1}{x} \\
&= \frac{1}{\left(\frac{1}{11}\right)} + \frac{1}{\left(\frac{1}{7}\right)} + \frac{1}{\left(\frac{1}{3}\right)} && \text{substituting } x = \tfrac{1}{3}, y = \tfrac{1}{7}, z = \tfrac{1}{11} \\
&= 11 + 7 + 3 \\
&= 21
\end{aligned}$$

2. Determine the x-intercepts and y-intercepts of the graph with equation

$$y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4).$$

Solution: It is tempting to expand each product above and add the two resulting cubic polynomials. Since the result will be a quadratic polynomial (notice that the x^3 terms will cancel) you can proceed like this to determine the intercepts. However, a useful observation to make first is that both cubic polynomials have a common factor of $(x-2)(x-3)$ and so you can completely avoid expanding the products. The equation of the graph can be rewritten as

$$\begin{aligned}
y &= (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4) \\
&= (x-2)(x-3)[(x-1) - (x-4)] \\
&= 3(x-2)(x-3)
\end{aligned}$$

This is a quicker way to get to this resulting quadratic, and it is already in factored form! From the factorization above, we can see that the x-intercepts are 2 and 3, and the y-intercept, which is the value of y when $x = 0$, is $3(-2)(-3) = 18$.

3. Solve the following system of equations

$$x^2 + x + y - y^2 = 20$$
$$(x+y)^2(x-y) = 75$$

Solution: You might think of either starting by expanding the second equation, or by factoring the first equation. It turns out that factoring is the best approach here, but exactly how to factor may not be immediately clear. One useful observation to make first is that, rather than trying to solve for x and y, we should try to solve for $x+y$ and $x-y$. How did we realize this? First, notice that the second equation involves a product of two copies of $x+y$ and one copy of $x-y$. Furthermore, with a bit of rearrangement and factoring of the first equation, we obtain:

$$\begin{aligned} x^2 + x + y - y^2 &= (x^2 - y^2) + (x+y) \\ &= (x+y)(x-y) + (x+y) \\ &= (x+y)(x-y+1) \end{aligned}$$

Thus, the first equation may be rewritten as $(x+y)(x-y+1) = 20$, which is starting to look similar to the second equation. Let $A = x+y$ and let $B = x-y$. With this substitution, we are now trying to solve the following related system in A and B:

$$\begin{aligned} A(B+1) &= 20 \\ A^2B &= 75 \end{aligned}$$

Notice that the second equation tells us that $A \neq 0$ and $B \neq 0$. Since B occurs only to the exponent 1 in both equations, we will solve for B in each equation. Solving for B in the first equation gives $B = \frac{20}{A} - 1$ and solving for B in the second equation gives $B = \frac{75}{A^2}$, and so we must have

$$\frac{20}{A} - 1 = \frac{75}{A^2}$$

Multiplying both sides by A^2 gives us the (equivalent) equation

$$20A - A^2 = 75 \text{ or } A^2 - 20A + 75 = 0$$

By looking at the different ways to factor 75, or using the quadratic formula, we can obtain the two solutions to the quadratic equation: $A = 5$ and $A = 15$.

When $A = 5$, the equation $A^2B = 75$ tells us that $B = 3$ and when $A = 15$, the same equation tells us that $B = \frac{1}{3}$.

Now that we have solved for A and B we need to solve for x and y. Remember that $A = x+y$ and $B = x-y$.

When $A = 5$ and $B = 3$ we have that $x+y = 5$ and $x-y = 3$. Adding the two equations we get $2x = 8$ and so $x = 4$. This means $y = 1$.

When $A = 15$ and $B = \frac{1}{3}$ we have that $x+y = 15$ and $x-y = \frac{1}{3}$. Adding the two equations we get $2x = \frac{46}{3}$ and so $x = \frac{23}{3}$. This means $y = \frac{22}{3}$.

Therefore, there are two solutions to the system: $(4,1)$ and $(\frac{23}{3}, \frac{22}{3})$.

You should verify that these two pairs are indeed solutions to the original system.

CEMC at Home

Grade 11/12 - Wednesday, April 29, 2020
Comparison Machine Strikes Again

Your last mission was a success. So once again, you have been asked to develop algorithms to complete tasks involving the relative order of n distinct integers.

If you are a newcomer to the mission or a veteran that needs some reminders about the set-up and details, see the April 22 CEMC at Home resource for Grade 11/12 called Comparison Machine.

The Tasks

Complete the tasks below. For each task, you are told the number of integers you will be given and the limit on the number of times you can use the machine.

Remember that the machine will take in the indices of two integers in the list and will output the index of the larger integer.

n	Task	Limit
4	List the integers from smallest to largest.	5
9	You are given the additional information that • $a_1 < a_2$ and, • $a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$. List all 9 integers from smallest to largest.	6
5	Determine the median integer.	6

A Python computer program has also been built to help you test your solutions for this second mission.

Here are instructions for using the tool:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open this free online Python interpreter in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. You will interact with the tool using the right black panel, and you might want to widen this panel.
5. After completing a test, or if you encounter an error, you can hit *run* to begin another test. If you want to start over during a test, you can hit *stop* and then *run*.

More Info:

Check out the CEMC at Home webpage on Wednesday, May 6 for a solution to Comparison Machine Strikes Again.

CEMC at Home

Grade 11/12 - Wednesday, April 29, 2020
Comparison Machine Strikes Again - Solutions

Summary of the Tasks

Develop algorithms to complete tasks involving the relative order of n distinct integers.

- The integers are random and unknown to you. All you know is that they are named $a_1, a_2, \ldots a_n$.
- For each task, your approach must work no matter what the order of the integers is.
- A helpful machine M is available. The machine knows the relative order of these integers. To use it, you enter the index of two integers into the machine and it will tell you which of the two corresponding integers is larger.
- For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.
- Your memory is perfect and you can remember (or record) the result every time you use the machine.

Here are the three tasks:

n	**Task**	**Limit**
4	List the integers from smallest to largest.	5
9	You are given the additional information that • $a_1 < a_2$ and, • $a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$. List all 9 integers from smallest to largest.	6
5	Determine the median integer.	6

Solution for Task 1

Compute $M(1, 2)$ and record the answer as a. If $a = 1$, then set $b = 2$. Otherwise set $b = 1$.
That is, we set a equal to the index of the larger integer (or "winner") and set b equal to the index of the smaller integer (or "loser").
Compute $M(3, 4)$ and record the answer as c. If $c = 3$, then set $d = 4$. Otherwise set $d = 3$.
Compute $M(a, c)$ and record the answer as e. If $e = a$, then set $f = c$. Otherwise set $f = a$.
Compute $M(b, d)$ and record the answer as g. If $g = b$, then set $h = d$. Otherwise set $h = b$.
At this point we have used the machine 4 times. We know the integer at index e is the largest overall because it is the "winner of the winners". We know the integer at index h is the smallest overall because it is the "loser of the losers". Finally, compute $M(f, g)$ to determine the order of the other two integers. This process allows us to list all four integers from smallest to largest and we have used the machine 5 times.

Solution for Task 2

First we find the correct place for a_1 among the items in the ordered list $a_3, a_4, a_5, a_6, a_7, a_8, a_9$. That is, we find an index i such that $a_i < a_1 < a_{i+1}$. To do this, we compute M(1,6). If we learn that $a_1 > a_6$, then we compute $M(1, 8)$. Otherwise, we compute $M(1, 4)$. Now we have used the machine twice and we know one of following four things is true (and we know which one is true):

- $a_1 > a_6$ and $a_1 > a_8$, or
- $a_1 > a_6$ and $a_1 < a_8$, or
- $a_1 < a_6$ and $a_1 > a_4$, or
- $a_1 < a_6$ and $a_1 < a_4$.

If the first case above is true, then we know that either $a_8 < a_1 < a_9$ or $a_1 > a_9$. We next compute $M(1, 9)$ to determine which of these is true. In particular, we determine which of $a_3, \ldots, a_8, a_9, a_1$ or $a_3, \ldots, a_8, a_1, a_9$ is a list of integers from smallest to largest. The other three cases shown above are similar. In summary, we can place a_1 correctly using the machine 3 times. Now we simply repeat this process to find the correct place for a_2 in the ordered list $a_3, a_4, a_5, a_6, a_7, a_8, a_9$. Finally, since we know that $a_1 < a_2$, we can list all 9 integers from smallest to largest and we have used the machine $3 + 3 = 6$ times.

(Note that the process of placing a_2 can use the machine less than 6 times in some cases by only placing it within the integers among $a_3, a_4, a_5, a_6, a_7, a_8$ and a_9 that are greater than a_1. However, in the worst-case, we will still need to use the machine 6 times.)

Part of this algorithm is well-known and very important. It is called binary search. Its key idea is to find the correct place for an item in an ordered list by repeatedly comparing it to a "middle item".

Solution for Task 3

A critical piece of this solution is to note that we can use the first 3 comparisons of the solution to Task 1 in order to find the largest of any four integers. Specifically, this involves comparing the integers in pairs and then determining the "winner of the winners". We will call this our *subroutine*.

We begin by using our subroutine with any four of the five integers. The largest of these four integers cannot be the median of the five integers (because it is larger than at least three others) so we discard it. We now need to find the second largest of the remaining four integers since this integer must be the median integer.

Use the subroutine with these four integers. However, from our first use of the subroutine, we already know how two of these integers compare so we can reuse this comparison as our first comparison in the second use of the subroutine. So to complete the second use of the subroutine we only need to do 2 new comparisons. We can discard the largest of the remaining four integers as before. Three integers remain. The largest of these three integers must be the median of the original five integers.

As before, from our previous use of the subroutine, we already know how two of the three remaining integers compare. So we compare the third integer with the larger of these two integers to determine the largest of these three integers. So we have completed our task using the machine $3 + 2 + 1 = 6$ times.

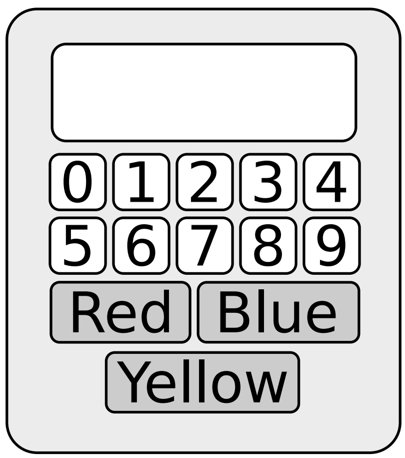
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 30, 2020
More Digit Swapping

Ali programs three buttons in a machine to swap digits in a 4-digit integer.

Red button:	swaps the thousands and tens digits
Blue button:	swaps the thousands and hundreds digits
Yellow button:	swaps the hundreds and units (ones) digits

Ali types the integer 1234 into the machine. Using only the Red, Blue, and Yellow buttons, determine all outputs she can produce with exactly 5 more button presses that she cannot produce using fewer than 5 more button presses.

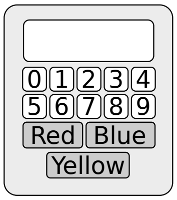


More Info:

Check the CEMC at Home webpage on Thursday, May 7 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, May 7.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem E and Solution

More Digit Swapping

Problem

Ali programs three buttons in a machine to swap digits in a 4-digit integer.

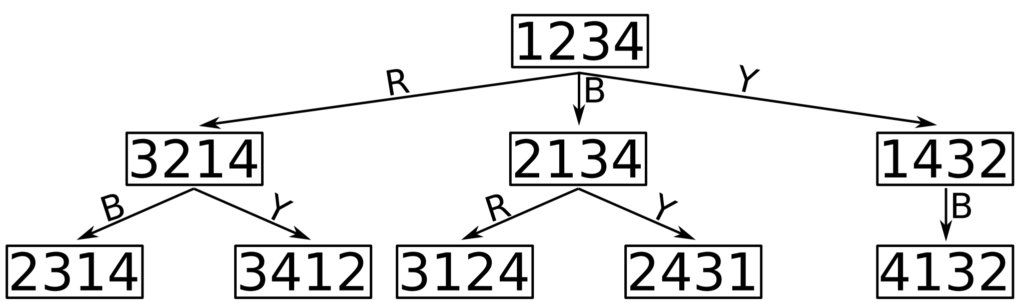
Red button:	swaps the thousands and tens digits
Blue button:	swaps the thousands and hundreds digits
Yellow button:	swaps the hundreds and units (ones) digits

Ali types the integer 1234 into the machine. Using only the Red, Blue, and Yellow buttons, determine all outputs she can produce with exactly 5 more button presses that she cannot produce using fewer than 5 more button presses.

Solution

We make a tree diagram to show all the possible outputs. We create this tree diagram one row at a time, moving from left to right. When we get a number that already exists in the tree, we do not write it down so our tree does not contain any duplicates. This will ensure that the tree shows only the shortest path to reach each of the possible different outputs.

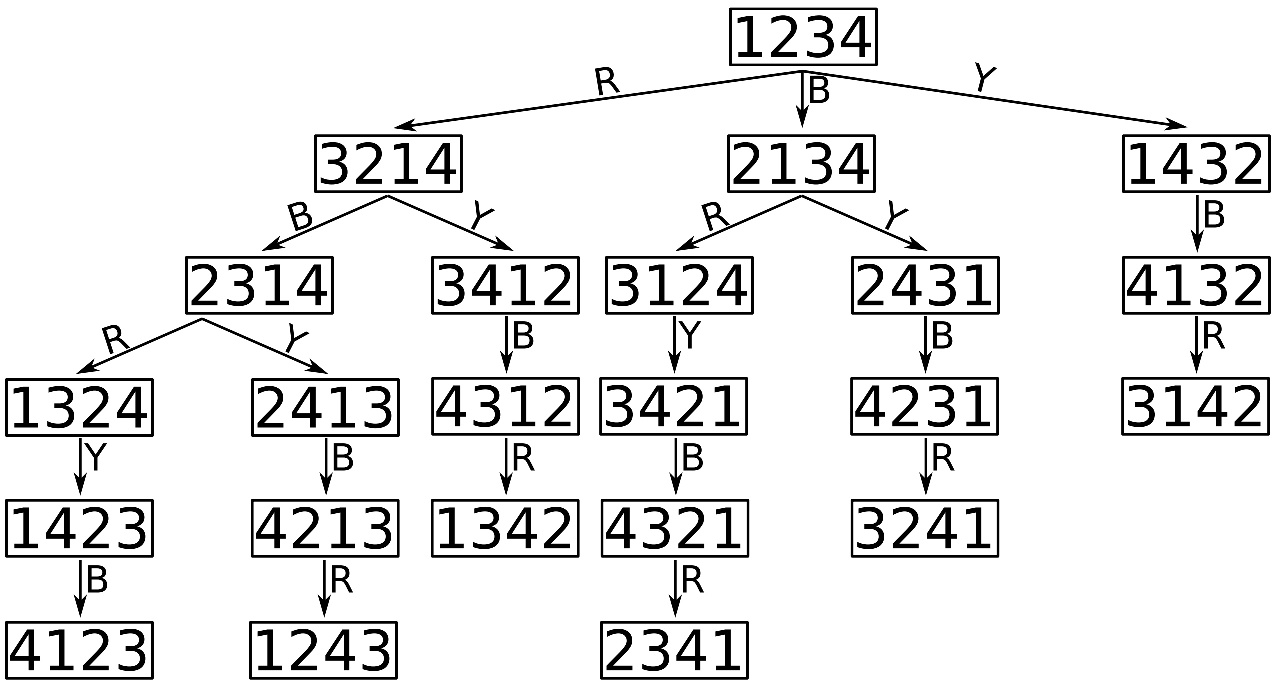
We can then use this tree diagram to find all the outputs that require exactly 5 more button presses to produce and cannot be produced in less.



Notice that we did not write down the output for pressing the Red button twice. This is because we would have ended up with 1234, which the number we were at before the first Red button push, and so is already in our tree. The same argument applies for pressing any button twice in a row.

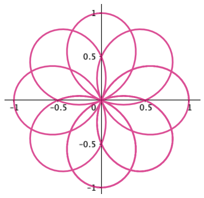
Notice also that we did not write down the output for pressing the Yellow button followed by the Red button. This is because we would have ended up with 3412, which is already in our tree.

We continue with the tree diagram, stopping after we have gone through all possible outputs from pressing 5 buttons.



Therefore, the outputs which can be produced by exactly five more button presses and cannot be produced with fewer are 4123, 1243, and 2341.

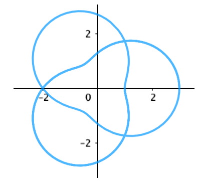
Extension: For the input of 1234, which output requires the most presses of the Red, Blue, and Yellow buttons to produce?



CEMC at Home

Grade 11/12 - Friday, May 1, 2020

Polar Coordinates



The *Cartesian Coordinate System* is the most familiar system that we use to represent points in the plane. Today, we will learn about a different system, the *Polar Coordinate System*. Next Friday, we will learn how to graph interesting curves like the two above. Equations for graphs like these are often very complicated using Cartesian coordinates, but can be much simpler using polar coordinates.

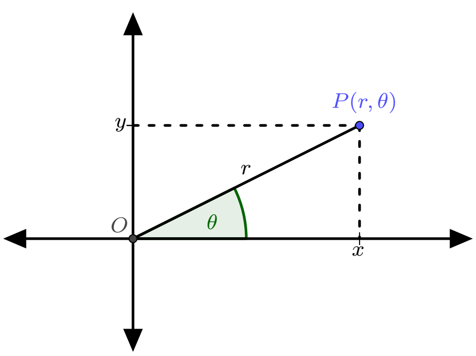
We will be using radians in this activity. If you have never measured angles in radians before, then either see the last page of this resource for an introduction to radians, or check out this lesson from the CEMC courseware.

Polar Coordinates

In Cartesian coordinates, a point P in the plane is given as $P(x, y)$, where x and y are real numbers.

Remind yourself of exactly what the values x and y represent here.

The point P can also be described using *polar coordinates* (r, θ). Here, r is the distance between the point P and the origin O. Also, θ is the angle (in radians) measured from the x-axis. (*Like when we look at the unit circle, positive angles are measured counter-clockwise from the positive x-axis.*) In polar coordinates, we call the positive x-axis the *polar axis*.



Suppose that P is in the first quadrant. Consider the right-angled triangle formed by the point P, the origin O, and the vertical line from P to the x-axis. This triangle has base x, height y and hypotenuse r.

By the Pythagorean Theorem, $r^2 = x^2 + y^2$.

We have $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$ from the definitions of sine and cosine in right-angled triangles.

Manipulating these equations, we obtain the three equations below that help us to relate the polar and Cartesian coordinates of the point P.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Example

Consider the Cartesian point $Q(1, 1)$. Since $x = 1$ and $y = 1$, then $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Also, the line segment joining the origin O to Q makes an angle of $\frac{\pi}{4}$ with the positive x-axis. This means that polar coordinates for Q are $(\sqrt{2}, \frac{\pi}{4})$.

Try drawing a picture and clearly labelling x, y, r, and θ. Make sure you understand why these values of r and θ are correct.

Question 1

Plot the points with Cartesian coordinates $A(8\sqrt{3}, 8)$ and $B(\frac{5}{4}, \frac{5\sqrt{3}}{4})$ and then convert them to polar coordinates.

Question 2

Plot the points with Cartesian coordinates $C(8, -8\sqrt{3})$ and $D(-\frac{5\sqrt{3}}{4}, -\frac{5}{4})$ and then convert them to polar coordinates.

Example

Consider the point with polar coordinates $\left(4, \frac{3\pi}{2}\right)$. Since $r = 4$ and $\theta = \frac{3\pi}{2}$, we have that

$$x = r\cos\theta = 4\cos\left(\frac{3\pi}{2}\right) = 4(0) = 0$$

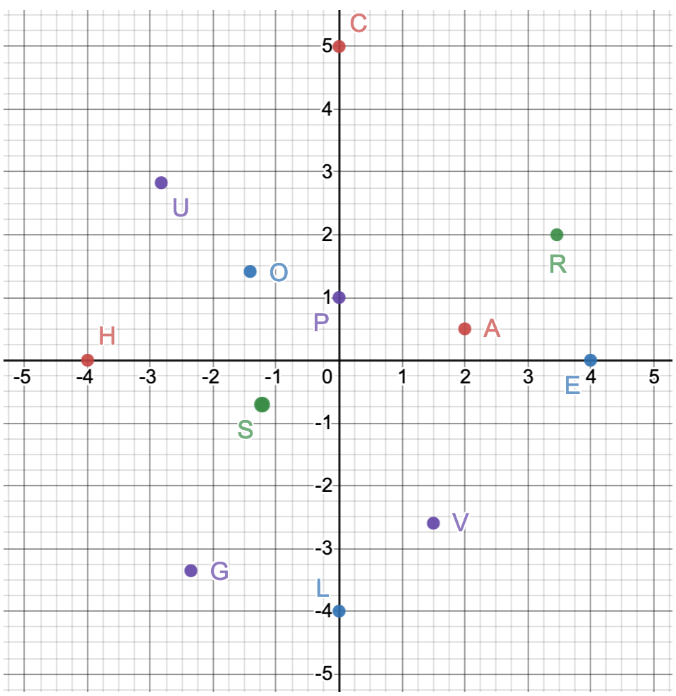
$$y = r\sin\theta = 4\sin\left(\frac{3\pi}{2}\right) = 4(-1) = -4$$

This means that the Cartesian coordinates of the point are $(0, -4)$.

Can you see why these must be the correct Cartesian coordinates by visualizing the point?

Activity

Consider the polar coordinates (r, θ), with $0 \le \theta < 2\pi$, of each of the 12 points plotted in the graph below. Exactly one of these points satisfies each of the following properties, and each point is labelled with a different letter. Determine which point best matches each property and use this information to complete the phrase below.



1. This point has polar coordinates $(4, 0)$.
2. This point has polar coordinates $(4, \frac{3\pi}{2})$.
3. This point has polar coordinates $(4, \frac{3\pi}{4})$.
4. This point could also be described using polar coordinates $(2, \frac{11\pi}{4})$.
5. This point's first coordinate, r, satisfies $r^2 = 2$.
6. This point has the largest first coordinate, r, out of all of the points.
7. This point has the smallest *positive* second coordinate, θ, out of all of the points.
8. This point's second coordinate, θ, satisfies $2\sin\theta = 1$.
9. This point's second coordinate, θ, satisfies $\cos\theta = -1$.
10. This point's first coordinate, r, satisfies $r = 3$.
11. This point's coordinates satisfy $r = \sin\theta$. *Remember that* $-1 \le \sin\theta \le 1$.
12. This point's coordinates satisfy $r = \theta$.

In next Friday's activity we will learn how to...

___	___	___	___	___	___	___	___	___	___	___	___	___	___	___	___	!
12	8	7	11	9	11	4	2	7	8	6	3	8	10	1	5	

More Info:

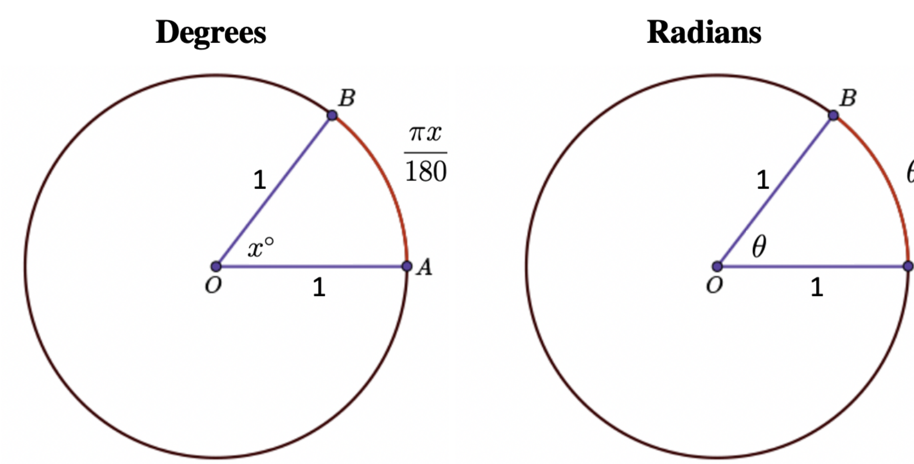
Check the CEMC at Home webpage on Friday, May 8 for a solution to Polar Coordinates.

Radians

When we first learn about angles, we write their measures (that is, their "sizes") using degrees. For example, a complete circular angle measures $360°$, a straight angle measures $180°$, and a right angle measures $90°$. Angles like $30°$, $45°$, and $60°$ are also familiar.

A second way of measuring angles is in *radians*. In this case, a complete circular angle measures 2π. What connection can you see between 2π and the unit circle? The circumference of the unit circle is 2π. Radians are defined so that an angle of measure $x°$ measures $\frac{\pi x}{180}$ radians.

The value $\frac{\pi x}{180}$ is actually the arc length of a sector of the unit circle defined by the angle with measure $x°$ so radians are in some sense measuring the arc length corresponding to the angle, which is one way of measuring the angle itself.



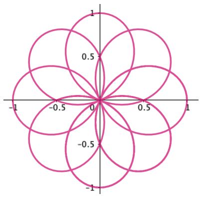
Questions:

(a) Convert the angles with the following measures from degrees to radians: $180°$, $90°$, $60°$, $45°$, $30°$, $48°$.

(b) Convert the angles with the following measures from radians to degrees: $\frac{\pi}{5}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$, $\frac{7\pi}{4}$.

(c) Complete the chart below. *The angles are given in radians.*

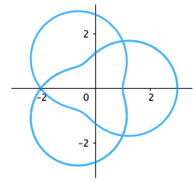
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$					
$\cos\theta$					



CEMC at Home

Grade 11/12 - Friday, May 1, 2020

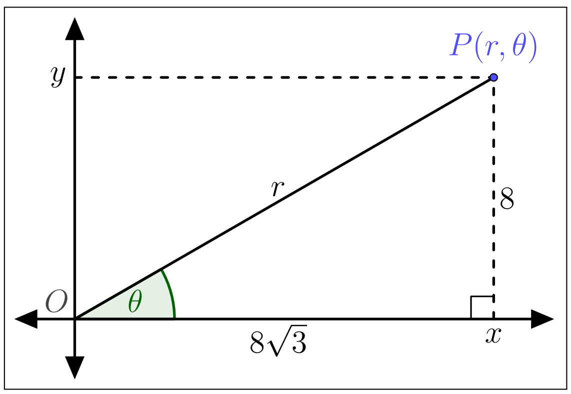
Polar Coordinates - Solution



Question 1

Plot the points with Cartesian coordinates $A\big(8\sqrt{3}, 8\big)$ and $B\big(\frac{5}{4}, \frac{5\sqrt{3}}{4}\big)$ and then convert them to polar coordinates.

Solution: We first plot the point $A\big(8\sqrt{3}, 8\big)$ in the plane.



Since $x = 8\sqrt{3}$ and $y = 8$, we have

$$r = \sqrt{x^2+y^2} = \sqrt{\big(8\sqrt{3}\big)^2 + 8^2} = \sqrt{192+64} = 16$$

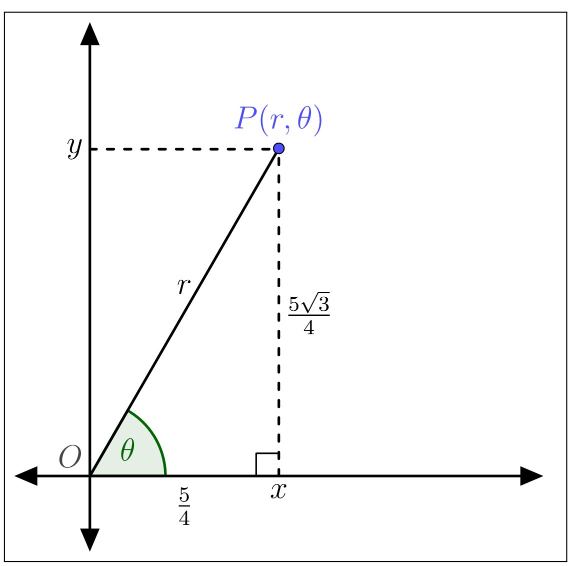
From the right-angled triangle in the diagram, we see we are looking for an angle θ in the first quadrant that satisfies

$$\sin(\theta) = \frac{y}{r} = \frac{8}{16} = \frac{1}{2}$$

One possible choice is $\theta = \frac{\pi}{6}$.

This means the point with Cartesian coordinates $(x, y) = \big(8\sqrt{3}, 8\big)$ can be described using polar coordinates $(r, \theta) = \big(16, \frac{\pi}{6}\big)$.

Now we plot the point $B\big(\frac{5}{4}, \frac{5\sqrt{3}}{4}\big)$ in the plane.



Since $x = \frac{5}{4}$ and $y = \frac{5\sqrt{3}}{4}$, we have

$$x^2+y^2 = \left(\tfrac{5}{4}\right)^2 + \left(\tfrac{5\sqrt{3}}{4}\right)^2 = \tfrac{25}{16} + \tfrac{75}{16} = \tfrac{100}{16} = \tfrac{25}{4}$$

and so $r = \sqrt{x^2+y^2} = \frac{5}{2}$. From the right-angled triangle in the diagram, we see we are looking for an angle θ in the first quadrant that satisfies

$$\cos(\theta) = \frac{x}{r} = \frac{\left(\frac{5}{4}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{2}$$

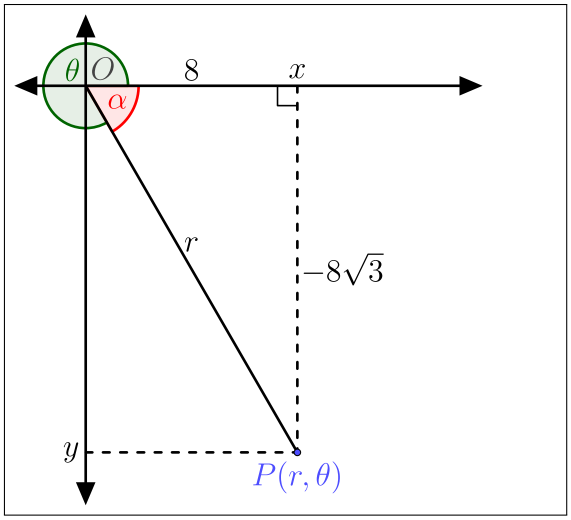
One possible choice is $\theta = \frac{\pi}{3}$.

This means the point with Cartesian coordinates $(x, y) = \big(\frac{5}{4}, \frac{5\sqrt{3}}{4}\big)$ can be described using polar coordinates $(r, \theta) = \big(\frac{5}{2}, \frac{\pi}{3}\big)$.

Question 2

Plot the points with Cartesian coordinates $C\left(8, -8\sqrt{3}\right)$ and $D\left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ and then convert them to polar coordinates.

Solution: We first plot the point $C\left(8, -8\sqrt{3}\right)$ in the plane.



Since $x = 8$ and $y = -8\sqrt{3}$, we have

$$r = \sqrt{x^2 + y^2} = \sqrt{8^2 + \left(-8\sqrt{3}\right)^2} = \sqrt{64 + 192} = 16$$

From the right-angled triangle in the diagram, we see we are looking for an angle θ in the fourth quadrant for which the associated acute angle α satisfies

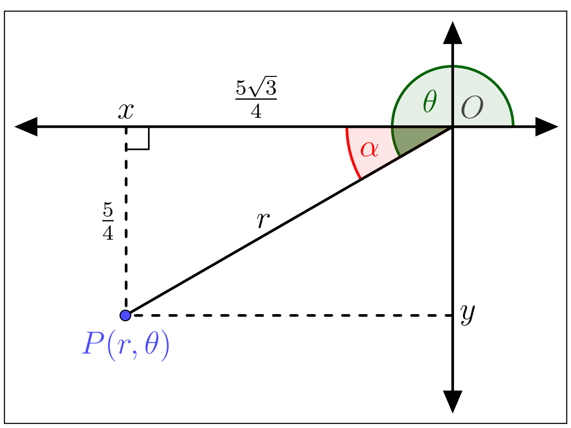
$$\cos(\alpha) = \frac{8}{16} = \frac{1}{2}$$

This means $\alpha = \frac{\pi}{3}$ and so one possible choice is $\theta = \frac{5\pi}{3}$.

This means the point with Cartesian coordinates $(x, y) = \left(8, -8\sqrt{3}\right)$ can be described using polar coordinates $(r, \theta) = \left(16, \frac{5\pi}{3}\right)$.

Note: We could have instead observed that point C is related to point A. They are the same distance from the origin, and their angles are complementary.

Now we plot the point $D\left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ in the plane.



Since $x = -\frac{5\sqrt{3}}{4}$ and $y = -\frac{5}{4}$, we have

$$x^2 + y^2 = \left(-\frac{5\sqrt{3}}{4}\right)^2 + \left(-\frac{5}{4}\right)^2 = \frac{75}{16} + \frac{25}{16} = \frac{25}{4}$$

and so $r = \sqrt{x^2 + y^2} = \frac{5}{2}$. From the right-angled triangle in the diagram, we see we are looking for an angle θ in the third quadrant for which the associated acute angle α satisfies

$$\sin(\alpha) = \frac{\left(\frac{5}{4}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{2}$$

This means $\alpha = \frac{\pi}{6}$ and one possible choice is $\theta = \frac{7\pi}{6}$.

This means the point with Cartesian coordinates $(x, y) = \left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ can be described using polar coordinates $(r, \theta) = \left(\frac{5}{2}, \frac{7\pi}{6}\right)$.

Activity Answers:

In next Friday's activity we will learn how to...

G	R	A	P	H		P	O	L	A	R		C	U	R	V	E	S	!
12	8	7	11	9		11	4	2	7	8		6	3	8	10	1	5	

See the next page for an explanation of each matching.

1. This point has polar coordinates $(4, 0)$. $\boldsymbol{(E)}$

Since $r = 4$ and $\theta = 0$, this is a point that is 4 units from the origin and lies on the ray defined by $\theta = 0$ which is the positive x-axis. This describes only point E.

2. This point has polar coordinates $\left(4, \frac{3\pi}{2}\right)$. $\boldsymbol{(L)}$

This is a point that is 4 units from the origin and lies on the ray defined by $\theta = \frac{3\pi}{2}$ which is the negative y-axis. This describes only point L.

3. This point has polar coordinates $\left(4, \frac{3\pi}{4}\right)$. $\boldsymbol{(U)}$

This is a point that is 4 units from the origin lies on the ray defined by $\theta = \frac{3\pi}{4}$. This describes only point U.

4. This point could also be described using polar coordinates $\left(2, \frac{11\pi}{4}\right)$. $\boldsymbol{(O)}$

Note that $\frac{11\pi}{4}$ and $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$ are equivalent angles. So we are looking for the point with polar coordinates $\left(2, \frac{3\pi}{4}\right)$. This is on the same ray as U above, but 2 units from the origin. This describes only point O.

5. This point's first coordinate, r, satisfies $r^2 = 2$. $\boldsymbol{(S)}$

This means $r = \pm\sqrt{2} \approx \pm 1.4$. It looks like the only point that is around 1.4 units from the origin is S. You can draw a circle of radius 1.4 on the graph to confirm. This is describing point S.

6. This point has the largest first coordinate, r, out of all of the points. $\boldsymbol{(C)}$

The point with the largest first coordinate will be the farthest from the origin. The point C is 5 units away and every other point appears to be closer than that. You can draw a circle of radius 5 on the graph to confirm! This is describing point C.

7. This point has the smallest *positive* second coordinate, θ, out of all of the points. $\boldsymbol{(A)}$

The point with the smallest positive second coordinate will make the smallest angle with the positive x-axis. This describes the point A.

8. This point's second coordinate, θ, satisfies $2\sin\theta = 1$. $\boldsymbol{(R)}$

If $0 \le \theta < 2\pi$ and $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. The only point that lies on the ray defined by $\theta = \frac{\pi}{6}$ is R and there are no points that lie on the ray defined by $\theta = \frac{5\pi}{6}$. This describes R.

9. This point's second coordinate, θ, satisfies $\cos\theta = -1$. $\boldsymbol{(H)}$

If $0 \le \theta < 2\pi$ and $\cos\theta = -1$, then $\theta = \pi$. The only point that lies on the ray defined by $\theta = \pi$ (the negative x-axis) is H.

10. This point's first coordinate, r, satisfies $r = 3$. $\boldsymbol{(V)}$

The only point that appears to be 3 units from the origin is V. You can draw a circle of radius 3 on the graph to confirm. This is describing point V.

11. This point's coordinates satisfy $r = \sin\theta$. $\boldsymbol{(P)}$

Since $-1 \le \sin\theta \le 1$, any coordinates that satisfy this equality must have $-1 \le r \le 1$. The only point within 1 unit of the origin is P. In fact, Pappears to have polar coordinates $r = 1$ and $\theta = \frac{\pi}{2}$ which do satisfy $\sin\theta = \sin\left(\frac{\pi}{2}\right) = 1 = r$.

12. This point's coordinates satisfy $r = \theta$. $\boldsymbol{(G)}$

We are now left with one property (12) and one point (G). This means G must be the point satisfying $r = \theta$. Using the distance formula, you can check that G is around 4 units from the origin. The ray through G is near the ray defined by $\theta = \frac{5\pi}{4} \approx 4$, which provides some evidence that $r \approx \theta$. (The actual point plotted has $r = \theta = 4.1$.)

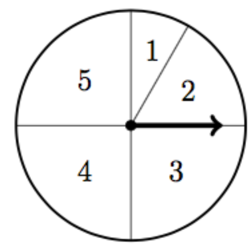
CEMC at Home

Grade 11/12 - Monday, May 4, 2020
Contest Day 1

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Individual Problem #4

A spinner was created by drawing five radii from the centre of a circle. The first four radii divide the circle into four equal wedges. The fifth radius divides one of the wedges into two parts, one having twice the area of the other. The five wedges are labelled as pictured with the wedge labeled by 2 having twice the area of the wedge labeled by 1. Determine the probability of spinning an odd number.



2020 Euclid Contest, #4(a)

The positive integers a and b have no common divisor larger than 1. If the difference between b and a is 15 and $\frac{5}{9} < \frac{a}{b} < \frac{4}{7}$, what is the value of $\frac{a}{b}$?

More Info:

Check out the CEMC at Home webpage on Monday, May 11 for solutions to the Contest Day 1 problems.

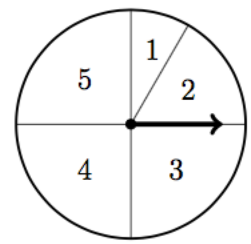
CEMC at Home

Grade 11/12 - Monday, May 4, 2020
Contest Day 1 - Solution

Solutions to the two contest problems are provided below, including a video for the second problem.

2020 Canadian Team Mathematics Contest, Individual Problem #4

A spinner was created by drawing five radii from the centre of a circle. The first four radii divide the circle into four equal wedges. The fifth radius divides one of the wedges into two parts, one having twice the area of the other. The five wedges are labelled as pictured with the wedge labeled by 2 having twice the area of the wedge labeled by 1. Determine the probability of spinning an odd number.



Solution:

The odd numbers on the spinner are 1, 3, and 5. The wedges labelled by 3 and 5 each take up $\frac{1}{4}$ of the spinner and so each will be spun with a probability of $\frac{1}{4}$. If we let the probability of spinning 1 be x, then we have that the probability of spinning 2 is $2x$. The probability of spinning either 1 or 2 is $\frac{1}{4}$, which means $x + 2x = \frac{1}{4}$ or $3x = \frac{1}{4}$ so $x = \frac{1}{12}$.

Therefore, the probability of spinning an odd number is $\frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{7}{12}$.

2020 Euclid Contest, #4(a)

The positive integers a and b have no common divisor larger than 1. If the difference between b and a is 15 and $\dfrac{5}{9} < \dfrac{a}{b} < \dfrac{4}{7}$, what is the value of $\dfrac{a}{b}$?

Solution:

Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.

Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.

Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.

We multiply both sides of the left inequality by $9(a+15)$ (which is positive) to obtain $5(a+15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.

From this, we see that $a > \dfrac{75}{4} = 18.75$.

Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a+15)$ (which is positive) to obtain $7a < 4(a+15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.

Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

Video

Visit the following link for a discussion of two different approaches to solving the second contest problem: https://youtu.be/phNdHo5mE2g.

CEMC at Home

Grade 11/12 - Tuesday, May 5, 2020
Factoring Polynomials without Division

When solving problems we may encounter a polynomial with integer coefficients that needs to be factored. You may have learned some techniques for factoring polynomials that use long division of polynomials. In this activity we will factor some polynomials without using long division.

Definition: Suppose we have a polynomial in the variable x. If the polynomial evaluates to 0 when $x = a$, then we say that a is a *root* of the polynomial.

The Factor Theorem: If a is a root of a polynomial, then $x - a$ is a factor of the polynomial.

Example 1

The number $x = 3$ is a root of the polynomial $x^2 - x - 6$ since $3^2 - 3 - 6 = 0$. The factor theorem tells us that the polynomial $x - 3$ is a factor of the polynomial $x^2 - x - 6$. We can check that indeed $x^2 - x - 6 = (x - 3)(x + 2)$.

Example 2

The number $x = -1$ is a root of the polynomial $x^3 + 5x^2 + 8x + 4$ since $(-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0$. The factor theorem tells us that the polynomial $x - (-1) = x + 1$ is a factor of the polynomial $x^3 + 5x^2 + 8x + 4$. We can check that indeed $x^3 + 5x^2 + 8x + 4 = (x + 1)(x^2 + 4x + 4)$.

How might we find this other quadratic factor?

In this activity, we will focus on factoring polynomials for which all but possibly two of the roots of the polynomial are integers; however, the techniques for factoring that we present below can also be useful in other situations.

Factoring Method

How do we go about factoring a polynomial with integer coefficients?

Let's say we are factoring the cubic polynomial $2x^3 - x^2 - 7x + 6$. If $x = a$ is a root of this polynomial, then $x - a$ is a factor of the polynomial. If a is an integer, then when we factor out $x - a$, we are left with some quadratic polynomial $Ax^2 + Bx + C$ with A, B, and C integers as shown

$$2x^3 - x^2 - 7x + 6 = (x - a)(Ax^2 + Bx + C)$$

If we expand the product on the right side and compare its terms to the like terms on the left side, we observe the following:

- The only term on the right side without an x in it will be the term $-aC$. This means $6 = -aC$. Since a and C are both integers, a must be a factor of 6.
- The only term on the right side with a power of x^3 comes from multiplying the term x by the term Ax^2. This means the term $2x^3$ must be equal to the term Ax^3 and so $A = 2$.
- There are two terms on the right side with a power of x^2, and they come from multiplying the term x by the term Bx and the term $(-a)$ by the term Ax^2. This means the term $-x^2$ on the left must be equal to $(x)(Bx) + (-a)(Ax^2)$ or $Bx^2 - aAx^2$.

Using these three observations, we can factor the polynomial completely! Start by testing all of the factors of 6 to find an integer root $x = a$, and then use this value of a along with the other two observations to solve for the coefficients A, B, and C. The full process is outlined in the examples below.

Example 3: Factor the cubic polynomial $2x^3 - x^2 - 7x + 6$.

$2x^3 - x^2 - 7x + 6$ $= (x-1)(Ax^2 + Bx + C)$	The factors of 6 are $\pm1, \pm2, \pm3$ and ±6. Using these factors, we determine that 1 is a root of the polynomial and so $x - 1$ is a factor. When we factor out $x - 1$ we will be left with a quadratic which we will call $Ax^2 + Bx + C$.
$= (x-1)(2x^2 + Bx - 6)$	The $2x^3$ term from our original polynomial comes from multiplying x by Ax^2. Since $2x^3$ equals Ax^3 we must have $A = 2$. The constant term 6 from our original polynomial comes from multiplying -1 by C and so $C = -6$.
$= (x-1)(2x^2 + x - 6)$	The $-x^2$ from our original polynomial comes from multiplying x by Bx and adding it to -1 times $2x^2$. Since $-x^2$ equals $Bx^2 - 2x^2$, we must have $B = 1$. *Note that we didn't use the $-7x$ term from our original polynomial, but it can be used to check that we didn't make a mistake.*
$= (x-1)(x+2)(2x-3)$	Finally, we factor the resulting quadratic using standard factoring techniques.

Example 4: Factor the quartic polynomial $6x^4 - 7x^3 - 13x^2 + 4x + 4$.

$6x^4 - 7x^3 - 13x^2 + 4x + 4$ $= (x-2)(Ax^3 + Bx^2 + Cx + D)$	The factors of 4 are $\pm1, \pm2$ and ±4. Using these factors we determine that 2 is a root of our polynomial and so $x - 2$ is a factor.
$= (x-2)(6x^3 + Bx^2 + Cx - 2)$	We use the $6x^4$ term from our original polynomial to determine that $A = 6$ and the constant term 4 from our original polynomial to determine that $D = -2$. *Notice that $6x^4$ equals $(x)(Ax^3)$ and 4 equals $(-2)(D)$.*
$= (x-2)(6x^3 + 5x^2 + Cx - 2)$	We use the $-7x^3$ term from our original polynomial to determine that $B = 5$. *Notice that $-7x^3$ equals $(-2)(6x^3) + (x)(Bx^2)$.*
$= (x-2)(6x^3 + 5x^2 - 3x - 2)$	We use the $4x$ term from our original polynomial to determine that $C = -3$. *Notice that $4x$ equals $(x)(-2) + (-2)(Cx)$.*
$= (x-2)(x+1)(Ex^2 + Fx + G)$	For the rest of our solution we ignore the $(x - 2)$ factor and focus on factoring the cubic $6x^3 + 5x^2 - 3x - 2$. *Remember that we have already discussed how to factor a cubic.* The factors of -2 are ±1 and ±2. Using these factors we determine that -1 is a root of this cubic and so $x + 1$ is a factor, and then we proceed as in Example 3.

Use these ideas to solve the following problems.

1. Factor $x^3 + 7x^2 + 11x + 5$.
2. Factor $x^4 + 5x^3 - 3x^2 - 17x - 10$. *Hint. Start by verifying that $x = 2$ is a root.*
3. Factor $4x^4 - 16x^3 + x^2 + 39x - 18$.
4. Given that $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$, determine the values of A, B, C and D.

More Info:

Check out the CEMC at Home webpage on Tuesday, May 12 for a solution to Factoring Polynomials without Division.

When finding the roots of these polynomials we looked at a special case of the Rational Roots Theorem. To learn more about the Rational Roots Theorem check out the lesson Factoring Polynomials Using the Factor Theorem from the CEMC Advanced Functions and Pre-Calculus courseware.

CEMC at Home

Grade 11/12 - Tuesday, May 5, 2020
Factoring Polynomials without Division - Solution

We use the strategy outlined in the activity to factor the first cubic.

Question 1: Factor $x^3 + 7x^2 + 11x + 5$

Solution:

$$x^3 + 7x^2 + 11x + 5$$
$$= (x + 1)(Ax^2 + Bx + C)$$

The factors of 5 are ± 1 and ± 5. Using these factors, we determine that -1 is a root and so $x - (-1) = x + 1$ is a factor.
Note that -5 is also a root and so we could instead start with the factor $x + 5$.

$$= (x + 1)(x^2 + Bx + 5)$$

The x^3 term from our original polynomial comes from multiplying x by Ax^2 and so $A = 1$. The constant term 5 from our original polynomial comes from multiplying 1 by C and so $C = 5$.

$$= (x - 1)(x^2 + 6x + 5)$$

The $7x^2$ term from our original polynomial comes from multiplying x by Bx and adding the result to 1 times x^2. In other words, the term $7x^2$ equals $Bx^2 + x^2$ or $(B + 1)x^2$. Comparing coefficients gives $7 = B + 1$ or $B = 6$.

$$= (x + 1)(x + 1)(x + 5)$$

Factor the resulting quadratic.

We use a similar strategy to factor a quartic polynomial with integer coefficients. In this case, once we find an integer root a, and factor out the corresponding linear factor $x - a$, we will be left with a cubic polynomial $Ax^3 + Bx^2 + Cx + D$ with A, B, C, and D integers for which we can solve.

Question 2: Factor $x^4 + 5x^3 - 3x^2 - 17x - 10$

Solution:

$$x^4 + 5x^3 - 3x^2 - 17x - 10$$
$$= (x - 2)(Ax^3 + Bx^2 + Cx + D)$$

Remember that we were told in the question that 2 is a root. You should verify this. This means that $x - 2$ is a factor. We factor this term out and are left with a cubic polynomial as shown.

$$= (x - 2)(x^3 + Bx^2 + Cx + 5)$$

The x^4 term from our original polynomial comes from multiplying x by Ax^3 and so $A = 1$. The constant term -10 from our original polynomial comes from multiplying -2 by D and so $D = 5$.

$$= (x - 2)(x^3 + 7x^2 + Cx + 5)$$

The $5x^3$ term from our original polynomial comes from multiplying x by Bx^2 and adding the result to -2 times x^3. In other words, the term $5x^3$ equals $(x)(Bx^2) + (-2)(x^3)$ or $Bx^3 - 2x^3$. Comparing coefficients, we have $5 = B - 2$ and so $B = 7$.

$$= (x - 2)(x^3 + 7x^2 + 11x + 5)$$

The $-17x$ term from our original polynomial comes from multiplying x by 5 and adding the result to -2 times Cx. In other words, the term $-17x$ equals $(x)(5) + (-2)(Cx)$ or $5x - 2Cx$. This means $-17 = 5 - 2C$ and so $C = 11$.

$$= (x - 2)(x + 1)(x + 1)(x + 5)$$

Notice that this cubic is identical to the cubic from Question 1 and so we can use the factorization found earlier to finish factoring this quartic.

We use a similar strategy to factor the next quartic polynomial, but this time we need to find a root on our own. Again, it can be shown that if a is an integer root of the polynomial, then a must be a factor of the constant term, -18.

Question 3: Factor $4x^4 - 16x^3 + x^2 + 39x - 18$.

Solution:

$4x^4 - 16x^3 + x^2 + 39x - 18$ $= (x-2)(Ax^3+Bx^2+Cx+D)$	The factors of -18 are $\pm1, \pm2, \pm3, \pm6, \pm9$, and ±18. Using these factors, we determine that 2 is a root and so $x-2$ is a factor.
$= (x-2)(4x^3+Bx^2+Cx+9)$	The $4x^4$ term from our original polynomial comes from multiplying x by Ax^2 and so $A = 4$. The constant term -18 from our original polynomial comes from multiplying -2 by D and so $D = 9$.
$= (x-2)(4x^3-8x^2-15x+9)$	The $-16x^3$ term from our original polynomial comes from multiplying x by Bx^2 and adding the result to -2 times $4x^3$. Therefore, $B = -8$. The $39x$ term from our original polynomial comes from multiplying x by 9 and adding the result to -2 times Cx. Therefore, $C = -15$.
$= (x-2)(x-3)(Ex^2+Fx+G)$	Now we factor the cubic. The factors of 9 are $\pm1, \pm3$, and ±9. Using these factors, we determine that 3 is a root and so $x-3$ is a factor of the cubic.
$= (x-2)(x-3)(4x^2+4x-3)$	Using the $4x^3$ term, we determine that $E = 4$ and using the constant term 9, we determine that $G = -3$. Using the $-8x^2$ term, we determine that $F = 4$.
$= (x-2)(x-3)(2x-1)(2x+3)$	Factor the resulting quadratic.

Finally, we use what we have learned to solve the following problem.

Question 4: Given that $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$, determine the values of A, B, C and D.

Solution:

Consider the equality $(Ax^2 + Bx + C)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$.

The $6x^4$ term on the right side must come from multiplying the term Ax^2 by the term $3x^2$ on the left side. Since $6x^4$ is equal to $3Ax^4$ we must have $3A = 6$ and so $A = 2$.

The constant term 4 on the right side must come from multiplying C by -2 on the left side. Since 4 is equal to $-2C$ we must have $C = -2$.

This means we have $(2x^2 + Bx - 2)(3x^2 + Dx - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$.

The $3x^3$ term must come from multiplying the term $2x^2$ by the term Dx and adding the result to the product of Bx and $3x^2$. Since $2x^2$ is equal to $2Dx^3 + 3Bx^3$, we must have $2D + 3B = 3$.

Similarly, we can show that $2x$ must be equal to $(Bx)(-2) + (-2)(Dx) = -2Bx - 2Dx$. This means $-2B - 2D = 2$.

Since $2D+3B = 3$ and $-2B-2D = 2$, adding the two equations gives $B = 5$. We can then determine that $D = -6$.

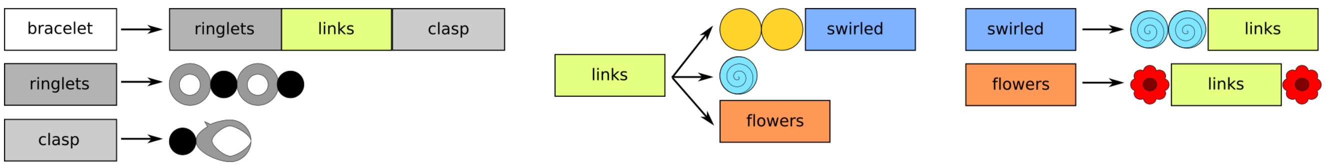
This means we have $A = 2$, $B = 5$, $C = -2$, $D = -6$, and

$$(2x^2 + 5x - 2)(3x^2 - 6x - 2) = 6x^4 + 3x^3 - 40x^2 + 2x + 4$$

CEMC at Home

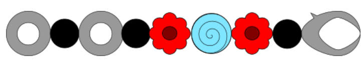
Grade 11/12 - Wednesday, May 6, 2020
Bracelets

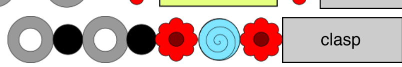
Stephen makes bracelets using the six replacement rules below.



Stephen always starts his pattern with the symbol bracelet. Then, one at a time, he replaces a symbol in the current pattern with a new sequence of symbols based on the rules above. Any symbol that appears on the left side of an arrow can be replaced with a sequence that appears on the right side of a connected arrow. In some, but not all cases, he has a choice about which particular replacement he could make at a particular stage in the process.

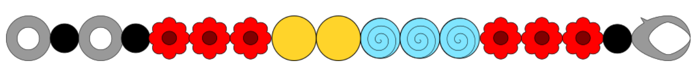
Example

Stephen could make the bracelet  following these steps:

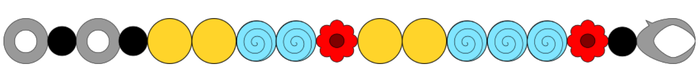
Step	Current Pattern	Explanation
1	bracelet	Stephen always starts with this symbol
2	ringlets links clasp	bracelet is replaced by ringlets links clasp
3		ringlets is replaced by 
4		links is replaced by flowers
5		flowers is replaced by 
6		links is replaced by 
7		clasp is replaced by 

Problems

1. Give a sequence of steps that Stephen could follow in order to produce the following bracelet:



2. Consider the three bracelets below. Stephen can make exactly two of the three bracelets using the rules. Explain how Stephen can make two of these bracelets, and explain why the remaining bracelet cannot be made using any sequence of steps.

(a) 

(b) 

(c) 

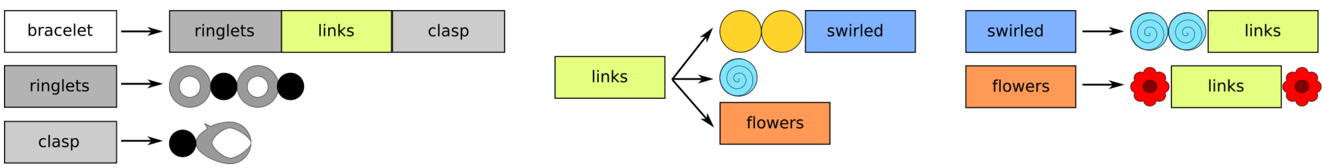
More Info:

Check out the CEMC at Home webpage on Wednesday, May 13 for a solution to Bracelets.

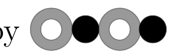
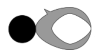
CEMC at Home

Grade 11/12 - Wednesday, May 6, 2020
Bracelets - Solution

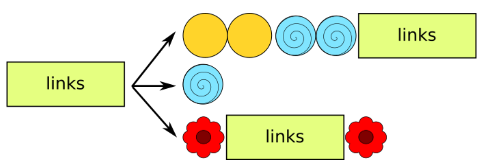
Before we look at particular sequences of symbols, we simplify the original six rules reproduced below.



Observe that five of the six rules do not involve a choice of which pattern to produce.

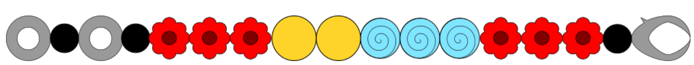
Recall that Stephen always starts with the symbol [bracelet] and so the next pattern must then be [ringlets][links][clasp]. At some point, the leftmost symbol will be replaced by  and the rightmost symbol will be replaced by . This means that to determine if it is possible to make a certain bracelet, we only need to determine if it is possible to create the "middle portion" by starting with [links] and following the three rules shown above on the right.

Notice that we can substitute the rightmost two rules into the replacement rule for [links] to give us the following single rule equivalent to the rightmost three original rules.

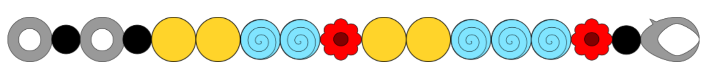


We will call the three choices in the rule above A (top), B (middle), and C (bottom).

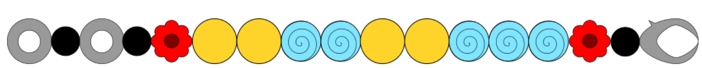
1. Stephen can make the bracelet shown below by making the choices C, C, C, A, B (in that order).



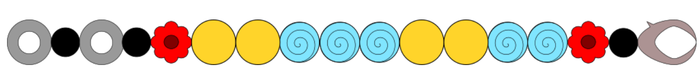
2. Stephen can make bracelets (a) and (c) but not (b).
   Stephen can make the bracelet shown below by making the choices A, C, A, B (in that order).

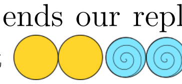
(a) 

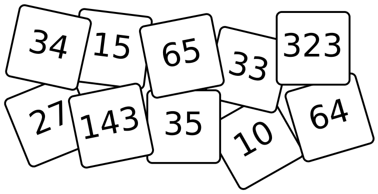
Stephen can make the bracelet shown below by making the choices C, A, A, B (in that order).

(c) 

Stephen cannot make the bracelet shown below.

(b) 

To justify this, we read from left to right to see that generating this bracelet would require us to make choice C first and then A. At this point, the next symbol is  which can only be achieved by choosing B next. This ends our replacement choices, but we have the wrong bracelet. (We are missing the rightmost  sequence.)



Problem of the Week
Problem E and Solution
The Factor Flip

Problem

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up. Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up. List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

Solution

We will start by writing down the prime factors for each of the numbers on the cards.

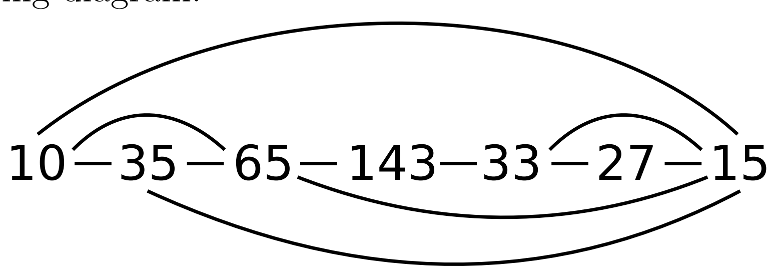
Number	Prime Factors
10	2,5
15	3,5
27	3
33	3,11
34	2,17

Number	Prime Factors
35	5,7
64	2
65	5,13
143	11,13
323	17,19

Notice the number 323 shares a prime factor with only the number 34. That means we must start (or end) with 323.

If we start with 323, then the next number must be 34. From 34, the next number could be 64 or 10. If the next number were 10, then in order to eventually flip over the 64 card, the 64 must follow the 10, and at this point no more cards can be flipped. Therefore, in order to flip all the cards, the first four numbers flipped must be 323, 34, 64, and then 10.

From this point, we can draw lines between numbers that share prime factors to create the following diagram.



We now just need to figure out the number of paths through the diagram, starting at 10 that use each number exactly once.

After 10, the next number can be 35, 65, or 15. In each case, we carefully trace through all possible paths.

Case 1: The number 35 follows the number 10. That gives us the following five possible paths.

35, 15, 65, 143, 33, 27
35, 15, 27, 33, 143, 65
35, 65, 15, 27, 33, 143
35, 65, 143, 33, 27, 15
35, 65, 143, 33, 15, 27

Case 2: The number 65 follows the number 10. That gives us the following two possible paths.

65, 143, 33, 27, 15, 35
65, 35, 15, 27, 33, 143

Case 3: The number 15 follows the number 10. That gives us the following two possible paths.

15, 27, 33, 143, 65, 35
15, 35, 65, 143, 33, 27

Starting with the card numbered 323, we have found that there is a total of nine orders for flipping the cards:

323, 34, 64, 10, 35, 15, 65, 143, 33, 27
323, 34, 64, 10, 35, 15, 27, 33, 143, 65
323, 34, 64, 10, 35, 65, 15, 27, 33, 143
323, 34, 64, 10, 35, 65, 143, 33, 27, 15
323, 34, 64, 10, 35, 65, 143, 33, 15, 27
323, 34, 64, 10, 65, 143, 33, 27, 15, 35
323, 34, 64, 10, 65, 35, 15, 27, 33, 143
323, 34, 64, 10, 15, 27, 33, 143, 65, 35
323, 34, 64, 10, 15, 35, 65, 143, 33, 27

Each of these can be reversed, so the total number of possible orders is 18.

Therefore, there are 18 orders in which Dani can flip the cards so that all cards get flipped over. They are the 9 orders listed above and the reverse of each.

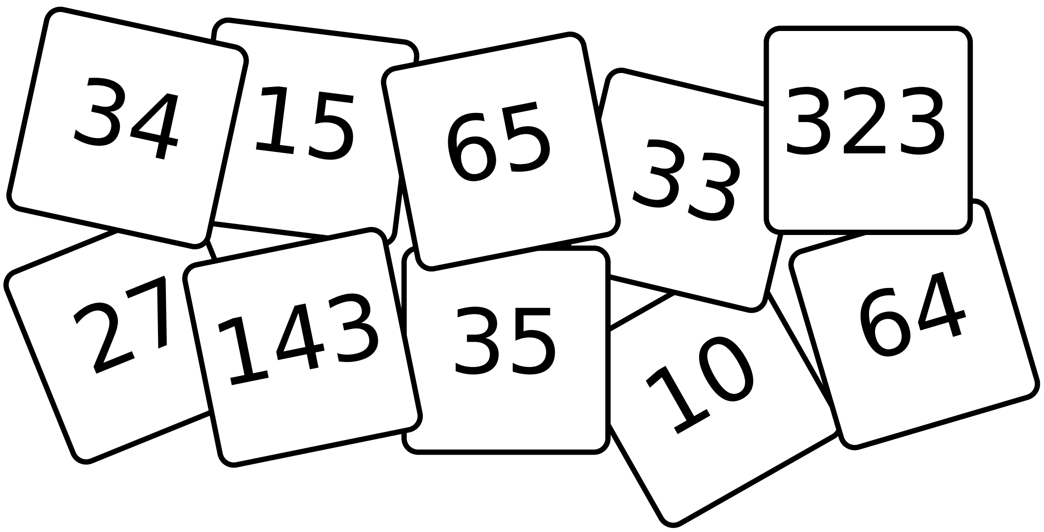
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, May 7, 2020
The Factor Flip

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up.

Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up.

List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

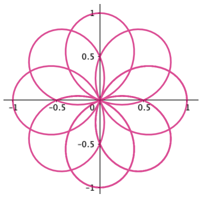


More Info:

Check the CEMC at Home webpage on Thursday, May 14 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution emailed to you on Thursday, May 14.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

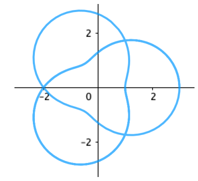
To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



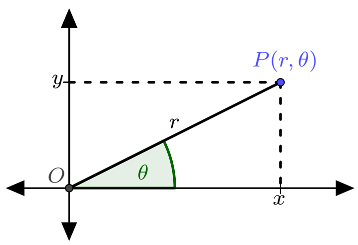
CEMC at Home

Grade 11/12 - Friday, May 8, 2020

Polar Curves



Last week we learned about a different coordinate system for the plane: the *Polar Coordinate System*. Remind yourself about how to work with polar coordinates before you try this activity.



Relationships between Cartesian coordinates and polar coordinates of a point in the plane

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

Why might we want to view the plane through the lens of polar coordinates? One reason is that simple equations of the form $r = f(\theta)$ involving polar coordinates can lead to interesting graphs!

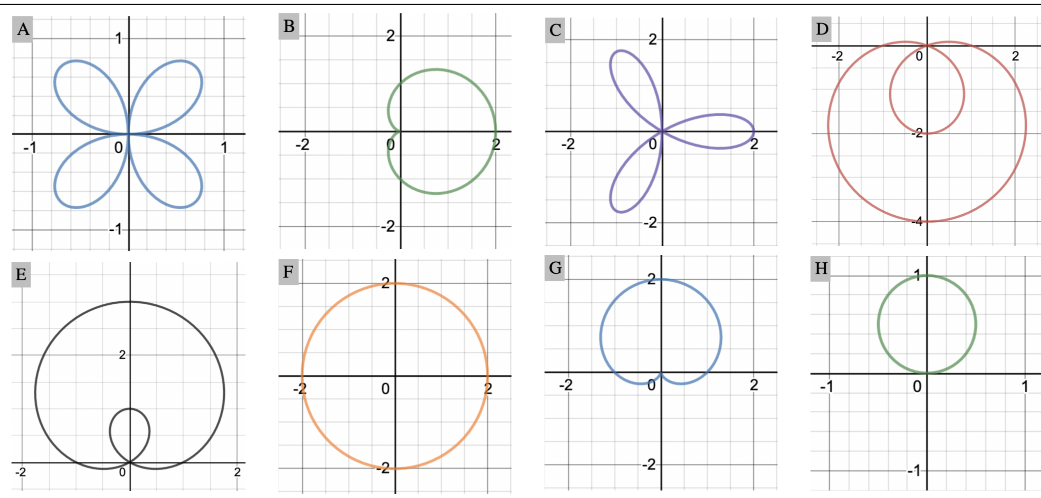
Let f be a function on the real numbers. The graph of the polar equation $r = f(\theta)$ consists of all points in the plane that have polar coordinates, (r, θ), that satisfy the relation $r = f(\theta)$.

Activity

Consider the following polar equations and the graphs below. Exactly one of the graphs corresponds to each equation. Can you match each equation with its graph? Think about the following techniques:

- Plot some key points on the curve. For example, when $\theta = \frac{\pi}{2}$, what is the value of r?
- Remember that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$. What does this mean for the range of r?
- Think about how r changes as θ changes. (See the next pages for help with this.)
- How are points with a negative r-coordinate plotted? (See the next pages for help with this.)

1. $r = 2$
2. $r = \sin\theta$
3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
5. $r = 1 + 2\sin\theta$
6. $r = 1 - 3\sin\theta$
7. $r = \sin(2\theta)$
8. $r = 2\cos(3\theta)$



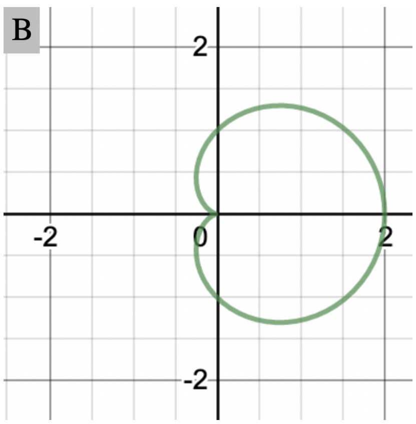
Example 1: Look at graph F. You should recognize this as a circle centred at the origin with radius 2. The points on this curve must be the points having polar coordinates that look like $(2, \theta)$ for some θ (2 units from the origin, at any angle). This means graph F must be matched with equation $r = 2$.

Note that we could also determine what the graph of $r = 2$ must look like by transforming this polar equation into a Cartesian equation. Since $r = \sqrt{x^2 + y^2}$, a point's polar coordinates satisfy the equation $r = 2$ exactly when its Cartesian coordinates satisfy the equation $\sqrt{x^2 + y^2} = 2$. Squaring both sides reveals the equation $x^2 + y^2 = 4$ which describes the circle shown!

Can you match each of the eight graphs with one of the eight equations without actually trying to sketch the complete graphs of the polar equations? Read the following example to get you started on possible matching strategies that do not involve graphing the polar equations.

Example 2

Consider graph B. Given that this graph is **matched with one of the five equations below**, can you figure out which one by eliminating all but one equation?



1. $r = 2$
2. $r = \sin\theta$
3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
7. $r = \sin(2\theta)$

Let's see if we can use only the range of r to eliminate several possibilities.

1. Graph B cannot be the graph of $r = 2$: We have already determined that $r = 2$ is matched with another graph.
2. Graph B cannot be the graph of $r = \sin\theta$: Since $\sin\theta$ cannot be larger than 1, no points on the graph of this polar equation can be more than 1 unit from the origin. Graph B has at least one point 2 units from the origin.
3. Graph B might be the graph of $r = 1 + \cos\theta$: Since $-1 \leq \cos\theta \leq 1$, we have $0 \leq 1 + \cos\theta \leq 2$ and so the points on this graph should all be within 2 units of the origin or exactly 2 units from the origin. This is true of the graph B.
4. Graph B might be the graph of $r = 1 + \sin\theta$: Similar reasoning as in 3.
7. Graph B cannot be the graph of $r = \sin(2\theta)$: Similar reasoning as in 2.

By considering the range of r we have narrowed down the choices to two equations: $r = 1 + \cos\theta$ and $r = 1 + \sin\theta$.

Can you see which one must be the correct equation for Graph B? Try plotting a few points.

For equation 3: When $\theta = 0$ we have $r = 1 + \cos 0 = 2$. This matches the graph above.

For equation 4: When $\theta = 0$ we have $r = 1 + \sin 0 = 1$. This does not match the graph above.

This tells us that the equation must be 3: $r = 1 + \cos\theta$.

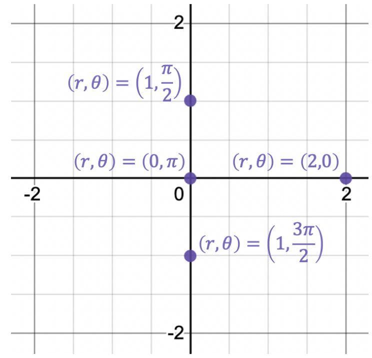
On the next page we will discuss how to sketch the graph of the polar equation $r = 1 + \cos\theta$ to see exactly why Graph B above matches this equation. You do not need to sketch this graph to complete the activity, but you may still want to spend some time thinking about why this is the correct graph.

For many of the eight equations, there are pairs (r, θ) with $r < 0$ that satisfy the equation. We discuss how to interpret negative r-coordinates on the last pages of the resource.

Example 3: Sketch the graph of the polar equation $r = 1 + \cos(\theta)$.

Plot a few key points.

- When $\theta = 0$, $r = 2$.
- When $\theta = \frac{\pi}{2}$, $r = 1$.
- When $\theta = \pi$, $r = 0$.
- When $\theta = \frac{3\pi}{2}$, $r = 1$.
- When $\theta = 2\pi$, $r = 2$.

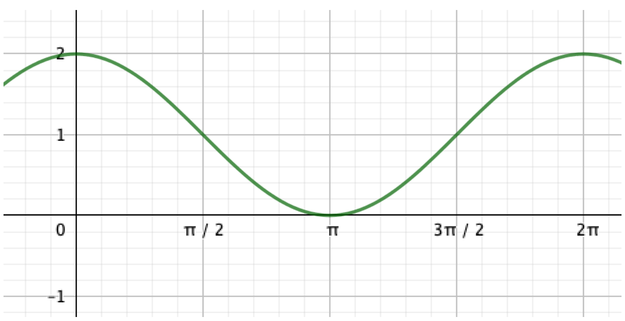


Think about the range of r.

Since $-1 \leq \cos\theta \leq 1$, we must have $0 \leq 1 + \cos\theta \leq 2$. This means all points on the graph must be at most 2 units from the origin.

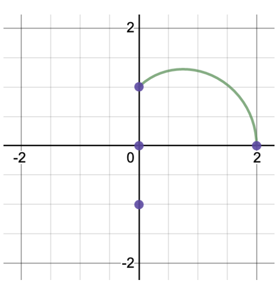
Think about how r changes as θ changes.

Can you describe what happens to r as θ ranges from 0 to 2π? We sketch the graph of $y = 1 + \cos x$ drawn in the usual Cartesian plane. Can you see how to use this information to make the the table?

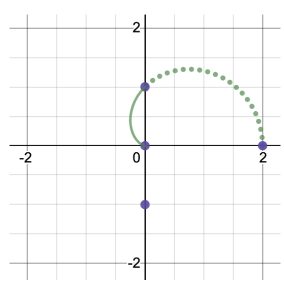


θ	$r = 1 + \cos(\theta)$	Polar Point
0	2	$(2, 0)$
0 to $\frac{\pi}{2}$	r decreases from 2 to 1	
$\frac{\pi}{2}$	1	$(1, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 1 to 0	
π	0	$(0, \pi)$
π to $\frac{3\pi}{2}$	r increases from 0 to 1	
$\frac{3\pi}{2}$	1	$(1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to 2π	r increases from 1 to 2	
2π	2	$(2, 2\pi)$

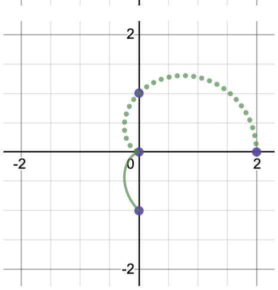
Draw a rough sketch of the curve



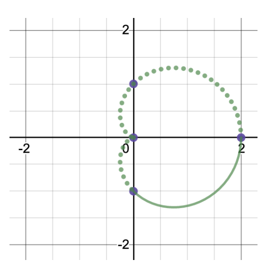
As θ increases from 0 to $\frac{\pi}{2}$, r decreases from 2 to 1. So we connect the polar points $(2, 0)$ and $(1, \frac{\pi}{2})$ through the first quadrant.



As θ increases from $\frac{\pi}{2}$ to π, r decreases from 1 to 0. So we connect the polar points $(1, \frac{\pi}{2})$ and $(0, \pi)$ through the second quadrant.



As θ increases from π to $\frac{3\pi}{2}$, r increases from 0 to 1. So we connect the polar points $(0, \pi)$ and $(1, \frac{3\pi}{2})$ through the third quadrant.



As θ increases from $\frac{3\pi}{2}$ to 2π, r increases from 1 to 2. So we connect the polar points $(1, \frac{3\pi}{2})$ and $(2, 2\pi)$ through the fourth quadrant.

Can you convince yourself that the sketch will take this curved shape? We used technology to plot many points in order to get an accurate curve. Since the function $\cos\theta$ repeats with period 2π, plotting points for more values of θ will just result in drawing this same curve over again!

Example 4: Consider the polar equation $r = 1 + 2\sin\theta$.

Notice that there are values of θ for which the corresponding r is negative. For example, when $\theta = \frac{3\pi}{2}$, we have

$$r = 1 + 2\sin\left(\tfrac{3\pi}{2}\right) = 1 + 2(-1) = -1$$

What does this mean in terms of our graphing activity?

Can we plot points with polar coordinates with negative values of r?

We can extend the definition of polar coordinates to include negative values of r.
How do we interpret the polar coordinates $(1, \frac{\pi}{2})$ versus the polar coordinates $(-1, \frac{\pi}{2})$?

- The fact that they both have the same angle $\frac{\pi}{2}$ tells us that they both describe points that lie on the line passing through the origin and making an angle of $\frac{\pi}{2}$ with the positive x-axis.
- The magnitude of the radii both being 1 tell us that they both describe points that are 1 unit from the origin.
- The different signs tell us that they describe points on opposite sides of the origin. The negative means that we move in the direction *opposite* to the direction defined the ray $\theta = \frac{\pi}{2}$. This means moving in the direction defined by the ray $\theta = \frac{3\pi}{2}$.

So the polar coordinates $(-1, \frac{\pi}{2})$ are equivalent to the polar coordinates $(1, \frac{3\pi}{2})$ and they both represent the Cartesian point $(0, -1)$. Indeed if we use the usual formulas to convert from polar coordinates to Cartesian coordinates, we get the following:

Polar coordinates $(-1, \frac{\pi}{2})$	*Polar coordinates* $(1, \frac{3\pi}{2})$
$x = r\cos\theta = (-1)\cos\left(\frac{\pi}{2}\right) = 0$	$x = r\cos\theta = 1\cos\left(\frac{3\pi}{2}\right) = 0$
$y = r\sin\theta = (-1)\sin\left(\frac{\pi}{2}\right) = -1$	$y = r\sin\theta = 1\sin\left(\frac{3\pi}{2}\right) = -1$

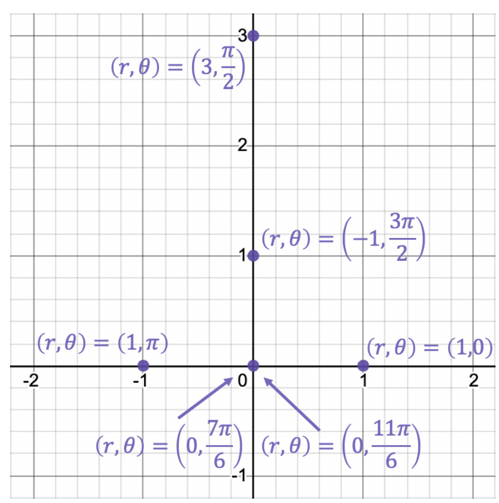
Example 5: Consider the graph of the polar equation $r = 1 + 2\sin\theta$.

Note that it will be important to know where r changes from negative to positive. To find these places, we solve the equation $r = 1 + 2\sin\theta = 0$. Two solutions are $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Plot a few key points.

- When $\theta = 0$ (or $\theta = 2\pi$), $r = 1$.
- When $\theta = \frac{\pi}{2}$, $r = 3$.
- When $\theta = \pi$, $r = 1$.
- When $\theta = \frac{7\pi}{6}$, $r = 0$.
- When $\theta = \frac{3\pi}{2}$, $r = -1$.

  Remember that this pair describes the same point as the pair $\theta = \frac{\pi}{2}$ and $r = 1$.
- When $\theta = \frac{11\pi}{6}$, $r = 0$.

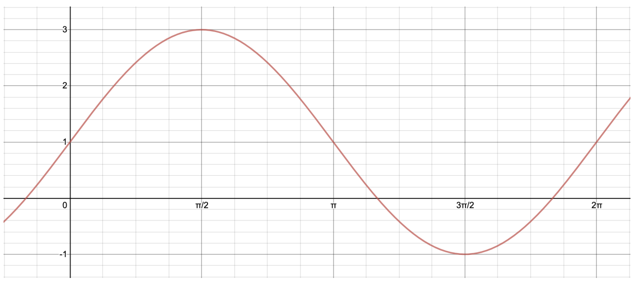


Think about the range of r.

Since $-1 \le \sin\theta \le 1$, we must have $-1 \le 1 + 2\sin\theta \le 3$. Since the magnitude of r must be at most 3, we know that all points on the graph must lie at most 3 units away from the origin.

Think about how r changes as θ changes.

Can you describe what happens to r as θ ranges from 0 to 2π?



θ	$r = 1 + 2\sin(\theta)$	Polar Point
0	1	$(1, 0)$
0 to $\frac{\pi}{2}$	r increases from 1 to 3	
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 3 to 1	
π	1	$(1, \pi)$
π to $\frac{7\pi}{6}$	r decreases from 1 to 0	
$\frac{7\pi}{6}$	0	$(0, \frac{7\pi}{6})$
$\frac{7\pi}{6}$ to $\frac{3\pi}{2}$	r decreases from 0 to -1	
$\frac{3\pi}{2}$	-1	$(-1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	r increases from -1 to 0	
$\frac{11\pi}{6}$	0	$(0, \frac{11\pi}{6})$
$\frac{11\pi}{6}$ to 2π	r increases from 0 to 1	
2π	1	$(1, 2\pi)$

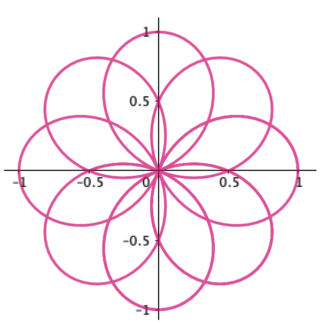
It is not easy to see how to translate the complete information from the table into a sketch of the graph. It takes most people a lot of time to get comfortable sketching these curves when they involve negative values of r. Luckily, you do not need to sketch the *whole* curve in order to figure out which graph matches the equation $r = 1 + 2\sin\theta$. If you can draw a few "pieces" of the graph for $r = 1 + 2\sin\theta$ then you should be able to pick its graph out of the list. In fact, you might be able to pick out the correct graph by using *only* the key points considered in this example!

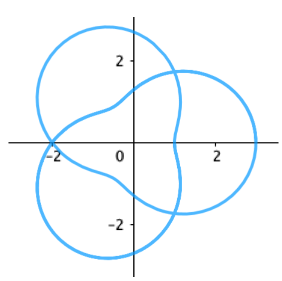
More Info:

Check out the CEMC at Home webpage on Friday, May 15 for a solution to Polar Curves.

You may also want to check out some of the free online graphing calculators for polar curves, like the ones offered by *WolframAlpha* or *Desmos* to verify your answers.

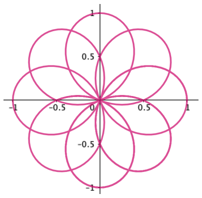
The graphs in the header of the first page of this activity each come from graphing one of the following polar equations. Which equation matches which graph and why?





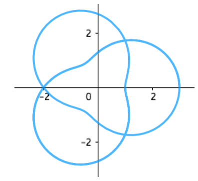
$$r = 2 + \cos\left(\frac{3\theta}{2}\right)$$

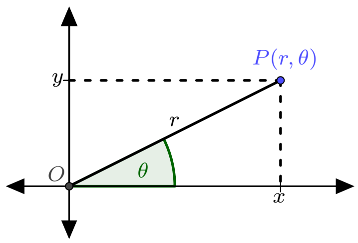
$$r = \cos\left(\frac{4\theta}{3}\right)$$



CEMC at Home

Grade 11/12 - Friday, May 8, 2020
Polar Curves - Solution





Relationships between Cartesian coordinates and polar coordinates of a point in the plane

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$

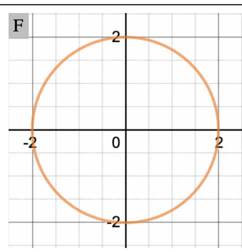
Let f be a function on the real numbers. The graph of the polar equation $r = f(\theta)$ consists of all points in the plane that have polar coordinates, (r, θ), that satisfy the relation $r = f(\theta)$.

Activity

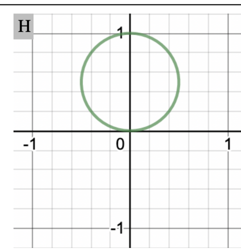
Consider the following polar equations and the graphs below. Exactly one of the graphs corresponds to each equation. Can you match each equation with its graph?

Answers (explanations provided on the pages that follow)

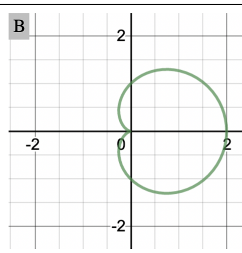
1. $r = 2$



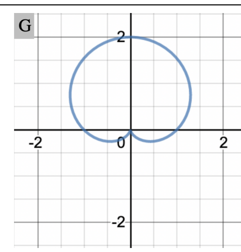
2. $r = \sin\theta$



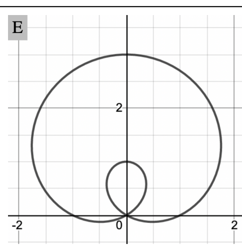
3. $r = 1 + \cos\theta$



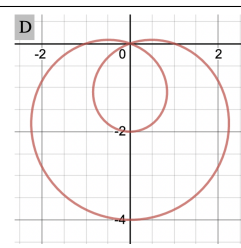
4. $r = 1 + \sin\theta$



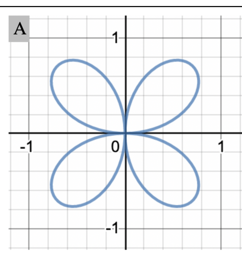
5. $r = 1 + 2\sin\theta$



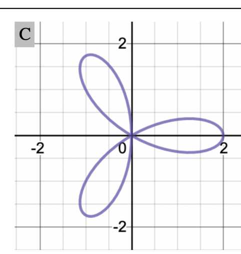
6. $r = 1 - 3\sin\theta$



7. $r = \sin(2\theta)$



8. $r = 2\cos(3\theta)$



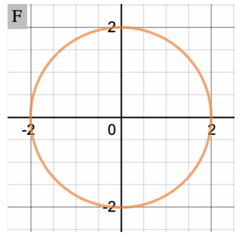
Step 1: Think about the range of r. *(This is only one of many possible first steps.)*

For the equations: Using the fact that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$, we can determine the range of r values for the polar functions $r = f(\theta)$.

For the graphs: We cannot determine the exact range of r values of the associated equation just by looking at the graph, but we can determine an upper bound on the *magnitude of r* from the graph.

1. $r = 2$

   Range: $2 \leq r \leq 2$



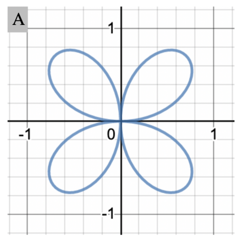
All points appear to be 2 units away from the origin.

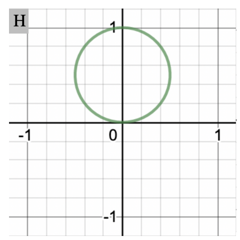
2. $r = \sin\theta$

   Range: $-1 \leq r \leq 1$

7. $r = \sin(2\theta)$

   Range of r: $-1 \leq r \leq 1$





The points that are farthest from the origin appear to be 1 unit away.

3. $r = 1 + \cos\theta$

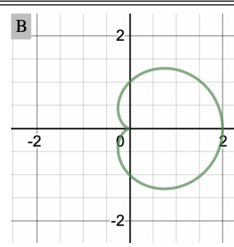
   Range of r: $0 \leq r \leq 2$

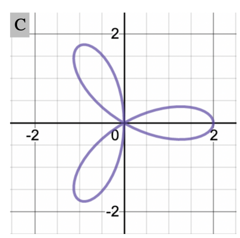
4. $r = 1 + \sin\theta$

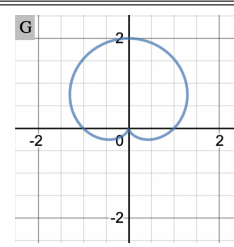
   Range of r: $0 \leq r \leq 2$

8. $r = 2\cos(3\theta)$

   Range of r: $-2 \leq r \leq 2$



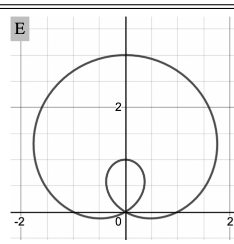




The points that are farthest from the origin appear to be 2 units away.

5. $r = 1 + 2\sin\theta$

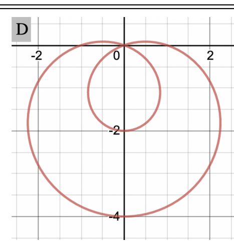
   Range of r: $-1 \leq r \leq 3$



The points that are farthest from the origin appear to be 3 units away.

6. $r = 1 - 3\sin\theta$

   Range of r: $-2 \leq r \leq 4$

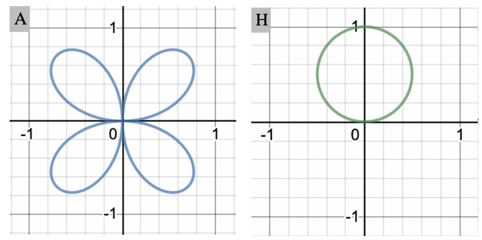


The points that are farthest from the origin appear to be 4 units away.

Step 2: Plot a few key points

Our work on the previous page matches equations 1, 5, and 6 with their graphs, and divides the remaining equations into two different groups as shown below. To determine which equation matches with which graph (within its group) we will think about plotting a few key points.

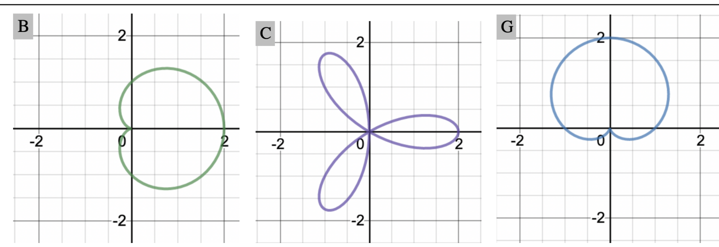
2. $r = \sin\theta$
7. $r = \sin(2\theta)$



Consider the point in the plane with Cartesian coordinates $(x, y) = (0, 1)$. Notice that this point is on graph H above but not on graph A. One way to describe this point using polar coordinates is $(r, \theta) = (1, \frac{\pi}{2})$.

Since $1 = \sin\left(\frac{\pi}{2}\right)$, this point must be on the graph of equation 2: $r = \sin\theta$. This means equation 2 must be matched with graph H. It follows that equation 7 must be matched with graph A.

3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
8. $r = 2\cos(3\theta)$



First, consider the point with Cartesian coordinates $(x, y) = (2, 0)$. Notice that this point is on graphs B and C above but not on graph G. One way to describe this point using polar coordinates is $(r, \theta) = (2, 0)$.

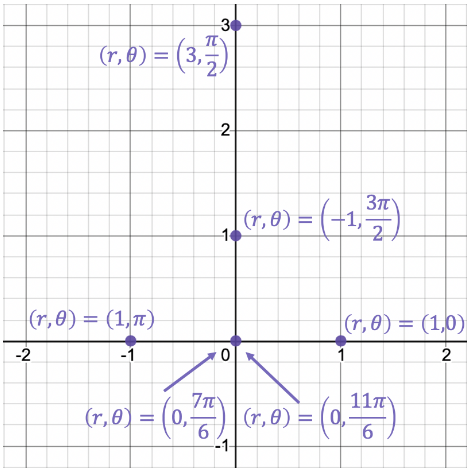
Since $2 = 1 + \cos(0)$ and $2 = 2\cos(3 \cdot 0)$, this point must be on the graphs of equation 3 ($r = 1 + \cos\theta$) and equation 8 ($r = 2\cos(3\theta)$). This means equations 3 and 8 must be matched with graphs B and C, in some order. It follows that equation 4 ($r = 1 + \sin\theta$) must be matched with graph G.

Now, consider the point with Cartesian coordinates $(x, y) = (0, 1)$ and polar coordinates $(r, \theta) = (1, \frac{\pi}{2})$. Notice that this point is on graph B but not on graph C.

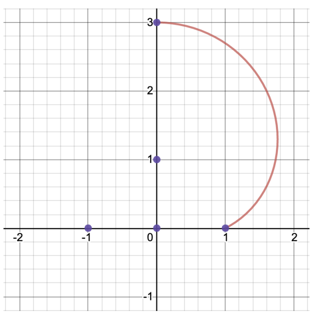
Since $1 = 1 + \cos\left(\frac{\pi}{2}\right)$, this point must be on the graph of equation 3: $r = 1 + \cos\theta$. This means equation 3 must be matched with graph B. It follows that equation 8 must be matched with graph C.

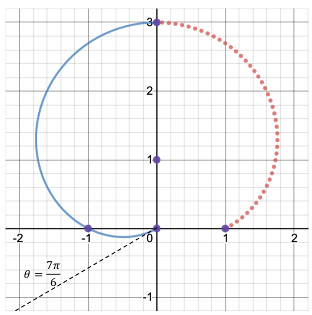
Notice that we have completed the matching activity without actually graphing any of the polar equations completely. We have just picked out certain characteristics of the equations and the graphs in order to find the right matches. We encourage you to follow the strategy outlined in the activity for how to sketch the graphs of the equations from scratch, and confirm the matchings that way as well. On the next page, we revisit the polar equation $r = 1 + 2\sin\theta$ *that was discussed in the activity, and outline how to sketch its graph.*

Using a few key points on the graph, and the table below, we sketch the graph of $r = 1 + 2\sin\theta$.



θ	$r = 1 + 2\sin(\theta)$	Polar Point
0	1	$(1, 0)$
0 to $\frac{\pi}{2}$	r increases from 1 to 3	
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 3 to 1	
π	1	$(1, \pi)$
π to $\frac{7\pi}{6}$	r decreases from 1 to 0	
$\frac{7\pi}{6}$ to $\frac{3\pi}{2}$	r decreases from 0 to -1	
$\frac{3\pi}{2}$	-1	$(-1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	r increases from -1 to 0	
$\frac{11\pi}{6}$ to 2π	r increases from 0 to 1	
2π	1	$(1, 2\pi)$

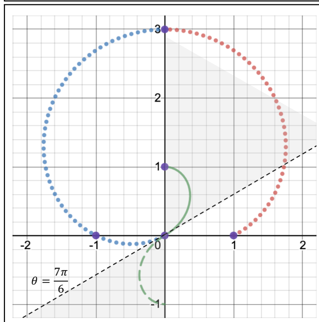




As θ increases from 0 to $\frac{\pi}{2}$, r increases continuously from 1 to 3. So we connect the polar points $(1, 0)$ and $(3, \frac{\pi}{2})$ through the first quadrant as shown in the leftmost image.

As θ increases from $\frac{\pi}{2}$ to π, r decreases from 3 to 1. So we connect polar points $(3, \frac{\pi}{2})$ and $(1, \pi)$ through the second quadrant.

As θ increases from π to $\frac{7\pi}{6}$, r decreases from 1 to 0. So we connect polar points $(1, \pi)$ and $(0, \frac{7\pi}{6})$ through the third quadrant as shown in the rightmost image.

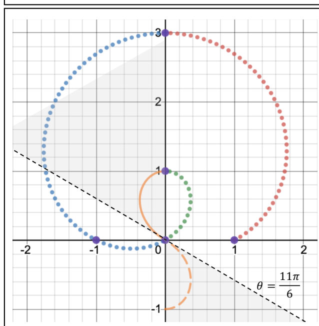


As θ increases from $\frac{7\pi}{6}$ to $\frac{3\pi}{2}$, r decreases from 0 to -1.

Since the point $(-1, \frac{3\pi}{2})$ is equivalent to the point $(1, \frac{\pi}{2})$, we know we are connecting the points $(0, \frac{7\pi}{6})$ and $(1, \frac{\pi}{2})$. But how?

Since r is negative between $\theta = \frac{7\pi}{6}$ and $\theta = \frac{3\pi}{2}$, the points we plot for this interval of θ will not actually lie in the third quadrant; they will lie in the first quadrant as indicated in the image. The magnitude of r tells us how far to plot the points from the origin; the negative sign attached to r tells us to plot the points on the "other side of the origin".

We connect the points $(0, \frac{7\pi}{6})$ and $(1, \frac{\pi}{2})$ through the first quadrant as shown.

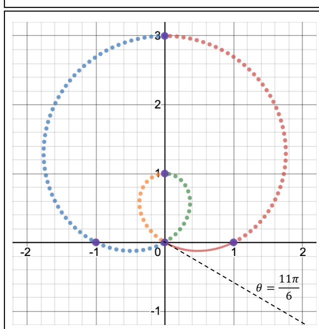


As θ increases from $\frac{3\pi}{2}$ to $\frac{11\pi}{6}$, r increases from -1 to 0.

Since the point $(-1, \frac{3\pi}{2})$ is equivalent to the point $(1, \frac{\pi}{2})$, we know we are connecting the points $(1, \frac{\pi}{2})$ and $(0, \frac{11\pi}{6})$. But how?

Since r is negative between $\theta = \frac{3\pi}{2}$ and $\theta = \frac{11\pi}{6}$, the points we plot for this interval of θ will not actually lie in the fourth quadrant; they will lie in the second quadrant as indicated in the image.

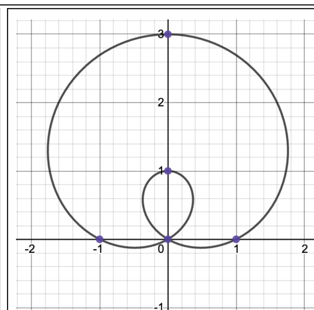
We connect the points $(1, \frac{\pi}{2})$ and $(0, \frac{11\pi}{6})$ through the second quadrant as shown.



Finally, we finish the sketch for θ between $\frac{11\pi}{6}$ and 2π, where r increases from 0 to 1.

Since the function $\cos\theta$ repeats with period 2π, plotting points for more values of θ will just result in drawing this same curve over again.

The completed graph is shown on the right.



CEMC at Home

Grade 11/12 - Monday, May 11, 2020
Contest Day 2

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Team Problem #7

What is the smallest four-digit positive integer that is divisible by both 5 and 9 and has only even digits?

2020 Euclid Contest, #4(b)

A geometric sequence has first term 10 and common ratio $\frac{1}{2}$.
An arithmetic sequence has first term 10 and common difference d.
The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence.
Determine all possible values of d.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 3, 6, 12 is a geometric sequence with three terms.)

More Info:

Check out the CEMC at Home webpage on Thursday, May 21 for solutions to the Contest Day 2 problems.

CEMC at Home

Grade 11/12 - Monday, May 11, 2020
Contest Day 2 - Solution

Solutions to the two contest problems are provided below, including a video for the first problem.

2020 Canadian Team Mathematics Contest, Team Problem #7

What is the smallest four-digit positive integer that is divisible by both 5 and 9 and has only even digits?

Solution 1:

Suppose the number is $abcd$ where a, b, c, and d are digits.
Since the number is divisible by 5, we must have that $d = 0$ or $d = 5$.
The digits are all even, which means $d \neq 5$, so $d = 0$.

The smallest that a can be is 2 since it must be even and greater than 0 (a four-digit number cannot have $a = 0$). So we will try to find such a number with $a = 2$.
In order to be divisible by 9, we must have $a + b + c + d$ divisible by 9. Substituting $a = 2$ and $d = 0$, this means $2 + b + c + 0 = 2 + b + c$ is divisible by 9.

Since b and c are even, $2 + b + c$ is even, which means it cannot equal 9. Thus, we will try to find b and c so that $2 + b + c = 18$, which is the smallest multiple of 9 that is greater than 9.
This equation rearranges to $b + c = 16$. Since b and c are even and satisfy $0 \leq b \leq 9$ and $0 \leq c \leq 9$, the only possibility is $b = c = 8$.

Solution 2:

An integer is divisible by both 5 and 9 exactly when it is divisible by 45.
Since we seek a number having only even digits, its last digit is one of $0, 2, 4, 6$, or 8, so the number itself is even.
Thus, we are looking for an even multiple of 45.

An even number is a multiple of 45 exactly when it is a multiple of 90.
This means we are looking for the smallest 4-digit multiple of 90 that has only even digits.
The smallest 4-digit multiple of 90 is 1080, but the first digit of this number is 1, which is odd.
Each of the next 10 multiples of 90 has a first digit equal to 1, so the number we seek must be larger than 2000.

The four-digit multiples of 90 that have a thousands digit of 2 are

$$2070, 2160, 2250, 2340, 2430, 2520, 2610, 2700, 2790, 2880, 2970$$

and the only number in this list that has all even digits is 2880.
Therefore, 2880 is the smallest 4-digit number that is a multiple of 5, a multiple of 9, and has only even digits.

Video

Visit the following link for a discussion of two different approaches to solving the first contest problem: https://youtu.be/hnksZR1etAg.

2020 Euclid Contest, #4(b)

A geometric sequence has first term 10 and common ratio $\frac{1}{2}$.
An arithmetic sequence has first term 10 and common difference d.
The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence.
Determine all possible values of d.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 3, 6, 12 is a geometric sequence with three terms.)

Solution:

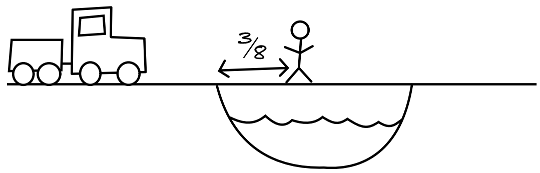
The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10,\ 10 + d,\ 10 + 2d,\ 10 + 3d,\ 10 + 4d,\ 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

CEMC at Home

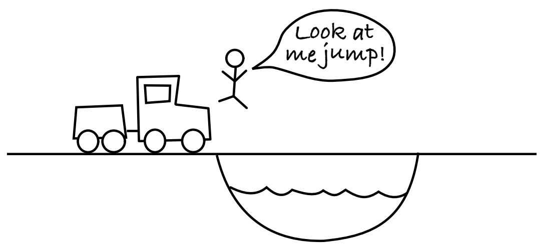
Grade 11/12 - Tuesday, May 12, 2020

Crocodiles!

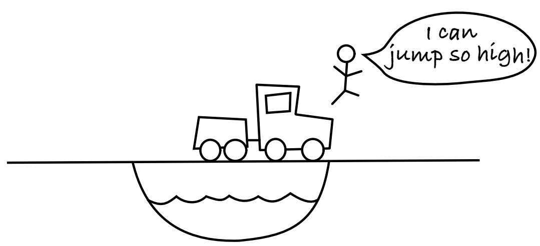
As part of an astounding race, you foolishly try to take a shortcut across a bridge over a river full of hungry crocodiles. When you are $\frac{3}{8}$ of the way across the bridge you notice something terrifying. This bridge is a railway bridge and there is a train travelling towards you!



You definitely aren't going to jump in the river with all those crocodiles. You quickly do some calculations and you realize that if you run towards the train, you will reach the end of the bridge just as the train reaches the bridge and you will have just enough time to jump off and be safe.



You also realize that if you run away from the train, you will reach the other end of the bridge just as the train catches up to you and again you will have just enough time to jump off and be safe.



The train is travelling at 40 km/h. How fast can you run?

Here are the answers to some questions you may want to ask. Yes, you do have enough information to solve the problem. No, we don't know how far the train is from the bridge at the beginning. Yes, we will assume our speed is constant (we can instantaneously run at our top speed!).

Next week we will present two different solutions to this problem. One solution will use algebra to solve the problem and the other will use a different type of reasoning. Can you figure out how to solve this problem in two very different ways?

More Info:

Check out the CEMC at Home webpage on Tuesday, May 19 for two different solutions to this problem. We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Tuesday, May 12, 2020

Crocodiles! - Solution

Solution 1:

Let d be initial distance from the train to the bridge, let x be the length of the bridge, and let s be your speed. The two different scenarios given in the problem give us two equations in terms of these variables. We will make use of the fact that

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

In the first scenario, we are told that the time it takes the train to reach the start of the bridge is equal to the time it takes you to run back to the start of the bridge. Therefore,

$$\frac{d}{40} = \frac{\frac{3}{8}x}{s},$$

which simplifies to

$$ds = 15x. \qquad (1)$$

In the second scenario, we are told that the time it takes the train to reach the start of the bridge and then cross it is equal to the time it takes you to run to the end of the bridge. Therefore,

$$\frac{d+x}{40} = \frac{\frac{5}{8}x}{s},$$

which simplifies to

$$ds + xs = 25x. \qquad (2)$$

Substituting equation (1) into equation (2), we get $15x + xs = 25x$ or $xs = 10x$. We know that $x \neq 0$ and so we divide both sides of the equation by x to obtain $s = 10$. Therefore, your speed is 10 km/h.

Solution 2:

Consider the scenario where you run away from the train. When the train reaches the start of the bridge, where are you? We know that if you run towards the train, then you will be at the start of the bridge when the train reaches the bridge. In other words, you run $\frac{3}{8}$ of the bridge in the time it takes for the train to reach the bridge. So if you run away from the train, when the train reaches the bridge you will be $\frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$ of the way across the bridge. You have $\frac{1}{4}$ of the bridge left to run. In the time it takes you to run what's left, the train will go the length of the bridge. Therefore, your speed is $\frac{1}{4}$ the speed of the train and so your speed is 10 km/h.

Discussion: Isn't the second solution here amazing? Many students are really excited when they see it! We often solve problems by introducing variables, creating equations and solving for unknowns like we did in the first solution. In many situations this process is necessary to solve the problem. However, in this problem, if we use the information in a different way, we can find a simple solution. As you continue to study mathematics you will add more and more tools to your problem solving toolbox. These tools are very powerful and very useful, but always be on the lookout for alternative ways to tackle problems.

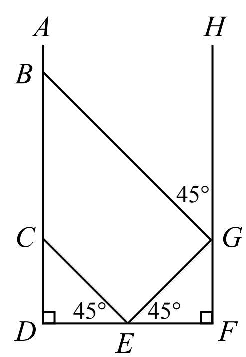
CEMC at Home

Grade 11/12 - Thursday, May 21, 2020
Contest Day 3

Today's resource features two questions from the recently released 2020 CEMC Mathematics Contests.

2020 Euclid Contest, #3(a)

Donna has a laser at C. She points the laser beam at the point E. The beam reflects off of DF at E and then off of FH at G, as shown, arriving at point B on AD. If $DE = EF = 1$ m, what is the length of BD, in metres?



2020 Euclid Contest, #5(b)

Determine all triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned}(x-1)(y-2) &= 0\\(x-3)(z+2) &= 0\\x+yz &= 9\end{aligned}$$

More Info:

Check out the CEMC at Home webpage on Monday, May 25 for solutions to the Contest Day 3 problems.

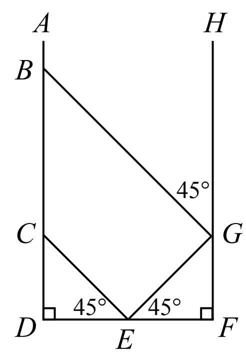
CEMC at Home

Grade 11/12 - Thursday, May 21, 2020
Contest Day 3 - Solution

Solutions to the two contest problems are provided below.

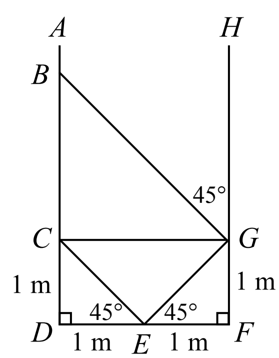
2020 Euclid Contest, #3(a)

Donna has a laser at C. She points the laser beam at the point E. The beam reflects off of DF at E and then off of FH at G, as shown, arriving at point B on AD. If $DE = EF = 1$ m, what is the length of BD, in metres?



Solution:

First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

See the next page for a solution to the second problem.

2020 Euclid Contest, #5(b)

Determine all triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned}(x-1)(y-2) &= 0\\(x-3)(z+2) &= 0\\x+yz &= 9\end{aligned}$$

Solution:

Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned}(1-3)(z+2) &= 0\\1+yz &= 9\end{aligned}$$

or

$$\begin{aligned}-2(z+2) &= 0\\yz &= 8\end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

$$\begin{aligned}(x-3)(z+2) &= 0\\x+2z &= 9\end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

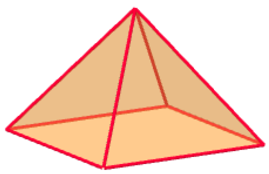
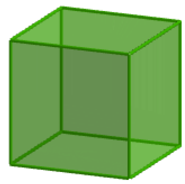
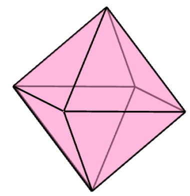
$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

CEMC at Home

Grade 11/12 - Friday, May 22, 2020
Euler Characteristic - Part 1

A *polyhedron* (plural: polyhedra or polyhedrons) is a three-dimensional solid with polygons for its faces. Below are three examples of polyhedra: a square-based pyramid, a cube, and an octahedron.

A polyhedron has vertices, edges, and faces. Given a polyhedron, we let V, E, and F denote the number of vertices, edges, and faces, respectively, of the polyhedron.

In this activity, we will explore the relationship between the values of V, E, and F.

Example
Verify that the square-based pyramid has 5 vertices, 8 edges, and 5 faces. If we calculate the value of $V - E + F$ then we get $5 - 8 + 5 = 2$.

Example
Verify that the cube has 8 vertices, 12 edges, and 6 faces. If we calculate the value of $V - E + F$ then we get $8 - 12 + 6 = 2$.

Question
What is the value of $V - E + F$ for the octahedron?
You should get an answer of 2, again! Confirm this for yourself.

It seems unlikely that it is just a coincidence that all three of these polyhedra produce the same value of $V - E + F$. Is there some reason to believe that this will always happen?

> The **Euler characteristic** of a polyhedron, denoted χ, is defined to be the value of $V - E + F$. It turns out that $\chi = 2$ for every (convex) polyhedron.

Explaining why the Euler characteristic is always 2 is challenging, and we will explore this idea a bit further in next week's activity. You can use this fact, when needed, to solve the following problems.

Problem 1
Verify directly that $\chi = 2$ for a tetrahedron, a dodecahedron, and an icosahedron.
You may need to look up one or two of these platonic solids first!

Problem 2
A particular polyhedron has 26 faces and has twice as many edges as vertices. How many edges must the polyhedron have?

Problem 3
An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

Problem 4

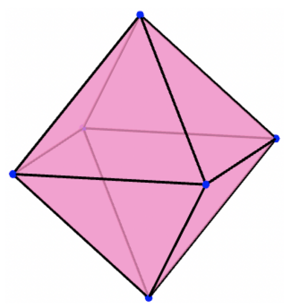
A polyhedron is formed with exactly P pentagons, exactly H hexagons, and no other polygons as its faces and has the property that three polygonal faces meet at each vertex.

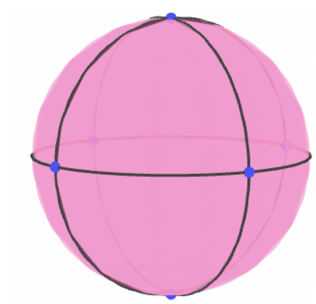
1. Explain why it is true that $E = \dfrac{5P+6H}{2}$.
2. Explain why it is true that $V = \dfrac{5P+6H}{3}$.
3. Using the fact that $\chi = 2$ for this polyhedron, show that it must be the case that $P = 12$.

A standard soccer ball can be thought of as an "inflation" of such a polyhedron.

Further Discussion

The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a convex polyhedron is always 2. Imagine "inflating" a polyhedron, as if its surface is elastic like a balloon. For example, if we "inflate" the octahedron, then we obtain a sphere as shown below. We visualize the vertices, edges, and faces of the octahedron on the spherical surface.





Can you visualize what would happen if we "inflated" the pyramid or the cube in a similar way?

It can be helpful to have a common way in which to view all possible polyhedra; we can identify each polyhedron with how it looks after it is "inflated" to form a sphere. Notice that the resulting figure looks like a "tiling" of a spherical surface that uses polygon-like shapes as the tiles. (Of course, these shapes are not flat as they tile a curved surface.) For this reason, we call the result of "inflating" a polyhedron a *polygonization of a sphere*. This type of model gives us a nice way to compare polyhedra and their values of χ. We will revisit this idea in next Friday's activity.

Extra Problems to Think About

- Choose one face in the polygonization of a sphere shown above. Add two new vertices on two different edges of this face and join the new vertices with a new edge. How does this alteration affect the values of V, E, and F? What is the value of χ for this new polygonization?
- Choose one vertex in the polygonization of a sphere shown above. Remove this vertex and all of the edges that are directly connected to it. How does this alteration affect the values of V, E, and F? What is the value of χ for this new polygonization? (What is the polyhedron corresponding to this new polygonization? Think about "deflating the sphere".)

More Info: Check out the CEMC at Home webpage on Friday, May 29 for solutions to the problems in this activity and further discussion of polygonizations of surfaces and the Euler characteristic.

CEMC at Home

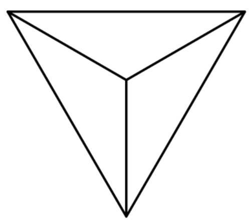
Grade 11/12 - Friday, May 22, 2020
Euler Characteristic - Part 1 - Solution

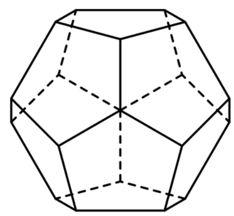
Problem 1

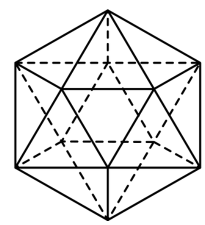
Verify directly that $\chi = 2$ for a tetrahedron, a dodecahedron, and an icosahedron.

Solution:

Here are diagrams outlining the shape of a tetrahedron, dodecahedron, and icosahedron. You can find many good images of these solids (three of the platonic solids) by searching online.







A tetrahedron has 4 triangular faces. We can count directly that $V = 4$, $F = 4$, and $E = 6$, and so $\chi = V - E + F = 4 - 6 + 4 = 2$.

A dodecahedron has 12 pentagonal faces and so we have $F = 12$. It is a bit harder to keep track of the vertices and edges as you count, and so instead we reason the numbers as follows:

There are 12 pentagons and each pentagon has 5 vertices. This means there are $12(5) = 60$ vertices among the faces. But 3 pentagons meet at every vertex and so the number 60 is triple counting the vertices. Therefore, $V = \frac{12(5)}{3} = 20$.

There are 12 pentagons and each pentagon has 5 sides. This means there are $12(5) = 60$ sides among the faces. But 2 pentagons meet at every edge of the polyhedron and so the number 60 is double counting the edges of the polyhedron. Therefore, $E = \frac{12(5)}{2} = 30$.

Putting it all together, we get $\chi = V - E + F = 20 - 30 + 12 = 2$, as desired.

Similar reasoning works for the icosahedron which has 20 triangular faces. We have $F = 20$, $V = \frac{20(3)}{5} = 12$ (since 5 faces meet at each vertex), and $E = \frac{20(3)}{2} = 30$ (since 2 faces meet at each edge). Therefore, $\chi = V - E + F = 12 - 30 + 20 = 2$.

One thing you may want to take away from this is that even though we had (2-dimensional) images of these tetrahedra in front of us, it was still easier to use combinatorics to determine E and V for the dodecahedron and icosahedron, rather than directly counting!

Problem 2

A particular polyhedron has 26 faces and has twice as many edges as vertices. How many edges must the polyhedron have?

Solution:

We are told that $F = 26$ and $E = 2V$. Using the fact that $\chi = 2$, we have that

$$2 = V - E + F = V - 2V + 26$$

which can be rearranged to give $V = 24$.

Problem 3

An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

Solution:

The given polyhedron has 37 faces, so $F = 37$.
The 15 square faces have $4(15) = 60$ sides in total, the 7 pentagonal faces have $5(7) = 35$ sides in total, and the 15 triangular faces have $3(15) = 45$ sides in total. Thus, the faces have a total of 140 sides.
Since these sides are paired up to form the edges of the polyhedron, we have $E = \frac{1}{2}(140) = 70$.
Since $V - E + F = 2$, we have $V = E - F + 2 = 70 - 37 + 2 = 35$, so the polyhedron has 35 vertices.

We have not included an image of this polyhedron here, but you can find an image on the internet if you are interested in seeing what it looks like.

Problem 4

A polyhedron is formed with exactly P pentagons, exactly H hexagons, and no other polygons as its faces and has the property that three polygonal faces meet at each vertex.

1. Explain why it is true that $E = \dfrac{5P + 6H}{2}$.

   Solution: There are P pentagons and H hexagons. Each pentagon has 5 sides and each hexagon has 6 sides. However, each edge is "shared" between two polygons. Therefore the total number of edges is $E = \dfrac{5P + 6H}{2}$.

2. Explain why it is true that $V = \dfrac{5P + 6H}{3}$.

   Solution: There are P pentagons and H hexagons. Each pentagon has 5 vertices and each hexagon has 6 vertices. However, each vertex is "shared" between three polygons (the statement of the question tells us this). Therefore the total number of vertices is $V = \dfrac{5P + 6H}{3}$.

3. Using the fact that $\chi = 2$ for this polyhedron, show that it must be the case that $P = 12$.

   Solution: We already have an expression for V and E. We also want an expression for the number of faces. Since P and H are, respectively, the number of pentagons and the number of hexagons, we have that $F = P + H$. Let's now calculate the Euler Characteristic of this polyhedron.

$$\begin{aligned} V - E + F &= \left(\frac{5P + 6H}{3}\right) - \left(\frac{5P + 6H}{2}\right) + (P + H) \\ &= \frac{(10P + 12H) - (15P + 18H) + (6P + 6H)}{6} \\ &= \frac{(10P - 15P + 6P) + (12H - 18H + 6H)}{6} \\ &= \frac{P}{6} \end{aligned}$$

   Since $\chi = V - E + F = 2$ we must have $\dfrac{P}{6} = 2$ and hence $P = 12$.

CEMC at Home

Grade 11/12 - Monday, May 25, 2020
Contest Day 4

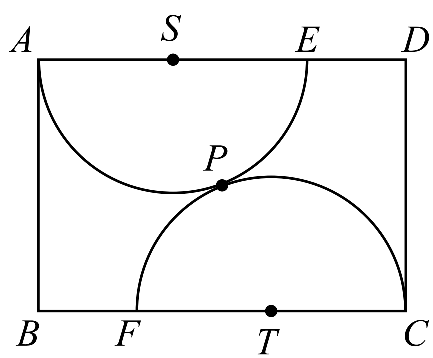
Today's resource features two questions from the recently released 2020 CEMC Mathematics Contests.

2020 Euclid Contest, #2(c)

Suppose that n is a positive integer and that the value of $\dfrac{n^2+n+15}{n}$ is an integer. Determine all possible values of n.

2020 Euclid Contest, #6(a)

Rectangle $ABCD$ has $AB = 4$ and $BC = 6$. The semi-circles with diameters AE and FC each have radius r, have centres S and T, and touch at a single point P, as shown. What is the value of r?



More Info:

Check out the CEMC at Home webpage on Monday, June 1 for solutions to the Contest Day 4 problems.

CEMC at Home

Grade 11/12 - Monday, May 25, 2020
Contest Day 4 - Solution

Solutions to the two contest problems are provided below, including a video for the first problem.

2020 Euclid Contest, #2(c)

Suppose that n is a positive integer and that the value of $\dfrac{n^2+n+15}{n}$ is an integer. Determine all possible values of n.

Solution:

First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n+1+\dfrac{15}{n}$.

This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n+1+\dfrac{15}{n}$ is an integer.

Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.

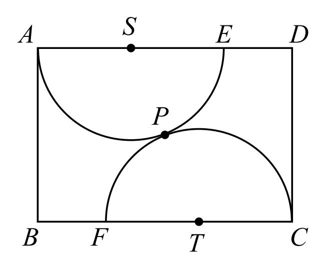
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.

Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

Video
Visit the following link for a discussion of a solution to the first contest problem:
https://youtu.be/MDV_HBu3-v4.

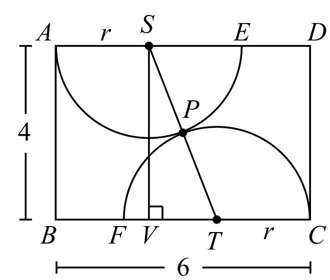
2020 Euclid Contest, #6(a)

Rectangle $ABCD$ has $AB = 4$ and $BC = 6$. The semi-circles with diameters AE and FC each have radius r, have centres S and T, and touch at a single point P, as shown. What is the value of r?



Solution:

Draw a perpendicular from S to V on BC. Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle. Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.



Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.

Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$. Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$. By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

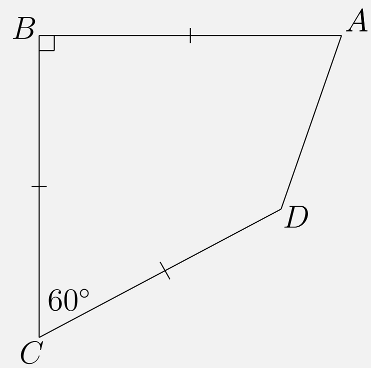
CEMC at Home

Grade 11/12 - Tuesday, May 26, 2020
Geometric Constructions

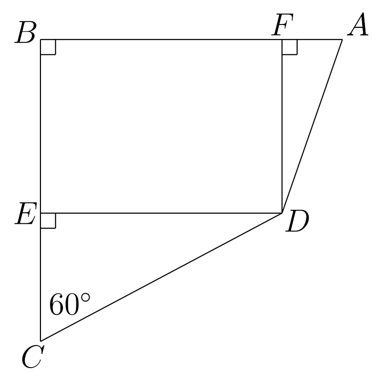
Sometimes when solving problems involving geometry, you can find a solution more easily by drawing something (eg. a point, a line, or an arc) that was not given in the original diagram. This is known as a *construction*. For example, consider the following problem.

Example

In the diagram, $ABCD$ is a quadrilateral with $AB = BC = CD = 6$, $\angle ABC = 90^\circ$, and $\angle BCD = 60^\circ$. Determine the length of AD.



One way to solve this problem is to start by dropping a perpendicular from D to E on BC and a perpendicular from D to F on BA. This gives us the diagram shown.



We start with $\triangle EDC$. We have that $ED = 6\sin 60^\circ = 3\sqrt{3}$ and $EC = 6\cos 60^\circ = 3$.

Consider rectangle $BFDE$. We have that $BE = BC - EC = 3$. Since $BFDE$ is a rectangle, we have that $BF = ED = 3\sqrt{3}$ and $FD = BE = 3$. Also, $FA = BA - BF = 6 - 3\sqrt{3}$.

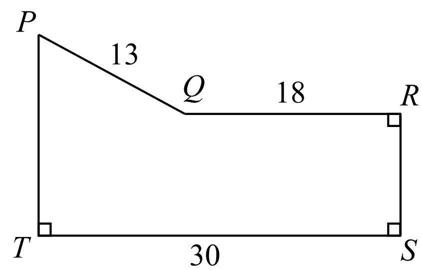
Finally, consider $\triangle FAD$. By the Pythagorean Theorem, we have that $AD^2 = FA^2 + FD^2 = (6 - 3\sqrt{3})^2 + 3^2 = 36(2 - \sqrt{3})$. Since $AD > 0$ we have $AD = 6\sqrt{2 - \sqrt{3}}$.

Notice how drawing in ED and FD made the problem more manageable by breaking it into smaller parts that were easier to work with.

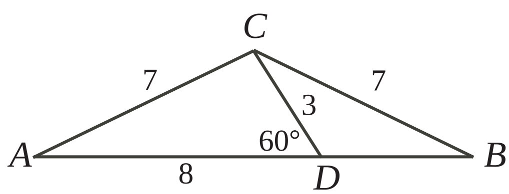
Each of the problems on the following page can be solved using a construction.

Problems

1. In the diagram, pentagon $PQRST$ has $PQ = 13$, $QR = 18$, $ST = 30$, and a perimeter of 82. Also, $\angle QRS = \angle RST = \angle STP = 90°$. Determine the area of pentagon $PQRST$.

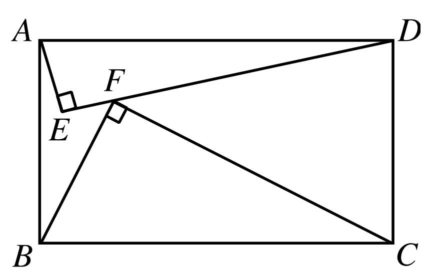


2. In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60°$, $AD = 8$, and $CD = 3$. Determine the length of BD.



Note: This problem can be solved without a construction (e.g. using the cosine law) but the solution becomes much simpler if you can find the right line to draw in the diagram!

3. In the diagram, right-angled triangles $\triangle AED$ and $\triangle BFC$ are constructed inside rectangle $ABCD$ so that F lies on DE. If $AE = 21$, $ED = 72$, and $BF = 45$, what is the length of AB?



More Info:

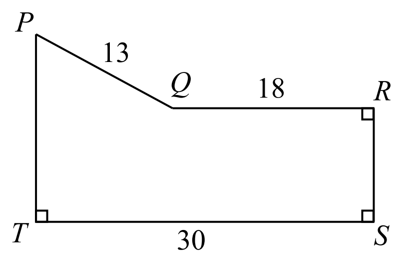
Check out the CEMC at Home webpage on Tuesday, June 2 for a solution to Geometric Constructions.

For more geometry problems check out this geometry unit in the CEMC Courseware.

CEMC at Home

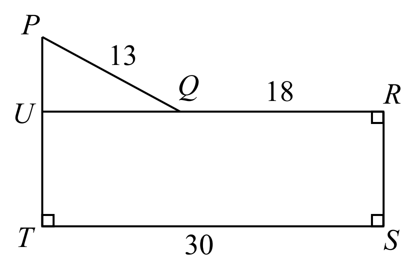
Grade 11/12 - Tuesday May 26, 2020
Geometric Constructions - Solution

1. In the diagram, pentagon $PQRST$ has $PQ = 13$, $QR = 18$, $ST = 30$, and a perimeter of 82. Also, $\angle QRS = \angle RST = \angle STP = 90^\circ$. Determine the area of pentagon $PQRST$.



Solution:

We extend RQ to the left until it meets PT at point U, as shown. Because quadrilateral $URST$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $UT = RS$ and $UR = TS = 30$.
Since $UR = 30$, then $UQ = UR - QR = 30 - 18 = 12$.
Now $\triangle PQU$ is right-angled at U.
By the Pythagorean Theorem, since $PU > 0$, we have

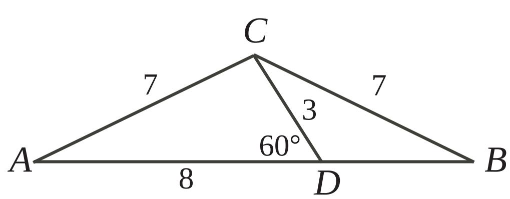


$$PU = \sqrt{PQ^2 - UQ^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$$

Since the perimeter of $PQRST$ is 82, then $13 + 18 + RS + 30 + (UT + 5) = 82$.
Since $RS = UT$, then $2 \times RS = 82 - 13 - 18 - 30 - 5 = 16$ and so $RS = 8$.
Finally, we can calculate the area of $PQRST$ by splitting it into $\triangle PQU$ and rectangle $URST$.
The area of $\triangle PQU$ is $\frac{1}{2} \times UQ \times PU = \frac{1}{2} \times 12 \times 5 = 30$.

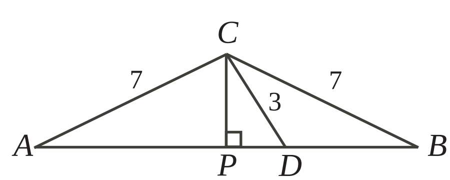
The area of rectangle $URST$ is $RS \times TS = 8 \times 30 = 240$.
Therefore, the area of pentagon $PQRST$ is $30 + 240 = 270$.

2. In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60^\circ$, $AD = 8$, and $CD = 3$. Determine the length of BD.

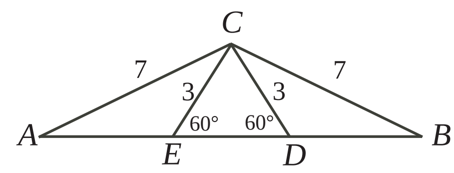


Solution 1:

Drop a perpendicular from C to P on AD. Since $\triangle ACB$ is isosceles, then $AP = PB$.
Since $\triangle CDP$ is a 30°-60°-90° triangle, then $PD = \frac{1}{2}(CD) = \frac{3}{2}$.
Thus, $AP = AD - PD = 8 - \frac{3}{2} = \frac{13}{2}$.
This tells us that $DB = PB - PD = AP - PD = \frac{13}{2} - \frac{3}{2} = 5$.

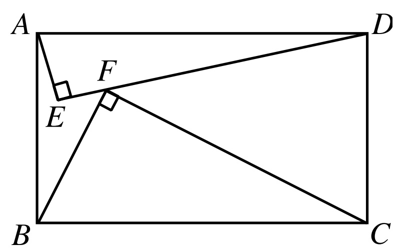


Solution 2:



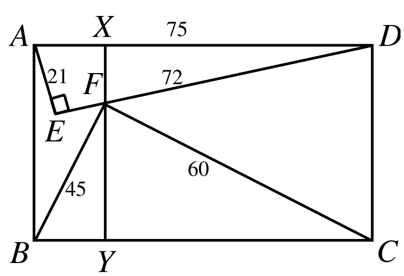
Since $\triangle ACB$ is symmetric about the vertical line through C, we can reflect CD in this vertical line, finding point E on AD with $CE = 3$ and $\angle CED = 60^\circ$.
Then $\triangle CDE$ has two 60° angles, so must have a third, and so is equilateral.
Therefore, $ED = CD = CE = 3$ and so $DB = AE = AD - ED = 8 - 3 = 5$.

3. In the diagram, right-angled triangles $\triangle AED$ and $\triangle BFC$ are constructed inside rectangle $ABCD$ so that F lies on DE. If $AE = 21$, $ED = 72$, and $BF = 45$, what is the length of AB?



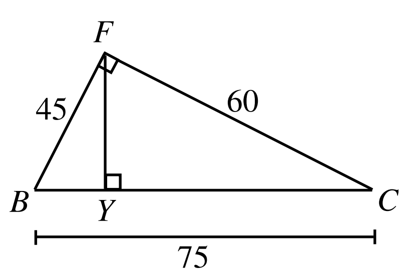
Solution:

By the Pythagorean Theorem in $\triangle AED$, $AD^2 = AE^2 + ED^2 = 21^2 + 72^2 = 5625$, so $AD = 75$.
Since $ABCD$ is a rectangle, $BC = AD = 75$. Also, by the Pythagorean Theorem in $\triangle BFC$, $FC^2 = BC^2 - BF^2 = 75^2 - 45^2 = 3600$, so $FC = 60$.
Draw a line through F parallel to AB, meeting AD at X and BC at Y.
To determine the length of AB, we can find the lengths of FY and FX.



<u>Step 1: Calculate the length of FY</u>

One way to do this is to calculate the area of $\triangle BFC$ in two different ways.
We know that $\triangle BFC$ is right-angled at F, so its area is equal to $\frac{1}{2}(BF)(FC)$ or $\frac{1}{2}(45)(60) = 1350$.
Also, we can think of FY as the height of $\triangle BFC$, so its area is equal to $\frac{1}{2}(FY)(BC)$ or $\frac{1}{2}(FY)(75)$.



Therefore, $\frac{1}{2}(FY)(75) = 1350$, so $FY = 36$.

Step 2: Calculate the length of FX

Since $FY = 36$, then by the Pythagorean Theorem,

$$BY^2 = BF^2 - FY^2 = 45^2 - 36^2 = 729$$

so $BY = 27$.
Thus, $YC = BC - BY = 48$.
Since $\triangle AED$ and $\triangle FXD$ are right-angled at E and X respectively and share a common angle D, then they are similar.
Since $YC = 48$, then $XD = 48$.
Since $\triangle AED$ and $\triangle FXD$ are similar, then $\dfrac{FX}{XD} = \dfrac{AE}{ED}$ or $\dfrac{FX}{48} = \dfrac{21}{72}$ so $FX = 14$.

Therefore, $AB = XY = FX + FY = 36 + 14 = 50$.

CEMC at Home

Grade 11/12 - Wednesday, May 27, 2020
Triangle of Integers

Here we revisit the idea of using computer science to solve previous CEMC math contest problems.

First, read Question 21 from the 2011 Fermat Contest and do your best to determine the correct answer. Read the solution to this question, and then answer the following related questions.

Problem Set 1

Suppose integers are arranged in a triangle as described in Question 21 of the 2011 Fermat Contest. Further, we will number the rows from top to bottom starting from 1 at the top.

1. What is the number of the row that contains the number 2020?
2. What is the sum of the numbers in the row immediately below the row that contains the number 500?

We will now look at a Python computer program that has been built to help you test your solutions to the questions above, and also to solve further related questions.

Here are instructions for using the program:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open the free CS Circles console in another tab.
3. Copy the code and paste it into the console of the interpreter.
4. Hit *Run program*.
5. You should see that the program outputs the correct answer to Question 21 of the 2011 Fermat Contest.

Try to understand how the program computes and displays the correct answer to the question. It is okay if you are new to computer programming or the language Python! The code is also included below for convenience and the notes below outline some of the details.

Python program

```
num = 400

# set n to be the number of the row containing num
n = 1
while n*(n+1) <= 2*num:
    n = n + 1

# compute the last numbers in rows n-1 and n
a = ((n-1)*n) // 2
b = (n*(n+1)) // 2

# add up the numbers in row n and display this sum
total = 0
for i in range(a+1,b+1):
    total = total + i
print(total)
```

When a Python program is run, anything beginning with a `#` character through to the end of the line is a *comment* which is ignored when the program is run.

When the `while` *loop* is reached, Python tests if the *condition* `n*(n+1) <= 2*num` is true. If it is true, the indented line `n = n + 1` is executed. This process is repeated until a test determines that the condition is false in which case the program continues after the indented line.

The line `n = n + 1` increases the value of the variable `n` by 1. For example, if the value of `n` is 4 when this line is reached, then after this line is executed, the value of `n` becomes $4 + 1 = 5$.

The two characters `//` form an *operator* used to divide two integers.

Do you see where the formulas for a and b above come from? See the solution to the contest problem.

Revisiting Problem Set 1

Let's revisit the questions on the previous page, and try to answer them using elements of our Python program. Suppose integers are arranged in a triangle as described in Question 21 of the 2011 Fermat Contest. Further, we will number the rows from top to bottom starting from 1 at the top.

1. What is the number of the row that contains the number 2020?

   To answer this question using the Python program, change the line `num = 400` to `num = 2020`, remove the lines after the while loop and add the line `print(n)` at the end. Your code should now look like the code in Program 1 below. Run the program. Does the answer given by this program agree with the answer you had calculated? Can you see why this program produces the desired output?

2. What is the sum of the numbers in the row immediately below the row that contains the number 500?

   Use Program 2 below to answer this question. Can you see what changes have been made to the original Python program to obtain Program 2? Run the program. Does the answer given by this program agree with the answer you had calculated? Can you see why this program produces the desired output?

Program 1

```
num = 2020
n = 1
while n*(n+1) <= 2*num:
    n = n + 1
print(n)
```

Program 2

```
num = 500
n = 1
while n*(n+1) <= 2*num:
    n = n + 1
n = n + 1
a = ((n-1)*n) // 2
b = (n*(n+1)) // 2
total = 0
for i in range(a+1,b+1):
    total = total + i
print(total)
```

Problem Set 2

Now, it's your turn! A solution to each of the following problems can be found that uses elements of the given Python program. Modify the given program, or use parts of it, to answer the questions below about the integers arranged in a triangle as described in the Fermat Contest question. Of course, you should feel free to use any other features that you know or learn about. The correct answers are provided so you can test your programs.

Of course, you can try to solve these problems by hand as well!

1. The sum of the numbers of a row is 34481. What is the number of this row?
   Answer: 41

2. For how many rows is the sum of the numbers in the row between 50000 and 90000?
   Answer: 10

More Info:

Check out the CEMC at Home webpage on Wednesday, June 3 for a solution to Triangle of Integers.

CEMC at Home

Grade 11/12 - Wednesday, May 27, 2020
Triangle of Integers - Solution

Recall that the questions in this resource are based around Question 21 from the 2011 Fermat Contest. Please remind yourself of the question and the solution.

Problem Set 1

1. What is the number of the row that contains the number 2020? (*Answer: 64*)

   Solution: After n rows, the total number of integers appearing in the pattern is

   $$1 + 2 + \cdots + +(n-1) + n = \tfrac{1}{2}n(n+1)$$

   In other words, the largest number in the nth row is $\frac{1}{2}n(n+1)$. To determine the row the number 2020 is in, we want to determine the smallest value of n such that $\frac{1}{2}n(n+1) \geq 2020$. If $n = 63$ then $\frac{1}{2}n(n+1) = 2016$ and if $n = 64$ then $\frac{1}{2}n(n+1) = 2080$. Therefore, 2020 appears in the 64th row.

2. What is the sum of the numbers in the row immediately below the row that contains the number 500? (*Answer: 17 985*)

   Solution: When $n = 31$ we have $\frac{1}{2}n(n+1) = 496$ and when $n = 32$ we have $\frac{1}{2}n(n+1) = 528$. This means that the number 500 must be in the 32nd row, and so we want the sum of the numbers in 33rd row. By the computation above, the largest number in the 32nd row is 528, and so the smallest number in the 33rd row is 529. Also, when $n = 33$ we have $\frac{1}{2}n(n+1) = 561$, which means that the largest number in the 33rd row is 561. Therefore, the 33rd row consists of the integers from 529 to 561. We calculate the sum of these numbers as follows:

   $$\begin{aligned} & 529 + 530 + \cdots + 560 + 561 \\ &= (1 + 2 + \cdots + 528 + 529 + 530 + \cdots + 560 + 561) - (1 + 2 + \cdots + 528) \\ &= \tfrac{1}{2}(561)(562) - \tfrac{1}{2}(528)(529) \\ &= 157\,641 - 139\,656 \\ &= 17\,985 \end{aligned}$$

Problem Set 2

On the next page you will find programs that compute and display the correct answers for the remaining two problems from the resource. Note that there are many different ways of using Python (or any other programming language) in each case. We have selected programs that are very similar to the program shown on the right that answers Question 21 from the 2011 Fermat Contest and was provided with the resource.

Original Python Program

```
num = 400
n = 1
while n*(n+1) <= 2*num:
    n = n + 1
a = ((n-1)*n) // 2
b = (n*(n+1)) // 2
total = 0
for i in range(a+1,b+1):
    total = total + i
print(total)
```

1. The sum of the numbers of a row is 34481. What is the number of this row?

```
n = 1
target = 34481
total = 0
while total < target:
    a = ((n-1)*n) // 2
    b = (n*(n+1)) // 2
    total = 0
    for i in range(a+1,b+1):
        total = total + i
    n = n + 1
print(n-1)
```

Answer: 41

2. For how many rows is the sum of the numbers in the row between 50000 and 90000?

```
n = 1
low = 50000
high = 90000
total = 0
while total < low:
    a = ((n-1)*n) // 2
    b = (n*(n+1)) // 2
    total = 0
    for i in range(a+1,b+1):
        total = total + i
    n = n + 1
numrows = 1
while total < high:
    a = ((n-1)*n) // 2
    b = (n*(n+1)) // 2
    total = 0
    for i in range(a+1,b+1):
        total = total + i
    n = n + 1
    numrows = numrows + 1
print(numrows-1)
```

Answer: 10

CEMC at Home

Grade 11/12 - Thursday, May 28, 2020
So Many Unknowns

In a sequence of twelve numbers, each number after the first three is equal to the sum of the previous three numbers.

The 3^{rd} number in the sequence is 6, the 6^{th} number in the sequence is 11, and the 11^{th} number in the sequence is 14.

Determine all twelve numbers in the sequence.

?	?	6	?	?	11	?	?	?	?	14	?

More Info:

Check out the CEMC at Home webpage on Friday, May 29 for a solution to So Many Unknowns.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Thursday, May 28, 2020
So Many Unknowns - Solution

Problem:

In a sequence of twelve numbers, each number after the first three is equal to the sum of the previous three numbers.

The 3$^{\text{rd}}$ number in the sequence is 6, the 6$^{\text{th}}$ number in the sequence is 11, and the 11$^{\text{th}}$ number in the sequence is 14.

Determine all twelve numbers in the sequence.

?	?	6	?	?	11	?	?	?	?	14	?

Solution:

Let a_1 be the first number in the sequence, a_2 be the second, a_4 be the fourth, and so on, until a_{12} which is the 12$^{\text{th}}$ number in the sequence. The twelve boxes are labelled in the following diagram.

a_1	a_2	6	a_4	a_5	11	a_7	a_8	a_9	a_{10}	14	a_{12}

Each number after the third number is equal to the sum of the previous three numbers. Therefore, looking at the 6$^{\text{th}}$ term, we have $11 = 6 + a_4 + a_5$, or $a_4 + a_5 = 5$.

Looking at the 7$^{\text{th}}$ term, $a_7 = a_4 + a_5 + 11 = 5 + 11 = 16$, since $a_4 + a_5 = 5$.

a_1	a_2	6	a_4	a_5	11	16	a_8	a_9	a_{10}	14	a_{12}

Looking at the 9$^{\text{th}}$ term, $a_9 = 11 + 16 + a_8 = 27 + a_8$.
Looking at the 10$^{\text{th}}$ term, $a_{10} = 16 + a_8 + a_9 = 16 + (a_8) + (a_8 + 27) = 2a_8 + 43$.
Looking at the 11$^{\text{th}}$ term, $a_{11} = a_8 + a_9 + a_{10} = (a_8) + (a_8 + 27) + (2a_8 + 43) = 4a_8 + 70$.
We are given that the 11$^{\text{th}}$ term is $a_{11} = 14$.
Therefore, $4a_8 + 70 = 14$, or $4a_8 = -56$, or $a_8 = -14$.
Therefore, $a_9 = a_8 + 27 = -14 + 27 = 13$ and $a_{10} = 2a_8 + 43 = 2(-14) + 43 = 15$.
Also, $a_{12} = a_9 + a_{10} + a_{11} = 13 + 15 + 14 = 42$.

So far, we know the sequence is

a_1	a_2	6	a_4	a_5	11	16	-14	13	15	14	42

Working backwards, $-14 = a_5 + 11 + 16$, so $a_5 = -41$.

From earlier, $a_4 + a_5 = 5$ and $a_5 = -41$, so $a_4 = 46$.

Continuing backwards, $a_5 = a_2 + 6 + a_4$, so $-41 = a_2 + 6 + 46$, or $a_2 = -93$.

And finally, $a_4 = a_1 + a_2 + 6$, so $46 = a_1 + (-93) + 6$, or $a_1 = 133$.

Therefore, the sequence of twelve numbers is

133	-93	6	46	-41	11	16	-14	13	15	14	42

We can indeed check that in this sequence each number after the first three numbers is equal to the sum of the previous three numbers.

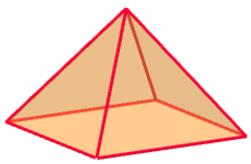
CEMC at Home

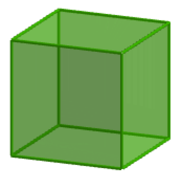
Grade 11/12 - Friday, May 29, 2020
Euler Characteristic - Part 2

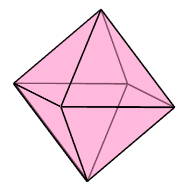
Last Friday we introduced the Euler Characteristic of a polyhedron in Part 1. Given a polyhedron, we let V, E, and F denote the number of vertices, edges, and faces, respectively, of the polyhedron.

> The **Euler characteristic** of a polyhedron, denoted χ, is defined to be the value of $V - E + F$. It turns out that $\chi = 2$ for every convex polyhedron.

The following are examples of *convex polyhedra*. We verified that each of these polyhedra have $\chi = 2$ directly.







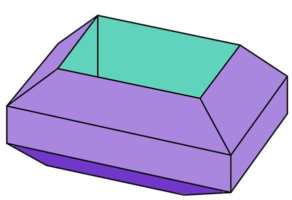
What makes a polyhedron convex?

> A polyhedron is said to be **convex** if for any two points we choose on the polyhedron, the line segment joining these two points is always on or inside polyhedron.

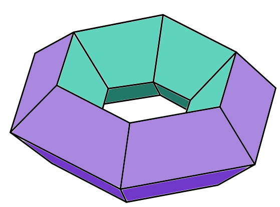
Looking at the images of the pyramid, the cube, and the octahedron, convince yourself that they are indeed convex polyhedra.

Examples of Non-Convex Polyhedra

The following two figures show three-dimensional objects with polygons for their faces. These objects are polyhedra by definition; however, they are not convex polyhedra. One way to form a non-convex polyhedron is to include a "hole". Notice that if you pick two points that are on opposite sides of the hole, then at least some portion of the line segment joining the two points must lie outside of the polyhedron.



Polyhedron 1



Polyhedron 2

Assume that the "bottom" of each of the polyhedra is identical to the "top", and that the "back" is identical to the "front".

Example

Note that Polyhedron 1 has 4 "inner faces" and 12 "outer faces". We can verify that Polyhedron 1 has $V = 16$, $E = 32$, and $F = 16$. Therefore, we can calculate that $\chi = V - E + F = 16 - 32 + 16 = 0$.

Notice that $\chi = 0$ for this polyhedron, rather than $\chi = 2$ as we observed for the convex polyhedra.

> **Problem 1**
> Determine the value of χ for Polyhedron 2.

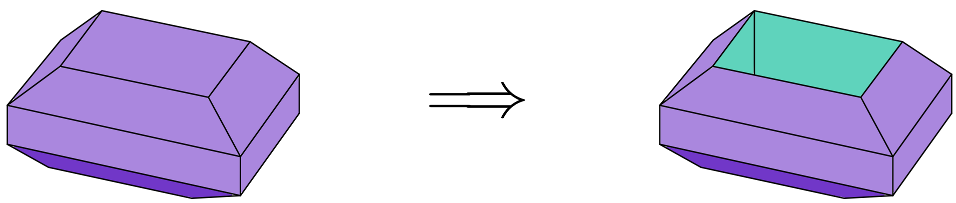
Solve this problem before moving on to the next page.

If you did the calculation correctly in Problem 1, you should have gotten an answer of $\chi = 0$, again! Is there something about these two polyhedra that make them have the same Euler characteristic, similar to how all convex polyhedra have $\chi = 2$?

It turns out that a polyhedron with "one hole" in its interior always has $\chi = 0$. The number of holes a polyhedron has will affect its value of χ. (Note that if a polyhedron has a hole, then it cannot be convex, but there are non-convex polyhedra without holes. Try to visualize one and think about its value of χ.)

Investigation: *What happens when we take a polyhedron and "cut a hole"?*

Consider the convex polyhedron shown below on the left. We obtain the non-convex polyhedron shown below on the right by removing a vertical chunk of the convex polyhedron. (Notice that this is Polyhedron 1 from the first page.) Doing so may change the number of vertices, edges, and/or faces. Can we use this construction to help us understand why the Euler characteristic changes from $\chi_{old} = 2$ to $\chi_{new} = 0$ when a hole is introduced?



Problem 2

Let $V_{old}, E_{old}, F_{old}$, and χ_{old} denote the vertices, edges, faces, and Euler characteristic for the polyhedron on the left above, and $V_{new}, E_{new}, F_{new}$, and χ_{new} denote these values for the polyhedron on the right above. Use the images above to determine the following values:

- $V_{new} - V_{old}$
- $E_{new} - E_{old}$
- $F_{new} - F_{old}$

Use this information to determine the relationship between χ_{new} and χ_{old}.

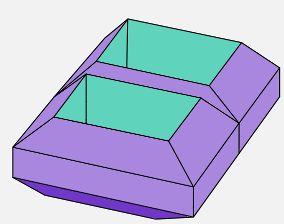
To get started, explain why $V_{new} - V_{old} = 0$. Next, do we "gain" or "lose" edges when we cut the hole?

Your work in Problem 2 should lead you to the fact that $\chi_{new} = \chi_{old} - 2$. This suggests that "adding a hole" may reduce the value of χ by 2. This would mean that if we started with a convex polyhedron, which has $\chi_{old} = 2$, and cut a hole in the interior, then we will be left with a non-convex polyhedron with $\chi_{new} = \chi_{old} - 2 = 2 - 2 = 0$. Of course, we already observed that $\chi = 0$ for Polyhedron 1 in the example on the first page.

What about polyhedra with more than one hole?

Problem 3

Consider the polyhedron shown to the right that has two holes. What is the value of χ for this polyhedron?



Assume that the "bottom" of this polyhedron is identical to the "top", and that the "back" is identical to the "front". If you imagine slicing this polyhedron down the centre, then you would essentially get two copies of Polyhedron 1 from the first page.

It turns out that if you know the number of holes that a polyhedron has, then you can easily calculate its Euler characteristic. In particular, if two polyhedra have the same number of holes, then they will have the same value of χ, regardless of what they look like. Proving this fact is challenging, but the formula for calculating χ based on the number of holes is simple!

We will make an attempt to outline the reasoning behind the fact above in the solutions for this resource, but the math involved in a formal proof is beyond the scope of this activity. Assuming the fact is true, try the following:

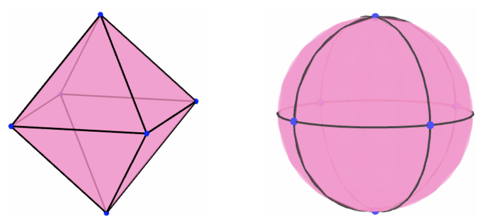
Extension: Suppose that a polyhedron has g holes, where g is a non-negative integer. Guess a formula for χ in terms of g. Your formula should give an answer of 2 when $g = 0$ and 0 when $g = 1$, and the value you calculated in Problem 3 when $g = 2$.

More Info: Check out the CEMC at Home webpage on Friday, June 5 for a solution to Part 2.

Further Discussion

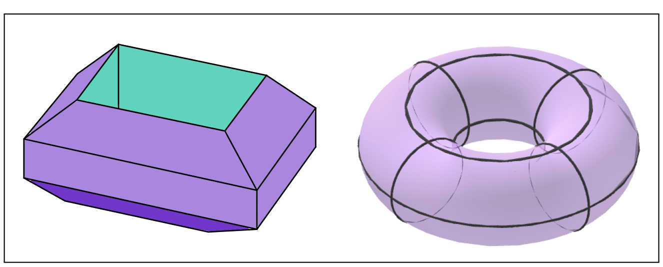
The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a convex polyhedron is always 2, and understand the formula for the Euler characteristic of a non-convex polyhedron with g holes.

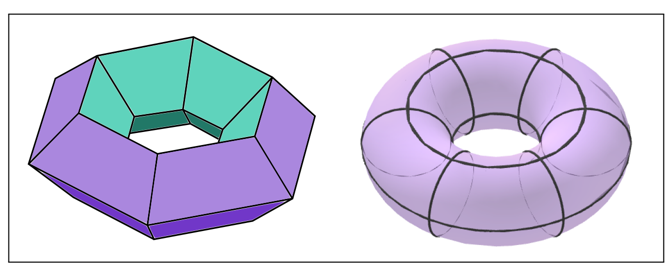
For example, if we "inflate" the octahedron, then we obtain a sphere as shown below. We visualize the vertices, edges, and faces of the octahedron on the spherical surface.



We call this result a *polygonization of a sphere*. Every convex polyhedron, when inflated, will produce such a polygonization. What happens when we "inflate" polyhedra with a hole?

We can imagine "inflating" Polyhedron 1 and Polyhedron 2 from the first page in a similar way to how we "inflated" the octahedron. The result is again a polygonization of a surface, but not of a sphere; these "inflate" to become a *polygonization of a torus*, which is a surface shaped like a donut as shown.





If you "inflated" the polyhedron with two holes from Problem 3, then you would get a tiling of a different surface. What would it look like? Can you describe it? If you have been to a water park that has inner tubes for more than one person, then you may already be familiar with the surface!

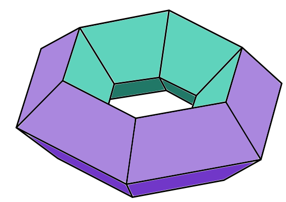
It turns out that every polyhedron that corresponds to a polygonization of a sphere (when "inflated") has Euler characteristic $\chi = 2$, and every polyhedron that corresponds to a polygonization of a torus (when "inflated") has Euler characteristic $\chi = 0$. There is a similar situation for g holes with $g \geq 2$.

CEMC at Home

Grade 11/12 - Friday, May 29, 2020
Euler Characteristic - Part 2 - Solution

Problem 1

Determine the value of χ for Polyhedron 2.



Polyhedron 2

Solution:

The polyhedron has a total of 24 vertices: 6 vertices at the "top level", 12 vertices at the "middle level" (6 on the outer surface and 6 on the inner surface) and 6 vertices at the "bottom level".

The polyhedron has a total of 48 edges: 6 edges running horizontally at the "top level", 12 edges running horizontally at the "middle level" (6 on the outer surface and 6 on the inner surface), 6 edges running horizontally at the "bottom level", and 24 slanted edges (12 on the outer surface and 12 on the inner surface).

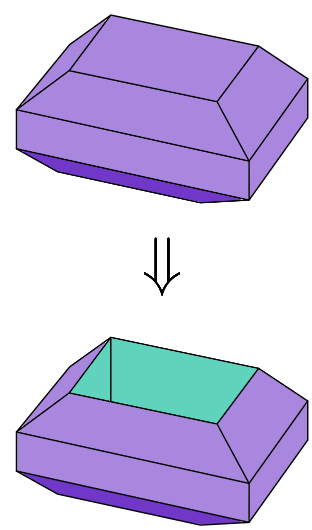
The polyhedron has a total of 24 faces: 6 top inner faces, 6 top outer faces, 6 bottom inner faces, and 6 bottom outer faces.

Therefore, $\chi = V - E + F = 24 - 48 + 24 = 0$.

Problem 2

Let $V_{old}, E_{old}, F_{old}$, and χ_{old} denote the vertices, edges, faces, and Euler characteristic for the convex polyhedron (top image), and $V_{new}, E_{new}, F_{new}$, and χ_{new} denote these values for the polyhedron after the vertical "hole" is introduced (bottom image). Use the images to determine the following values:

- $V_{new} - V_{old}$
- $E_{new} - E_{old}$
- $F_{new} - F_{old}$



Polyhedron 1

Use this information to determine the relationship between χ_{new} and χ_{old}.

Solution:

We could count all of the vertices, edges, and faces of each of the polyhedra, but instead we focus on how introducing the hole "changes" the values.

First, we observe that no vertices are gained or lost. Therefore, $V_{new} - V_{old} = 0$.

Second, we observe that 4 edges are gained: the 4 "vertical edges" of the inner surface of the object. Therefore, $E_{new} - E_{old} = 4$.

Finally, we observe that 2 faces are removed (the top and bottom rectangular faces) but 4 new faces are created (the 4 interior faces). This means a net gain of 2 faces and therefore, $F_{new} - F_{old} = 2$.

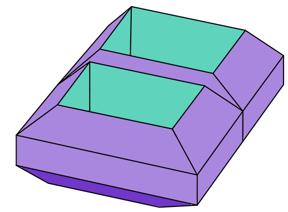
Using these four values, we determine the relationship between χ_{new} and χ_{old}:

$$\begin{aligned}
\chi_{new} - \chi_{old} &= (V_{new} - E_{new} + F_{new}) - (V_{old} - E_{old} + F_{old}) \\
&= (V_{new} - V_{old}) - (E_{new} - E_{old}) + (F_{new} - F_{old}) \\
&= 0 - 4 + 2 \\
&= -2
\end{aligned}$$

Therefore, introducing this hole results in $\chi_{new} = \chi_{old} - 2$.

Problem 3

Consider the polyhedron shown to the right that has two holes. What is the value of χ for this polyhedron?



Assume that the "bottom" of this polyhedron is identical to the "top", and that the "back" is identical to the "front". If you imagine slicing this polyhedron down the centre, then you would essentially get two copies of Polyhedron 1 from the first page.

Solution:

We could directly count up vertices, edges, and faces for this polyhedron, but we will instead use what we already know about Polyhedron 1.

Let V_{old}, E_{old}, and F_{old} denote the number of vertices, edges, and faces of Polyhedron 1. We can think of the polyhedron in this question as two copies of Polyhedron 1 "glued together" at rectangular faces on the side as shown in the image above.

Let V_{new}, E_{new}, and F_{new} denote the number of vertices, edges, and faces of the given polyhedron.

When we "glue" the two copies of Polyhedron 1 together, we lose two faces: the two rectangular faces on the sides where the polyhedra are "glued". Therefore, $F_{new} = 2F_{old} - 2$.

We also lose 4 edges. The two rectangular faces which are "glued" start with 8 edges in total, which is reduced to 4 once they are "glued" together. Therefore, $E_{new} = 2E_{old} - 4$.

Similarly, we have $V_{new} = 2V_{old} - 4$.

Note that we already know from earlier that $\chi_{old} = V_{old} - E_{old} + F_{old} = 0$. Therefore,

$$\begin{aligned}
\chi_{new} &= V_{new} - E_{new} + F_{new} \\
&= (2V_{old} - 4) - (2E_{old} - 4) + (2F_{old} - 2) \\
&= 2(V_{old} - E_{old} + F_{old}) - 4 + 4 - 2 \\
&= 2(0) - 4 + 4 - 2 \\
&= -2
\end{aligned}$$

Extension: Suppose that a polyhedron has g holes, where g is a non-negative integer. Guess a formula for χ in terms of g. Your formula should give an answer of 2 when $g = 0$ and 0 when $g = 1$, and the value you calculated in Problem 3 when $g = 2$.

Solution:

All polyhedra we have seen with 0 holes have $\chi = 2$, the two polyhedra we have seen with 1 hole have $\chi = 0$, and the polyhedron we just saw with 2 holes has $\chi = -2$. These findings suggest a possible pattern: every hole added reduces the Euler characteristic by 2. Note that our work in Problem 2 (where we showed $\chi_{new} = \chi_{old} - 2$) also provides evidence of this pattern.

If we guess that this pattern continues, then we would arrive at the following formula:

If a polyhedron has g holes, then $\chi = 2 - 2g$.

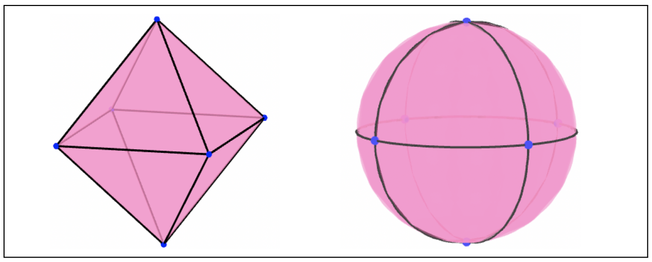
Fun fact: This formula is indeed true! Mathematicians often use the value of χ of a polyhedron to <u>*define*</u> *the number of holes the polyhedron has. That is, we define a polyhedron to have $\frac{2-\chi}{2}$ holes.*

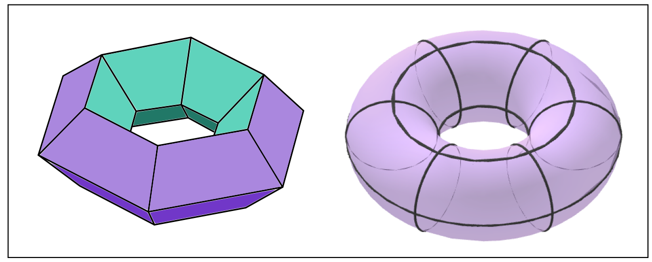
See the next page for an outline of an argument for why $\chi = 2$ for all polyhedra with 0 holes. (Since every convex polyhedron has 0 holes, in particular this argues that $\chi = 2$ for all convex polyhedra.)

Further Discussion

The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a polyhedron with 0 holes is always 2, and understand the formula for the Euler characteristic of a polyhedron with g holes.

Every polyhedron with 0 holes will result in a polygonization of a sphere when "inflated", and every polyhedron with 1 hole will result in a polygonization of a torus when "inflated".





What type of surface would you get if you "inflated" a polyhedron with g holes for some $g \geq 2$?

We can use the idea of a polygonization to relate the Euler characteristic of all polyehedra with the same number of holes. We outline this for the polyhedra with 0 holes below.

Altering a polygonization

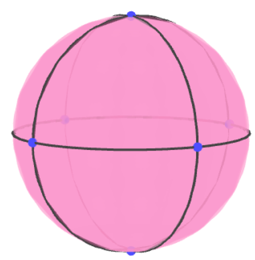
We can make a small alteration to a polygonization in many ways. One way to do so is to "divide one face into two faces". This is an example of a *refinement* of a polygonization.

Start with a polygonization of a sphere as shown to the right.

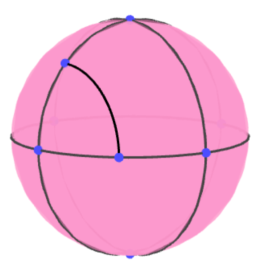
Recall that this polygonization arose from the octahedron.

To refine the polygonization, we can choose a "polygonal face", add 2 new vertices on 2 different edges of this face, and join the new vertices with a new edge.

An example of such a refinement is shown below on the right.



$$V_{old} = 6$$
$$E_{old} = 12$$
$$F_{old} = 8$$



$$V_{new} = 6 + 2$$
$$E_{new} = 12 + 3$$
$$F_{new} = 8 + 1$$

What are the new values of V, E, and F for the polyhedron corresponding to this new polygonization? We can determine these values by looking at the vertices, edges, and faces of the polygonal faces on the surface of the sphere.

We can verify that this polygonization corresponds to a polyhedron with 8 vertices, 15 edges, and 9 faces and so we have $\chi = V - E + F = 8 - 15 + 9 = 2$ as expected! However, the most important thing to note here is how the values of V, E, and F *changed*.

When we change the polygonization as we did above, it is not hard to see that we gain 2 vertices, and gain 1 face during the process. What happens to the number of edges? We actually gain 3 edges overall. Can you see why? We "drew" 1 new edge, but in adding the 2 vertices, we "split" 2 edges and so we gained another 2 edges over all.

We could have added these two vertices in different places on the edges of our chosen face above, or added the vertices on a different face altogether, but the result will always be the same. How the values of V, E, and F change is summarized below. Notice that these changes have no effect on χ.

$$\begin{aligned} V_{new} &= V_{old} + 2 \quad \text{(gain 2 vertices)} \\ E_{new} &= E_{old} + 3 \quad \text{(gain 3 edges)} \\ F_{new} &= F_{old} + 1 \quad \text{(gain 1 face)} \end{aligned}$$

$$\begin{aligned} \chi_{new} &= V_{new} - E_{new} + F_{new} \\ &= (V_{old} + 2) - (E_{old} + 3) + (F_{old} + 1) \\ &= V_{old} - E_{old} + F_{old} \\ &= \chi_{old} \end{aligned}$$

Exploration:

See what happens if instead of adding two vertices to the polygonization as we did above, you make one of the following alterations.

- Choose a polygonal face, add 1 new vertex in the interior of this face, and draw new edges from this vertex so that you get another polygonization.
- Choose a vertex, remove this vertex and all edges that are attached to it.

Determine how performing each of these alterations will change the values of V, E, and F. Confirm that in each case you have $\chi_{new} = \chi_{old}$.

All polyhedra we have seen with 0 holes have $\chi = 2$, and changing their polygonizations in the ways described above does not affect the value of χ. Using this idea, it can be shown that every polyhedron with 0 holes has $\chi = 2$. Explaining exactly how this is done is beyond the scope of this activity, but we encourage you to think about this and research this topic further on your own if you are interested. The rough idea is that given any two polyhedra that both “inflate” into polygonizations of a sphere, you can always get from one polygonization to the other by doing the operations described above, none of which change the value of χ. This means they must have the same Euler characteristic, and we know this must be $\chi = 2$.

The argument is similar for the polyhedra that correspond to polygonizations of the torus. If we already know χ on one of them is 0 (for example Polyhedron 1) then we can deduce that any polyhedron that is inflated onto a torus must have $\chi = 0$ as well. If you would like, you can complete the “altering a polygonization” activity again for the given polygonization of the torus, and convince yourself that making each of these alterations leaves the value of χ unchanged.

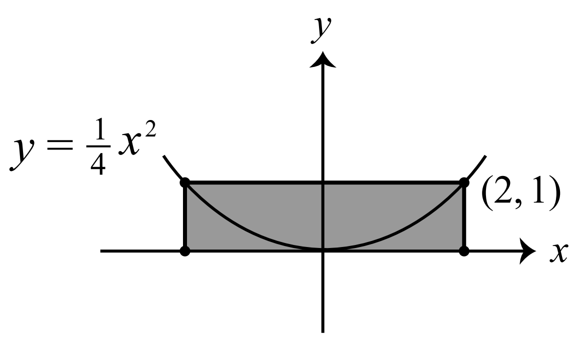
CEMC at Home

Grade 11/12 - Monday, June 1, 2020
Contest Day 5

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Hypatia Contest, #2

The parabola with equation $y = \frac{1}{4}x^2$ has its vertex at the origin and the y-axis as its axis of symmetry. For any point (p, q) on the parabola (not at the origin), we can form a *parabolic rectangle*. This rectangle will have one vertex at (p, q), a second vertex on the parabola, and the other two vertices on the x-axis. A parabolic rectangle with area 4 is shown.



(a) A parabolic rectangle has one vertex at $(6, 9)$. What are the coordinates of the other three vertices?

(b) What is the area of the parabolic rectangle having one vertex at $(-3, 0)$?

(c) Determine the areas of the two parabolic rectangles that have a side length of 36.

(d) Determine the area of the parabolic rectangle whose length and width are equal.

More Info:

Check out the CEMC at Home webpage on Monday, June 8 for a solution to the Contest Day 5 problem.

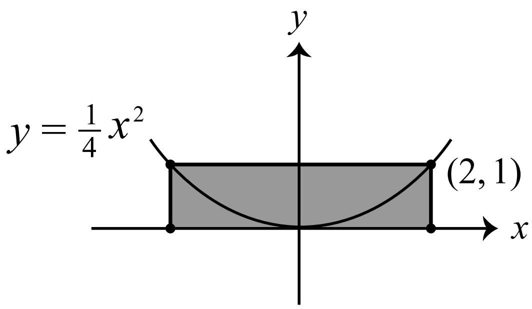
CEMC at Home

Grade 11/12 - Monday, June 1, 2020
Contest Day 5 - Solution

A solution to the contest problem is provided below.

2020 Hypatia Contest, #2

The parabola with equation $y = \frac{1}{4}x^2$ has its vertex at the origin and the y-axis as its axis of symmetry. For any point (p, q) on the parabola (not at the origin), we can form a *parabolic rectangle*. This rectangle will have one vertex at (p, q), a second vertex on the parabola, and the other two vertices on the x-axis. A parabolic rectangle with area 4 is shown.



(a) A parabolic rectangle has one vertex at $(6, 9)$. What are the coordinates of the other three vertices?

(b) What is the area of the parabolic rectangle having one vertex at $(-3, 0)$?

(c) Determine the areas of the two parabolic rectangles that have a side length of 36.

(d) Determine the area of the parabolic rectangle whose length and width are equal.

Solution:

(a) The parabola $y = \frac{1}{4}x^2$ and the parabolic rectangle are each symmetrical about the y-axis, and thus a second vertex of the rectangle lies on the parabola and has coordinates $(-6, 9)$.
A third vertex of the parabolic rectangle lies on the x-axis vertically below $(6, 9)$, and thus has coordinates $(6, 0)$.
Similarly, the fourth vertex also lies on the x-axis vertically below $(-6, 9)$, and thus has coordinates $(-6, 0)$.

(b) If one vertex of a parabolic rectangle is $(-3, 0)$, then a second vertex has coordinates $(3, 0)$, and so the rectangle has length 6.
The vertex that lies vertically above $(3, 0)$ has x-coordinate 3.
This vertex lies on the parabola $y = \frac{1}{4}x^2$ and thus has y-coordinate equal to $\frac{1}{4}(3)^2 = \frac{9}{4}$.
The width of the rectangle is equal to this y-coordinate $\frac{9}{4}$, and so the area of the parabolic rectangle having one vertex at $(-3, 0)$ is $6 \times \frac{9}{4} = \frac{54}{4} = \frac{27}{2}$.

(c) Let a vertex of the parabolic rectangle be the point $(p, 0)$, with $p > 0$.
A second vertex (also on the x-axis) is thus $(-p, 0)$, and so the rectangle has length $2p$.
The width of this rectangle is given by the y-coordinate of the point that lies on the parabola vertically above $(p, 0)$, and so the width is $\frac{1}{4}p^2$.
The area of a parabolic rectangle having length $2p$ and width $\frac{1}{4}p^2$ is $2p \times \frac{1}{4}p^2 = \frac{1}{2}p^3$.
If such a parabolic rectangle has length 36, then $2p = 36$, and so $p = 18$.
The area of this rectangle is $\frac{1}{2}(18)^3 = 2916$.
If such a parabolic rectangle has width 36, then $\frac{1}{4}p^2 = 36$ or $p^2 = 144$, and so $p = 12$ (since $p > 0$).
The area of this rectangle is $\frac{1}{2}(12)^3 = 864$.
The areas of the two parabolic rectangles that have side length 36 are 2916 and 864.

(d) Let a vertex of the parabolic rectangle be the point $(m, 0)$, with $m > 0$.
A second vertex (also on the x-axis) is thus $(-m, 0)$, and so the rectangle has length $2m$.
The width of this rectangle is given by the y-coordinate of the point that lies on the parabola vertically above $(m, 0)$, and so the width is $\frac{1}{4}m^2$.
The area of a parabolic rectangle having length $2m$ and width $\frac{1}{4}m^2$ is $2m \times \frac{1}{4}m^2 = \frac{1}{2}m^3$.
If the length and width of such a parabolic rectangle are equal, then

$$\begin{aligned}
\tfrac{1}{4}m^2 &= 2m \\
m^2 &= 8m \\
m^2 - 8m &= 0 \\
m(m-8) &= 0
\end{aligned}$$

Thus $m = 8$ (since $m > 0$), and so the area of the parabolic rectangle whose length and width are equal is $\frac{1}{2}(8)^3 = 256$.

CEMC at Home

Grade 11/12 - Tuesday, June 2, 2020

Famous Mathematicians

Throughout human history, many mathematicians have made significant contributions to the subject. These important historical figures often lead fascinating lives filled with interesting stories. Five of these mathematicians are listed below.

Euclid	Best known for his work in geometry, he was a mathematician from ancient Greece whose work the *Elements* may be the most influential writing in the history of mathematics.
Hypatia	From Alexandria, Egypt, she is considered by many to be the greatest mathematician of her time. She was also an astronomer and philosopher.
Srinivasa Ramanujan	His immense talent allowed him to accumulate an incredible amount of mathematical knowledge despite coming from very modest means in India and having very limited access to formal education.
John Fields	This Canadian mathematician is best known for establishing a global award for outstanding contributions to mathematics.
Terence Tao	He was a child prodigy and a recipient of the the prestigious Fields Medal. He is currently an active mathematician doing research in many areas including number theory and probability.

Choose two of these five mathematicians and for each one you choose:

1. Do some online research to determine an additional interesting fact about the mathematician.
2. Try to find a connection between something you have studied in a recent mathematics class and the mathematical work of this historical figure.
3. If you had the chance to go back in time and meet this mathematician, what question would you ask them?

More Info:

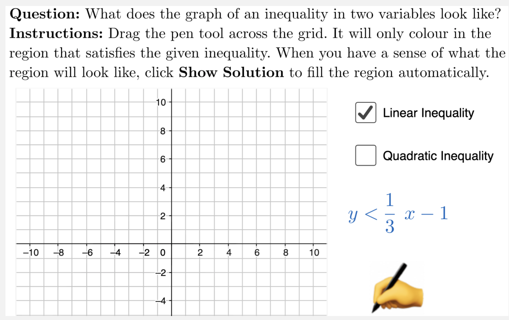
The CEMC Hypatia Math Contest and Euclid Math Contest are named in honour of two of these five mathematicians.

CEMC at Home

Grade 11/12 - Wednesday, June 3, 2020
Interact with Mathematics

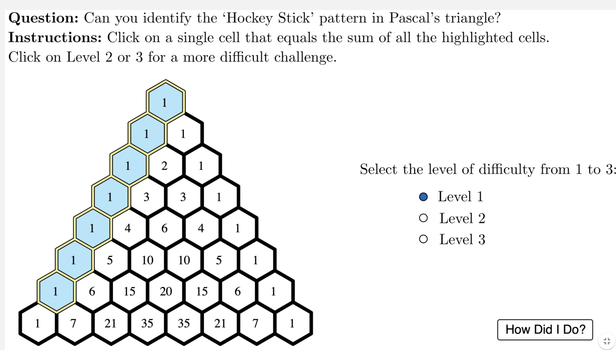
Technology can help us make mathematical discoveries and learn about mathematical objects. Three online examples of this from different areas of mathematics are featured below.

Inequalities: What does the graph of an inequality in two variables look like?



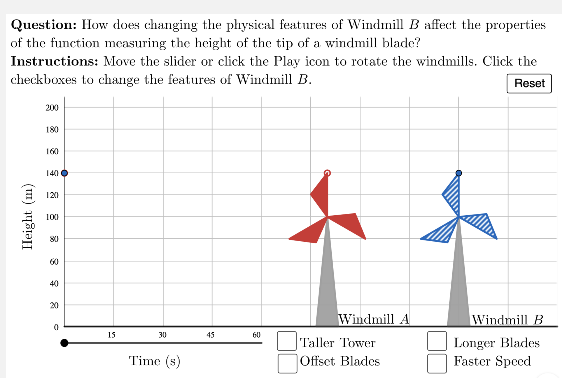
Link to App: https://www.geogebra.org/m/nqurv5yr

Pascal's Triangle: Explore the *Hockey Stick* pattern found in Pascal's Triangle.



Link to App: https://www.geogebra.org/m/bzvdfqtk

Modelling Periodic Behaviour: Discover how the physical properties of a windmill relate to the sinusoidal function that models the movement of its blades.



Link to App: https://www.geogebra.org/m/bmmrh2ap

More Info: CEMC courseware lessons feature hundreds of interactive mathematics applications. For the Grade 9/10/11 CEMC courseware, an interactive library has been built which allows you to perform a keyword search and/or display only the applications from a given strand, unit or lesson.

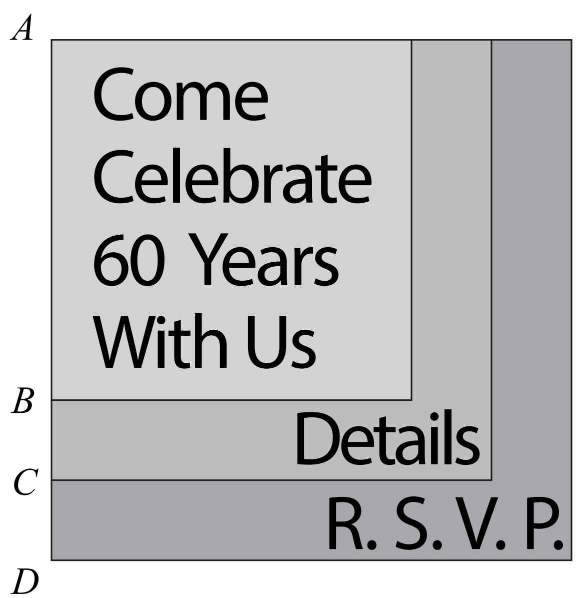
CEMC at Home

Grade 11/12 - Thursday, June 4, 2020
An Inviting Problem

An invitation to a 60^{th} anniversary party is made by overlapping three squares, as shown below.

Each square has a positive integer side length. Side AB of the smallest square lies along side AC of the middle square, which lies along side AD of the largest square. The area of the middle square not covered by the smallest square is 33 cm^2.

If $BC = CD$, determine all possible side lengths of each square.



More Info:

Check out the CEMC at Home webpage on Friday, June 5 for a solution to An Inviting Problem.

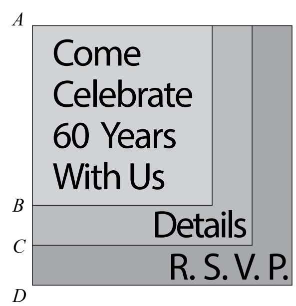
This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Thursday, June 4, 2020
An Inviting Problem - Solution

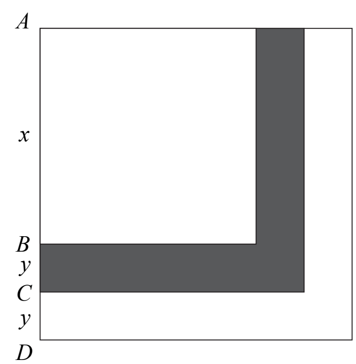
Problem:

An invitation to a 60^{th} anniversary party is made by overlapping three squares, as shown below. Each square has a positive integer side length. Side AB of the smallest square lies along side AC of the middle square, which lies along side AD of the largest square. The area of the middle square not covered by the smallest square is 33 cm^2.
If $BC = CD$, determine all possible side lengths of each square.



Solution:

Let $AB = x$ and $BC = y$. Therefore $CD = BC = y$.
Also, since the side lengths of the squares are integers, x and y are integers.



The shaded region has area 33. The shaded region is equal to the area of the square with side length AC minus the area of the square with side length AB. Since $AB = x$ and $AC = AB + BC = x + y$, we have

$$\begin{aligned} 33 &= (\text{area of square with side length } AC) - (\text{area of square with side length } AB) \\ &= (x+y)^2 - x^2 \\ &= x^2 + 2xy + y^2 - x^2 \\ &= 2xy + y^2 \\ &= y(2x+y) \end{aligned}$$

Since x and y are integers, so is $2x+y$. Therefore, $2x+y$ and y are two positive integers that multiply to give 33. Therefore, we must have $y = 1$ and $2x + y = 33$ or $y = 3$ and $2x + y = 11$ or $y = 11$ and $2x + y = 3$ or $y = 33$ and $2x + y = 1$. The last two would imply that $x < 0$, which is not possible. Therefore, $y = 1$ and $2x + y = 33$ or $y = 3$ and $2x + y = 11$.

When $y = 1$ and $2x + y = 33$, it follows that $x = 16$. Then the small square has side length $x = 16$ cm, the middle square has side length $x + y = 17$ cm, and the largest square has side length $x + 2y = 18$ cm.

When $y = 3$ and $2x + y = 11$, it follows that $x = 4$. Then the small square has side length $x = 4$ cm, the middle square has side length $x + y = 7$ cm, and the largest square has side length $x + 2y = 10$ cm.

Therefore, there are two possible sets of squares: 16 cm $\times$ 16 cm, 17 cm $\times$ 17 cm and 18 cm $\times$ 18 cm or 4 cm $\times$ 4 cm, 7 cm $\times$ 7 cm and 10 cm $\times$ 10 cm. Each set of squares satisfies the conditions of the problem.

CEMC at Home

Grade 11/12 - Friday, June 5, 2020
Math and CS in the News

Most weeks, our CEMC Homepage provides a link to a story in the media about mathematics and/or computer science. These stories show us how important mathematics and computer science are in today's world. They are a great source for discussions.

Using this article from Phys.org, think about the following questions. (URL also provided below.)

1. What is a cryptocurrency? Can you describe this to a friend? Can you name two examples of cryptocurrencies?

2. What advantages and disadvantages do you see to cryptocurrencies?

3. It turns out that the cryptocurrencies require enormous energy consumption. Do some research on this issue and think about the implications.

4. Predict the future: How much do you think cryptocurrencies will be used in 20 years?

URL of the article:
https://phys.org/news/2020-04-decrypting-cryptocurrencies.html

---

More Info:

A full archive of past posts can be found in our Math and CS in the News Archive. Similar resources for other grades may also be of interest.

CEMC at Home

Grade 11/12 - Monday, June 8, 2020
Contest Day 6

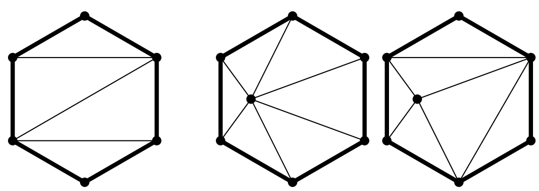
Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Hypatia Contest, #3

A *triangulation* of a regular polygon is a division of its interior into triangular regions. In such a division, each vertex of each triangle is either a vertex of the polygon or an interior point of the polygon. In a triangulation of a regular polygon with $n \geq 3$ vertices and $k \geq 0$ interior points with no three of these $n + k$ points lying on the same line,

- no two line segments connecting pairs of these points cross anywhere except at their endpoints, and
- each interior point is a vertex of at least one of the triangular regions.

Every regular polygon has at least one triangulation. The number of triangles formed by any triangulation of a regular polygon with n vertices and k interior points is constant and is denoted $T(n, k)$. For example, in every possible triangulation of a regular hexagon and one interior point, there are exactly 6 triangles. That is, $T(6, 1) = 6$.



$T(6, 0) = 4$ $\qquad$ $T(6, 1) = 6$

(a) What is the value of $T(3, 2)$?

(b) Determine the value of $T(4, 100)$.

(c) Determine the value of n for which $T(n, n) = 2020$.

More Info:

Check out the CEMC at Home webpage on Monday, June 15 for a solution to the Contest Day 6 problem.

CEMC at Home

Grade 11/12 - Monday, June 8, 2020
Contest Day 6 - Solution

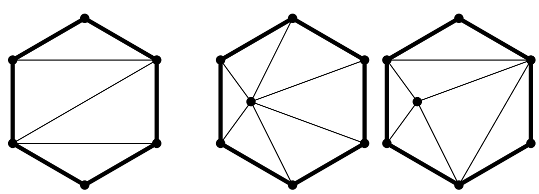
A solution to the contest problem is provided below.

2020 Hypatia Contest, #3

A *triangulation* of a regular polygon is a division of its interior into triangular regions. In such a division, each vertex of each triangle is either a vertex of the polygon or an interior point of the polygon. In a triangulation of a regular polygon with $n \geq 3$ vertices and $k \geq 0$ interior points with no three of these $n + k$ points lying on the same line,

- no two line segments connecting pairs of these points cross anywhere except at their endpoints, and
- each interior point is a vertex of at least one of the triangular regions.

Every regular polygon has at least one triangulation. The number of triangles formed by any triangulation of a regular polygon with n vertices and k interior points is constant and is denoted $T(n, k)$. For example, in every possible triangulation of a regular hexagon and one interior point, there are exactly 6 triangles. That is, $T(6, 1) = 6$.



$T(6, 0) = 4$ $\qquad$ $T(6, 1) = 6$

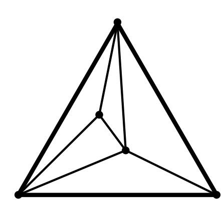
(a) What is the value of $T(3, 2)$?

(b) Determine the value of $T(4, 100)$.

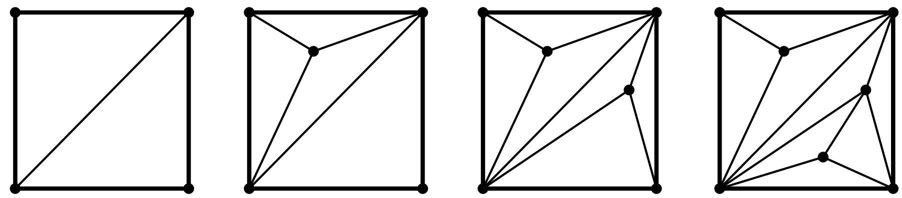
(c) Determine the value of n for which $T(n, n) = 2020$.

Solution:

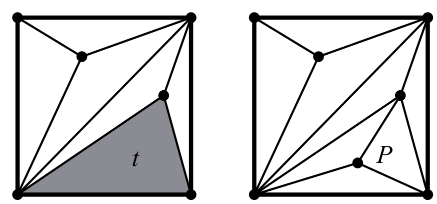
(a) For $n \geq 3$ and $k \geq 0$, the value of $T(n, k)$ is constant for all possible locations of the k interior points and all possible triangulations.
Thus, we may use the triangulation shown to determine that $T(3, 2) = 5$.



(b) We begin by drawing triangulations to determine the values of $T(4, k)$ for $k = 0, 1, 2, 3$.



Although we would obtain these same four answers by positioning the interior points in different locations, or by completing the triangulations in different ways, the diagrams above were created to help visualize a pattern.
From the answers shown, we see that $T(4, k + 1) = T(4, k) + 2$, for $k = 0, 1, 2$.
We must justify why this observation is true for all $k \geq 0$ so that we may use the result to determine the value of $T(4, 100)$.
Notice that each triangulation (after the first) was created by placing a new interior point inside the previous triangulation.
Further, each square is divided into triangles, and so each new interior point is placed *inside* a triangle of the previous triangulation (since no 3 points may lie on the same line).
For example, in the diagrams shown to the right, we observe that P lies in triangle t of the previous triangulation.



Also, each of the triangles outside of t is untouched by the addition of P, and thus they continue to contribute the same number of triangles (5) to the value of $T(4, 3)$ as they did to the value of $T(4, 2)$.
Triangle t contributes 1 to the value of $T(4, 2)$.
To triangulate the region defined by triangle t, P must be joined to each of the 3 vertices of triangle t (no other triangulation of this region is possible).
Thus, the placement of P divides triangle t into 3 triangles for every possible location of P inside triangle t.
That is, t contributes 1 to the value of $T(4, 2)$, but the region defined by t contributes 3 to the value of $T(4, 3)$ after the placement of P.
To summarize, the value of $T(4, k + 1)$ is 2 more than the value of $T(4, k)$ for all $k \geq 0$ since:

- the $(k + 1)^{st}$ interior point may be placed anywhere inside the triangulation for $T(4, k)$ (provided it is not on an edge)
- specifically, the $(k+1)^{st}$ interior point lies inside a triangle of the triangulation which gives $T(4, k)$
- this triangle contributed 1 to the value of $T(4, k)$
- after the $(k + 1)^{st}$ interior point is placed inside this triangle and joined to each of the 3 vertices of the triangle, this area contributes 3 to the value of $T(4, k + 1)$
- this is a net increase of 2 triangles, and thus $T(4, k + 1) = T(4, k) + 2$, for all $k \geq 0$.

$T(4, 0) = 2$ and each additional interior point increases the number of triangles by 2.
Thus, k additional interior points increases the number of triangles by $2k$, and so $T(4, k) = T(4, 0) + 2k = 2 + 2k$ for all $k \geq 0$.
Using this formula, we get $T(4, 100) = 2 + 2(100) = 202$.

(c) In the triangulation of a regular n-gon with no interior points, we may choose any one of the n vertices and join this vertex to each of the remaining $n-3$ non-adjacent vertices.
All such triangulations of a regular n-gon with no interior points creates $n-2$ triangles, and so $T(n,0) = n-2$ for all $n \geq 3$ (since $T(n,0)$ is constant).
The reasoning used in part (b) extends to any regular polygon having $n \geq 3$ vertices.
That is, each additional interior point that is added to the triangulation for $n \geq 3$ vertices and $k \geq 0$ interior points gives a net increase of 2 triangles.
Thus, $T(n,k+1) = T(n,k)+2$ for all regular polygons having $n \geq 3$ vertices and $k \geq 0$ interior points.
So then k additional interior points increases the number of triangles by $2k$, and so $T(n,k) = T(n,0) + 2k = (n-2) + 2k$ for all $k \geq 0$.
Using this formula $T(n,k) = (n-2) + 2k$, we get $T(n,n) = (n-2) + 2n = 3n-2$ and $3n-2 = 2020$ when $n = \frac{2022}{3} = 674$.

CEMC at Home

Grade 11/12 - Tuesday, June 9, 2020
It's True Because it Isn't False!

Background

What do you think of the following multiple choice question?

Exactly one of the following statements is true. Which one is it?

(a) 6 is a prime number.

(b) $x = 2$ is a solution to the equation $x + 3 = 4$.

(c) The point $(1, 2)$ lies on the line with equation $y = x + 10$.

(d) The CEMC was founded in 1995 with origins dating back to the 1960s.

(e) $2^0 = 0$

The correct answer is (d) and you probably got it right without knowing anything about the history of the CEMC. How did you do that? You may have used a useful trick for answering a multiple choice question: elimination!

Answer (a) is false because $6 = 2 \times 3$ so 6 is not prime.
Answer (b) is false because when $x = 2$, the value of $x + 3$ is $2 + 3 = 5$ and $5 \neq 4$.
Answer (c) is false because when $x = 1$, the value of $x + 10$ is $1 + 10 = 11$ and $11 \neq 2$.
Answer (e) is false because $2^0 = 1$ and $1 \neq 0$.

Now that we have eliminated four of the five answers, we know that (d) must be the true statement. Elimination works here because we are told that exactly one of the statements is true. Without that information, we cannot confidently answer the question (unless we are CEMC history buffs!).

Example 1: Consider the statement "There is no greatest positive integer."

How can we eliminate the possibility that this statement is false?

We know that the statement is either true or false, but let's say that we do not know which is the case. Let's suppose that the statement is false, and see where this leads.

Suppose that there is a greatest positive integer.
Let's call this greatest positive integer k.

Now, we proceed as we normally would in a mathematical argument, using sound logic and facts that we know to be true.

Consider the number $k + 1$.
We know that the number $k + 1$ must be a positive integer and must satisfy $k + 1 > k$.

Here we see that something is wrong. We have shown that if the given statement is false, then we can deduce that the following are both true about the number k:

- k is the greatest positive integer, and
- $k + 1$ is a positive integer that is greater than k.

But these cannot both be true of k. We claim that this means that the given statement could not possibly be false. Can you see why we can conclude this?

Explanation of a Proof Method (Proof by Contradiction)

In mathematics, we deal with sentences that have a definite state of being either true or false. We call these sentences *statements.* Since a statement is either true or false, we can use elimination to argue that a statement must be true by eliminating the possibility that it is false.

How can we eliminate the possibility that a statement is false? We can suppose that it is false and show that this assumption leads us to *a contradiction.* A contradiction is a combination of ideas that are opposed to one another, and hence cannot be simultaneously true. (For example, if we deduce that a number x must satisfy $x > 1$ *and* $x < -1$, then we have reached a contradiction.)

If we reach a contradiction in our argument, then we can be sure that there is at least one flaw in our argument. If the only possible error in our argument was our initial assumption ("the statement is false") then this is the only thing that could have caused us to reach a contradiction. If we are sure that the assumption "the statement is false" is wrong, then the only remaining option is that the statement is actually true!

We call this logical method a *proof by contradiction.*

Example 2: Here is an example of a proof by contradiction.

Statement: The sum of a rational number and an irrational number is an irrational number.

Proof:

Suppose, for a contradiction, that there is a rational number r and an irrational number α for which the sum $r + \alpha$ is rational.

Since r is rational, we can write $r = \frac{a}{b}$ for integers a, b $(b \neq 0)$.

Since $r+\alpha$ is rational, we can write $r+\alpha = \frac{c}{d}$ for integers c, d $(d \neq 0)$.

This means we have $\frac{a}{b} + \alpha = \frac{c}{d}$.

It follows that $\alpha = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{bd}$.

This means α is rational.

We have reached a contradiction.

Therefore, it cannot be the case that there is a rational number and an irrational number whose sum is rational.

We conclude that it must be the case that the sum of a rational number and an irrational number is an irrational number.

A rational number is a real number that can be expressed as a fraction of two integers (with non-zero denominator). An irrational number is a real number that cannot be expressed in this way.

We suppose that the given statement is false.

We proceed using logic and mathematical facts we know to be true.

We have deduced the following:

- α is irrational
- α is rational

But of course, they cannot both be true!

We have eliminated the possibility that our statement is false.

Practice: Your turn! Prove each of the following statements using a proof by contradiction approach.

1. There do not exist integers x and y such that $10x - 25y = 6$.
   Start by supposing that there do exist integers x and y that satisfy $10x - 25y = 6$. Then think about the factors of the integer $10x - 25y$.
2. If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.
   Start by supposing that there are positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.
3. Extra Challenge: If the parabola $y = ax^2 + bx + c$ (with a, b, c non-zero real numbers) touches or crosses the x-axis, then a, b, c cannot form a geometric sequence, in that order.

More Info:

Check out the CEMC at Home webpage on Tuesday, June 16 for proofs of the above statements.

CEMC at Home

Grade 11/12 - Tuesday, June 9, 2020
It's True Because it Isn't False! - Solution

Prove each of the following statements using a proof by contradiction approach.

1. There do not exist integers x and y such that $10x - 25y = 6$.

   Proof:

   Suppose, for a contradiction, that there do exist integers x and y such that $10x - 25y = 6$.

   Since x and y are integers, the quantity $10x - 25y$ is an integer as well.
   Since $10x - 25y = 5(2x - 5y)$, the integer 5 must be a factor of the integer $10x - 25y$.
   Since $10x - 25y = 6$, this means the integer 5 must also be a factor of 6.
   This is a contradiction because 5 is not a factor of 6.

   Therefore, we conclude that there cannot exist integers x and y such that $10x - 25y = 6$.

2. If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

   Proof:

   Suppose, for a contradiction, that there are positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   Squaring both sides of the equation gives

$$\begin{aligned}
\left(\sqrt{x+y}\right)^2 &= \left(\sqrt{x} + \sqrt{y}\right)^2 \\
x + y &= \left(\sqrt{x}\right)^2 + 2\sqrt{x}\sqrt{y} + \left(\sqrt{y}\right)^2 \\
x + y &= x + 2\sqrt{x}\sqrt{y} + y \\
0 &= 2\sqrt{xy} && \text{since } x, y > 0 \\
0 &= \sqrt{xy} \\
0 &= xy
\end{aligned}$$

   Since $xy = 0$ we must have $x = 0$ or $y = 0$. This is a contradiction. Since x and y are both positive real numbers, it cannot be the case that $x = 0$ and it cannot be the case that $y = 0$.

   This means there cannot be positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   Therefore, we conclude that if x and y are positive real numbers, then we must have $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

3. Extra Challenge: If the parabola $y = ax^2 + bx + c$ (with a, b, c non-zero real numbers) touches or crosses the x-axis, then a, b, c cannot form a geometric sequence, in that order.

   Solution:

   Let $y = ax^2 + bx + c$ be a parabola, with a, b, c non-zero real numbers, that crosses or touches the x-axis. We want to prove that a, b, c cannot form a geometric sequence, in that order.

   Suppose, for a contradiction, that a, b, c is a geometric sequence, in that order.

   Since a, b, c is a geometric sequence, it has a common ratio, r, such that $b = ar$ and $c = ar^2$.
   Since we know that $b \neq 0$, we know that $r \neq 0$.

Since the parabola crosses or touches the x-axis, the quadratic equation $ax^2 + bx + c = 0$ has at least one real solution. So the discriminant of the quadratic must be at least zero.

Since the discriminant is

$$b^2 - 4ac = (ar)^2 - 4a(ar^2) = (ar)^2 - 4(ar)^2 = -3(ar)^2$$

we must have $-3(ar)^2 \geq 0$.

However, since $a \neq 0$ and $r \neq 0$, we must have $ar \neq 0$. It follows that $(ar)^2 > 0$ and so $-3(ar)^2 < 0$. This is a contradiction since we cannot have both $-3(ar)^2 \geq 0$ and $-3(ar)^2 < 0$.

Therefore, it cannot be the case that a, b, c is a geometric sequence, in that order.

Discussion: Proving statements can be one of the most rewarding parts of mathematics, but it can also be challenging. There are a variety of proof techniques that we can use, one of which is proof by contradiction. While there are no fixed rules about which technique to use to prove a statement (one of the reasons why writing a proof can be challenging!), there are some statements that lend themselves well to a proof by contradiction approach. The statements from this resource fall into this category.

Our first problem here involved showing that a pair of integers with some property *cannot* exist, and the third problem here involved showing that a triple of real numbers *cannot* satisfy a certain condition. Taking a proof by contradiction approach allowed us to see what would happen if the pair of integers *did* exist and if the real numbers *did* satisfy the condition. We were able to do some algebra and, quite quickly, we discovered that these situations lead to contradictions and so we could rule them out.

Think about how you would prove these statements without taking a proof by contradiction approach. For example, for the first statement, you would need to argue directly that every possible pair of integers fails to satisfy the equality. How would you make this argument? Sometimes, it can be challenging to argue that objects *do not* exist or *do not* satisfy a certain condition *directly* (especially when there are infinitely many objects to rule out!). You may not need to do a proof by contradiction, but thinking about the problem in this way will often lead you to a nice argument.

In contrast, you might find it very natural to prove the second statement here without using a proof by contradiction approach. Given two positive real numbers x and y, can you check directly that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$?

Here is an example of a *direct proof of statement 2:*

Let x and y be positive real numbers. Then we have

$$(\sqrt{x+y})^2 - (\sqrt{x} + \sqrt{y})^2 = (x+y) - (x + 2\sqrt{x}\sqrt{y} + y) = -2\sqrt{x}\sqrt{y}$$

Since $\sqrt{x} > 0$ and $\sqrt{y} > 0$ we have $\sqrt{x}\sqrt{y} > 0$ and so $-2\sqrt{x}\sqrt{y} < 0$. It follows that

$$(\sqrt{x+y})^2 - (\sqrt{x} + \sqrt{y})^2 < 0$$

and hence

$$(\sqrt{x+y})^2 < (\sqrt{x} + \sqrt{y})^2$$

Since $\sqrt{x+y} > 0$ and $\sqrt{x} + \sqrt{y} > 0$, it must be the case that $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$. In particular, this means that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$ as desired.

CEMC at Home

Grade 11/12 - Wednesday, June 10, 2020
Trugs and Falths

You are an ambassador of The Republic of Logica and you have been sent on a mission to Duoterra, one of the islands of Logica. This island is entirely inhabited by two societies of people: the Trugs, who always tell the truth and the Falths, who always lie. You must figure out to which society the people you encounter belong. To do so you will have to use the skills you learned at the University of Logica.

One of these skills is the process of elimination which is more formally known as "Proof by Contradiction". To learn about this method, see the CEMC at Home Grade 11/12 activity from June 9.

On each day, you meet at least two new people on the island. Let's see what happens on Day 1.

Day 1: You meet Algorn and Birk.

Algorn: Birk and I are Trugs.
Birk: Algorn is a Falth.

From this information you must determine to which society Algorn and Birk belong.

Solution for Day 1: Algorn must be either a Trug or a Falth.

- *Suppose that Algorn is a Trug.* This means that Algorn is telling the truth and so both Algorn and Birk must be Trugs. Therefore, Birk is also telling the truth and so Algorn is a Falth. This contradicts our initial assumption that Algorn is a Trug. So Algorn must be a Falth.
- *Algorn is a Falth.* This means Algorn is lying, which is already clear because she said that both Birk and her are Trugs. Birk's statement is then true and so Birk must be a Trug.

In summary, Algorn is a Falth and Birk is a Trug.

Logical Connectives

During your travels you will encounter sentences that involve the words "and", "or", and "not". Here is a summary of how to interpret the truth of statements like this:

AND	A statement of the form "P and Q" is true if *both* P and Q are true. A statement of the form "P and Q" is false if *at least one* of P and Q is false.
OR	A statement of the form "P or Q" is true if *at least one* of P and Q is true. A statement of the form "P or Q" is false if *both* P and Q are false.

Note that the word "or" is sometimes used differently than described above in everyday English. Sometimes people interpret "P or Q" as being true when *exactly one* of P and Q is true. We will not use this interpretation in what follows. You will also encounter statements involving "not". This has the usual meaning of the "opposite". In particular, "not P" is true exactly when P is false.

Day 2: You meet Crozul and Dek.

Crozul: I am a Trug or Dek is a Falth.
Dek: Exactly one of Crozul and I is a Trug.

From this information you must determine to which society Crozul and Dek belong.

Solution for Day 2: Crozul must be either a Trug or a Falth.

- *Suppose that Crozul is a Trug.* Therefore, Crozul is telling the truth. Since Crozul is a Trug, then Dek could be a Trug or a Falth and Crozel's statement would still be true.
  - *Suppose Dek is a Trug.* Then they are both Trugs and Dek's statement is false, which is not possible since Dek is a Trug. So Dek is a Falth.
  - *Dek is a Falth.* Then exactly one of Crozul and Dek is a Trug and so Dek's statement is true, which is not possible since Dek is a Falth.

  So, if Crozul is a Trug, then there are no possibilities for Dek to belong to either society, which is not possible. So it must be the case that Crozul is a Falth.
- *Crozul is a Falth.* Then Crozul is lying and so his statement tells us that he is not a Trug (this is consistent with our assumption) and that Dek is not a Falth (he is a Trug). We need to verify that this assignment of societies is consistent with Dek's statement. Exactly one of Crozul and Dek is a Trug and so Dek is telling the truth, which is consistent with the fact that he is a Trug.

In summary, Cozul is a Falth and Dek is a Trug.

Your journey continues

On your third, fourth and fifth days you have three more interactions. Use these interactions to determine to which society the people you meet belong.

Day 3: You meet Gup and Hoken.

Gup: Hoken is a Falth.
Hoken: I am a Trug or Gup is a Trug.

Day 4: You meet Ized and Jeke.

Ized: Jeke is not a Falth and I am a Trug.
Jeke: Ized and I are from the same society.

Day 5: You meet Kip, Lolo and Moy.

Kip: I am not a Falth and Lolo is not a Falth.
Lolo: Kip is a Falth.
Moy: Lolo is a Trug.

More Info:

Check out the CEMC at Home webpage on Wednesday, June 17 for a solution to Trugs and Falths.

These problems are based on a famous type of logic puzzles called "Knights and Knaves". They were made popular in the late 1970s by Raymond Smullyan, an American mathematician. An internet search for "knights and knaves" will lead to many other problems of this type.

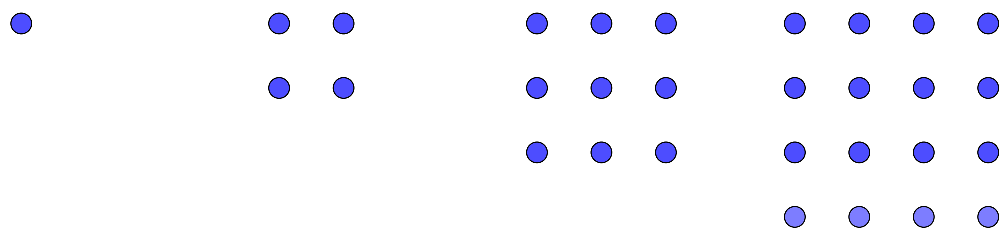
CEMC at Home

Grade 11/12 - Thursday, June 11, 2020
That Number Makes it Perfect

A *perfect square* is an integer that can be expressed as the product of two equal integers. The integer 25 is a perfect square since it can be expressed as the product 5×5 or 5^2.

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.



$1 \times 1 = 1^2 = 1$ $\quad 2 \times 2 = 2^2 = 4$ $\quad 3 \times 3 = 3^2 = 9$ $\quad 4 \times 4 = 4^2 = 16$

Did you know that the sum, S, of the positive integers from 1 to some positive integer n can be calculated using the formula $S = \dfrac{n \times (n+1)}{2}$?
For example,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{10 \times 11}{2} = 55.$$

This result may be helpful in this problem.

More Info:

Check out the CEMC at Home webpage on Friday, June 12 for a solution to That Number Makes it Perfect.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Thursday, June 11, 2020
This Number Makes it Perfect - Solution

Problem:

A *perfect square* is an integer that can be expressed as the product of two equal integers. The integer 25 is a perfect square since it can be expressed as the product 5×5 or 5^2.

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.

Solution:

We are asked to determine the smallest positive integer, n, such that

$$2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n \qquad (1)$$

is a perfect square.

Factoring (1), we obtain

$$\begin{aligned}
2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n &= 2n(1 + 2 + 3 + \cdots + 798 + 799 + 800) \\
&= 2n\left(\frac{800 \times 801}{2}\right) \\
&= n(800)(801) && (2) \\
&= n[(2)(2)(2)(2)(2)(5)(5)][(3)(3)(89)] \\
&= n(2^5)(5^2)(3^2)(89) && (3)
\end{aligned}$$

In going from (2) to (3), we have expressed the 800×801 as the product of prime factors.

We need to determine what additional factors are required to make the quantity in (3) a perfect square such that n is as small as possible.
A positive integer larger than one is a perfect square exactly when each of its prime factors occurs an even number of times in its prime factorization. In order for each prime in the prime factorization to occur an even number of times, we need n to be $2 \times 89 = 178$. Then the quantity in (3) becomes

$$n(2^5)(5^2)(3^2)(89) = (2)(89)(2^5)(5^2)(3^2)(89) = (2^6)(5^2)(3^2)(89^2) = \left[(2^3)(5)(3)(89)\right]^2,$$

a perfect square.

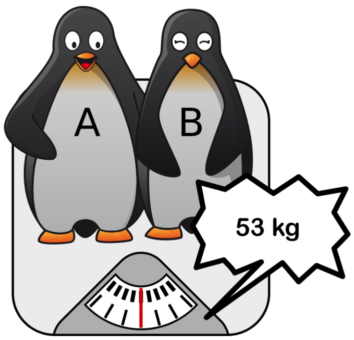
Therefore, the smallest positive integer is $n = 178$ and the perfect square is

$$178 \times 800 \times 801 = 114\,062\,400 = (10\,680)^2.$$

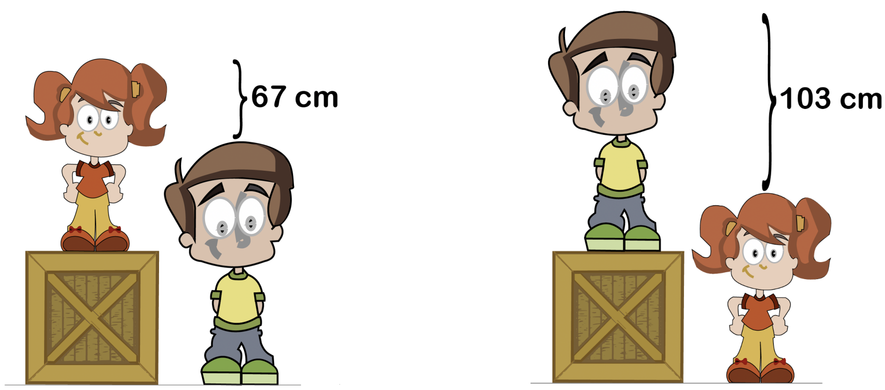
CEMC at Home

Grade 11/12 - Friday, June 12, 2020
Picture This

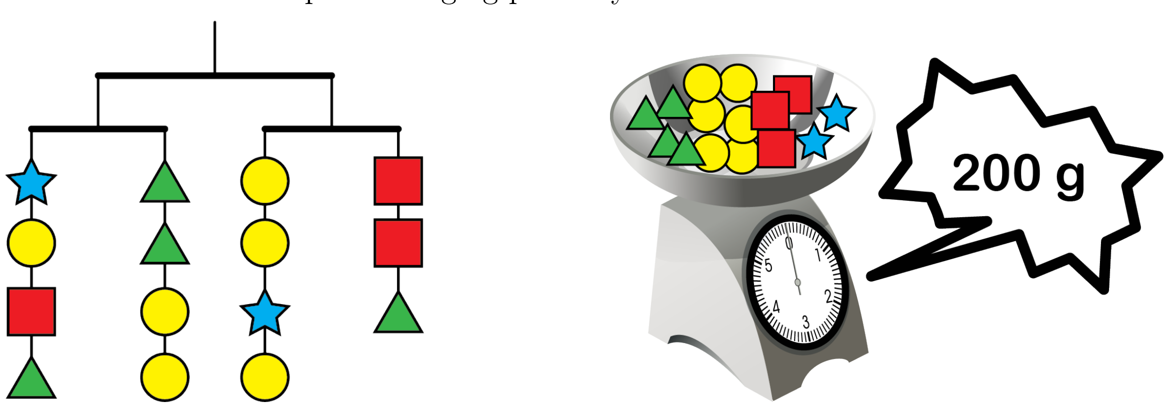
Problem 1: What is the mass of the baby penguin?



Problem 2: How tall is the crate?



Problem 3: A mobile of shapes is hanging perfectly balanced. What is the mass of each shape?



All of the individual shapes of the same type (star, circle, square, or triangle) have the same mass. All of the shapes from the mobile are placed on the scale.

More Info:

Check out the CEMC at Home webpage on Friday, June 19 for a solution to Picture This.

CEMC at Home

Grade 11/12 - Friday, June 12, 2020
Picture This - Solution

Problem 1: What is the mass of the baby penguin?

Solution 1:

Let A, B, and C represent the masses, in kg, of the three penguins (as indicated in the diagrams). The mass of the baby penguin is C kg. The three images give us the following three equations, in order:

$$A + B = 53$$
$$A + C = 37$$
$$B + C = 32$$

Adding these three equations together, we see that $2A + 2B + 2C = 53 + 37 + 32 = 122$. This means $2(A + B + C) = 122$ and hence $A + B + C = 61$. Finally, we have

$$C = (A + B + C) - (A + B) = 61 - 53 = 8$$

Therefore, the baby penguin has a mass of 8 kg.

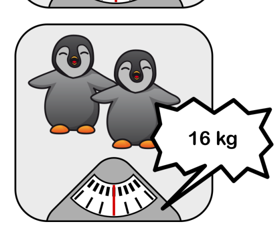
Solution 2:

First, combine the penguins from the last two images.

This produces a scale containing the two adult penguins and two "copies" of the baby penguin, for a total mass of $37 + 32 = 69$ kg.

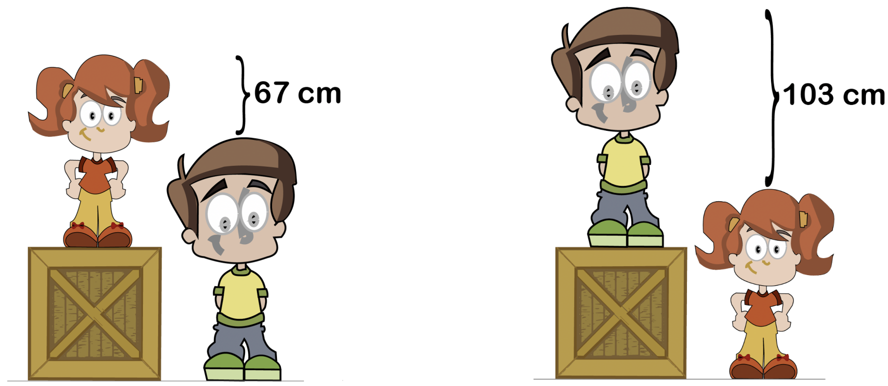
Now remove the penguins from the first image.

This produces a scale containing two baby penguins, with a combined mass of $69 - 53 = 16$ kg.



Therefore, one baby penguin has a mass of $\frac{16}{2} = 8$ kg.

Problem 2: How tall is the crate?



Solution 1:

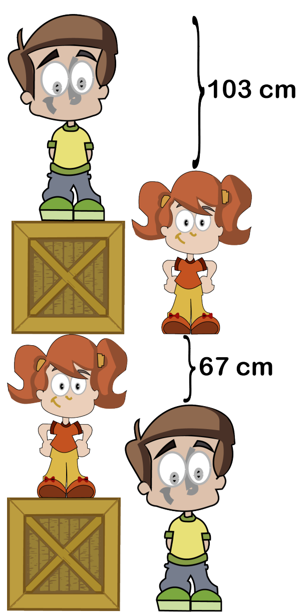
Let a denote the height of the child on the crate in the left image, b denote the height of the child on the crate in the right image, and c denote the height of the crate (in each image), all in cm.

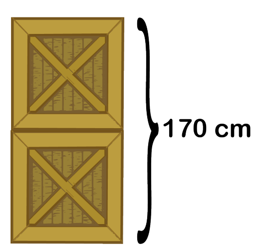
Looking at the first image, we have $c + a = b + 67$.
Looking at the second image, we have $c + b = a + 103$.

Rearranging the first equation gives $c + a - b = 67$, and rearranging the second equation gives $c + b - a = 103$. Adding the two equations together we get $2c + a - b + b - a = 103 + 67$ which simplifies to $2c = 170$. This means $c = 85$ and therefore, the height of the crate is 85 cm.

Solution 2:

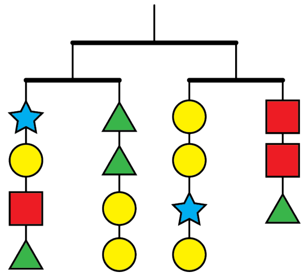
Stack the two images vertically as shown below on the left. The leftmost stack consists of two crates and both children. The rightmost stack consists both children and $67 + 103 = 170$ cm of "space". These two stacks must have the same total height. (Why?)

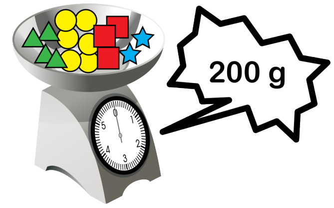




Now remove the children from each side of the stack. This tells us that two crates stacked one on top of the other have a height of 170 cm. Therefore, one crate is $\frac{170}{2} = 85$ cm tall.

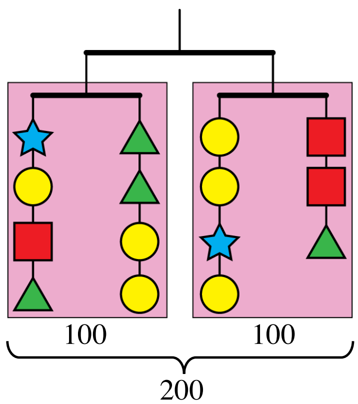
Problem 3: A mobile of shapes is hanging perfectly balanced. What is the mass of each shape?

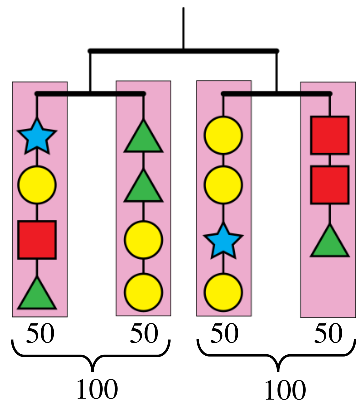




Solution 1:

Recall that the total mass of all of the shapes in the mobile is 200 grams. Since the mobile is perfectly balanced, these 200 grams must be evenly distributed between the left portion and the right portion of the mobile, making the mass of the shapes in each portion 100 grams as shown in the image below on the left. Using similar reasoning, we see that the total mass of the shapes on each of the four bottom strings must be 50 grams as shown in the image below on the right.





Let b be the mass of one blue star, y be the mass of one yellow circle, r be the mass of one red square, and g be the mass of one green triangle, all in grams. The first, second, third, and fourth strings show the following relationships, in order:

(1) $b + y + r + g = 50$ (2) $2g + 2y = 50$ (3) $3y + b = 50$ (4) $2r + g = 50$

There are many ways to proceed from here. One way (that may be hard to find!) is to notice the following relationship involving the four lefthand expressions above:

$$(2g + 2y) - 4(b + y + r + g) + 4(3y + b) + 2(2r + g) = 10y$$

Using equations (1), (2), (3), and (4), we have

$$(2g + 2y) - 4(b + y + r + g) + 4(3y + b) + 2(2r + g) = 50 - 4(50) + 4(50) + 2(50) = 150$$

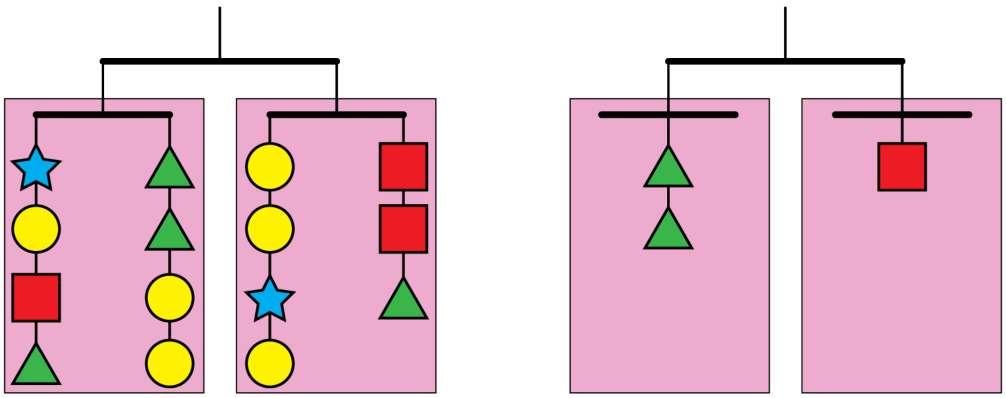
and so $10y = 150$ or $y = 15$.

Since $y = 15$ and $g + y = 25$, from (2), we have $g = 10$. Since $g = 10$ and $2r + g = 50$, from (4), we have $r = 20$. Since $y = 15$ and $3y + b = 50$, from (3), we have $b = 5$.

Therefore, each blue star has a mass of 5 grams, each yellow circle has a mass of 15 grams, each red square has a mass of 20 grams, and each green triangle has a mass of 10 grams.

Solution 2:

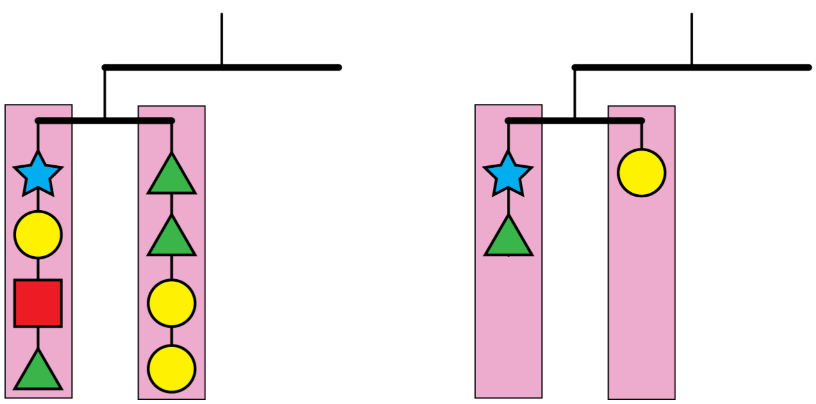
Since the mobile is perfectly balanced, the total mass of the shapes on the left portion of the mobile must be equal to the total mass of the shapes on the right portion. We can remove identical shapes from each side of the mobile, while still keeping it balanced.



After removing 1 , 3 , 1 and 1 from each side of the mobile, we are left with:

$$1 \; = 2 \qquad \text{(Equation 1)}$$

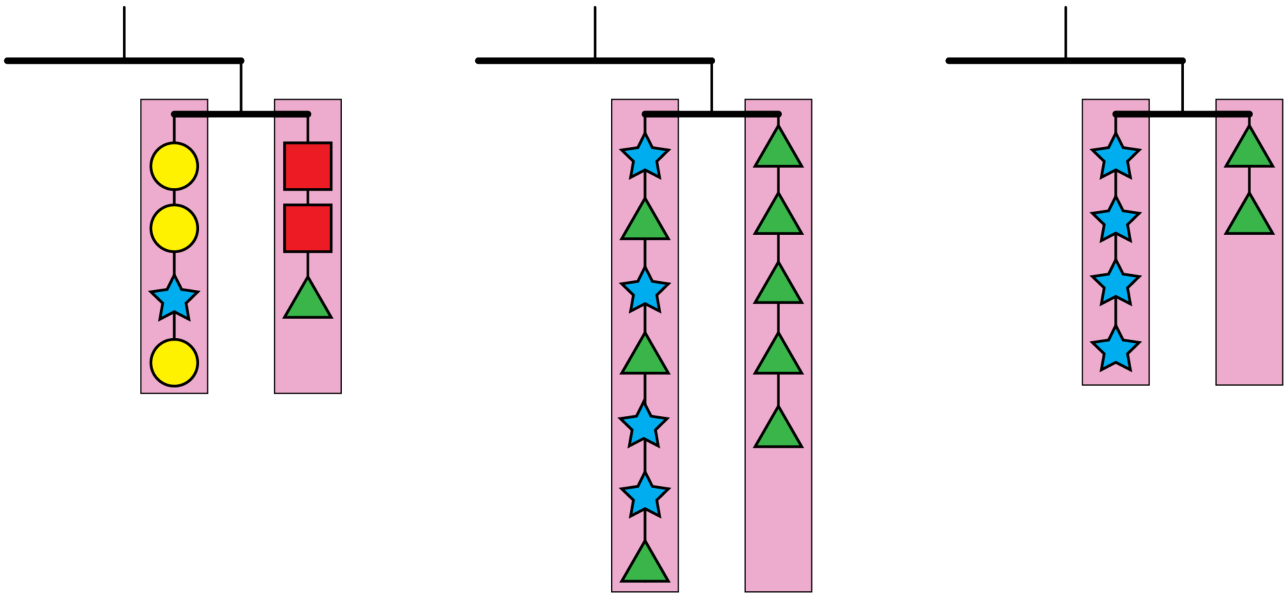
Next, since the left portion of the mobile is also perfectly balanced, the total mass of the shapes on the first string is equal to the total mass of the shapes on the second string. We can remove identical shapes, or groups of shape that have the same total mass, while still keeping it balanced.



We can remove 1 from each string, and we can also remove 1 from the first string and 2 from the second string since we have learned they are equivalent in mass. Now we are left with:

$$1 \; = 1 \; + 1 \qquad \text{(Equation 2)}$$

Similarly, the total mass of the shapes on the third string is equal to the total mass of the shapes on the fourth string. Since the strings don't have any shapes in common, let's first use Equations 1 and 2 to make some shape substitutions.



Using Equation 1, replace every ■ on the fourth string with 2 ▲. Using Equation 2, replace every ● on the third string with 1 ★ and 1 ▲. Then, after removing 3 ▲ from each string, we are left with:

$$4\,★ = 2\,▲ \text{ or } 2\,★ = 1\,▲ \quad \text{(Equation 3)}$$

Using our three equations we can express the mass of each shape in terms of its mass in ★:

Shape	Equivalent mass in ★	Reason
★	1 ★	by definition
▲	2 ★	Equation 3
●	3 ★	Equation 2 & 3
■	4 ★	Equation 1 & 3

Since the total mass of all the shapes is 200 g we can then conclude the following:

$$
\begin{aligned}
4\,\triangle + 6\,\bigcirc + 3\,\square + 2\,\bigstar &= 200\\
4\left(2\,\bigstar\right) + 6\left(3\,\bigstar\right) + 3\left(4\,\bigstar\right) + 2\,\bigstar &= 200\\
8\,\bigstar + 18\,\bigstar + 12\,\bigstar + 2\,\bigstar &= 200\\
40\,\bigstar &= 200\\
1\,\bigstar &= 5
\end{aligned}
$$

Since 1 $\bigstar$ has a mass of 5 g we can now determine the mass of each shape:

Shape	Equivalent mass in $\bigstar$	Mass in g
$\bigstar$	1 $\bigstar$	5 g
$\triangle$	2 $\bigstar$	10 g
$\bigcirc$	3 $\bigstar$	15 g
$\square$	4 $\bigstar$	20 g

CEMC at Home

Grade 11/12 - Wednesday, June 10, 2020
Trugs and Falths - Solution

Here are the solutions for the remaining days of your travels.

> **Day 3:** You meet Gup and Hoken.
>
> *Gup:* Hoken is a Falth.
> *Hoken:* I am a Trug or Gup is a Trug.

Solution:

Gup must be either a Trug or a Falth.

- *Suppose that Gup is a Trug.*
  This means that Gup is telling the truth and so Hoken is a Falth. Therefore, Hoken is lying and so Hoken is not a Trug and Gup is not a Trug. (Recall that if "P or Q" is false, then P is false and Q is false.) But Gup is a Trug and so we have reached a contradiction. Therefore, Gup cannot be a Trug and hence Gup must be a Falth.
- *Gup is a Falth.*
  This means that Gup is lying and so Hoken must be a Trug. Note that this information is consistent with Hoken's statement as well. Since Hoken is a Trug, Hoken must be telling the truth and indeed at least one of Gup and Hoken is a Trug. (Recall that if "P or Q" is true, then at least one of P and Q is true.)

In summary, Gup is a Falth and Hoken is a Trug.

> **Day 4:** You meet Ized and Jeke.
>
> *Ized:* Jeke is not a Falth and I am a Trug.
> *Jeke:* Ized and I are from the same society.

Solution:

Jeke must be either a Trug or a Falth.

- *Suppose that Jeke is a Trug.*
  This means that Jeke is telling the truth. Therefore, Ized and Jeke are from the same society and so Ized is also a Trug. Note that this is consistent with what Ized said. If Ized is a Trug, then Ized is telling the truth which means Jeke is not a Falth (and so is a Trug) and Ized is also a Trug.
  Note that we did not reach a contradiction here. This means we cannot eliminate the possibility that Jeke is a Trug. Does this mean we can be sure that Jeke is a Falth? Let's confirm that the other possibility leads to a contradiction.
- *Suppose that Jeke is a Falth.*
  This means that Jeke is lying and so Jeke and Ized are from different societies. Therefore, Ized is a Trug. If Ized is a Trug, Ized is telling the truth. But this means Jeke is not a Falth. This is a contradiction. Therefore, Jeke cannot be a Falth and so must be a Trug.

Since Jeke is a Trug, we already know from our work above that Ized must also be a Trug.
In summary, Jeke and Ized are both Trugs.

Day 5: You meet Kip, Lolo and Moy.

Kip: I am not a Falth and Lolo is not a Falth.
Lolo: Kip is a Falth.
Moy: Lolo is a Trug.

Solution:

Lolo must be either a Trug or a Falth.

- *Suppose that Lolo is a Falth.*
  This means Lolo is lying and so Kip is not a Falth and is hence a Trug. If Kip is a Trug, then Kip is telling the truth and so Lolo is not a Falth. This is a contradiction. Therefore, Lolo cannot be a Falth and hence must be a Trug.

- *Lolo is a Trug.*
  This means Lolo is telling the truth and so Kip is a Falth. It also means that Moy is telling the truth about Lolo's society and hence Moy is a Trug. Note that this information is consistent with Kip's statement as well. Since Kip is a Falth, Kip must be lying and indeed the first part of Kip's statement is false.

In summary, Kip is a Falth, and Lolo and Moy are both Trugs.

CEMC at Home

Grade 11/12 - Monday, June 15, 2020
History of Computing

Computers can be found on our desks, in our pockets and even in our refrigerators! This is remarkable because modern computers have been around for less than 100 years. During this time, there has been a constant stream of new discoveries and advances in technology.

> Use this online tool to arrange the following list of events in the history of computer science from earliest to most recent.

A. The ASCII is developed to create standard binary codes for 128 different characters.

B. Deep Blue is the first computer program to beat a human world chess champion.

C. Computers are used to determine that a perfect winning strategy does not exist for the game of checkers.

D. A robot named Elektro is built which responds to voice commands.

E. Konrad Zuse designs the Z3 electromechanical computer which is considered the first automatic programmable computer.

F. The Harvard Mark I mechanical computer is built and is used for military purposes during World War II.

G. An algorithm named *Quicksort* is developed to arrange objects in increasing or decreasing order.

H. Keyboard input is introduced as a way of entering data into a computer.

I. The Altair 8800 is the first personal computer to sell in large numbers.

J. Alan Turing uses the *halting problem* to establish a theoretical limit on the power of computers.

K. An international messaging service named Telex allows for the transfer of data and secure communications.

L. NASA and Grumman build the Apollo Guidance Computer, which is used during Apollo space missions.

M. Sun Microsystems develops the Java programming language.

N. Guido van Rossum creates and releases the Python programming language.

O. Animators create Cindy, the first human-like CGI (computer generated imagery) movie character.

More Info:

Our webpage Computer Science and Learning to Program is the best place to find the CEMC's computer science resources.

CEMC at Home

Grade 11/12 - Tuesday, June 16, 2020
Can You Find the Terms?

Can you find all of the given mathematics and computer science terms in the grid? Good Luck!

C	V	O	W	I	Y	J	I	V	L	Q	V	S	M	L	R	B	R	U	O
O	N	S	N	W	R	O	A	Z	S	F	G	M	D	D	G	A	Q	D	U
J	U	Y	S	W	T	Y	G	T	F	E	E	L	H	E	S	R	F	L	H
S	M	X	Y	A	E	V	A	Y	V	M	Q	U	S	R	H	I	N	D	U
U	N	M	C	H	M	Q	N	E	R	Y	Z	O	Z	I	B	L	Q	J	O
H	O	N	N	F	O	P	C	G	X	S	L	H	H	V	E	A	T	R	C
U	I	O	E	U	N	T	L	D	X	O	B	E	Y	A	C	G	O	E	S
K	T	I	I	R	O	O	S	I	G	I	W	X	W	T	N	S	V	T	U
Z	A	S	C	R	G	S	I	A	N	P	M	A	T	I	A	B	Z	E	T
L	R	R	I	M	I	K	R	T	E	G	J	D	J	V	T	H	N	M	J
T	E	U	F	V	R	I	U	R	A	V	L	E	W	E	I	O	A	A	I
G	T	C	F	J	T	W	M	J	J	Z	U	C	A	J	R	M	R	R	T
D	I	E	E	H	G	U	R	K	Y	E	I	I	Q	J	E	M	J	A	L
G	O	R	M	U	T	A	Q	F	U	Q	T	M	O	V	H	R	F	P	S
Y	Z	H	H	A	W	L	C	Q	D	L	Q	A	I	J	N	U	T	K	T
H	L	U	T	R	O	S	K	C	I	U	Q	L	N	T	I	M	W	H	A
M	J	I	P	E	I	R	A	D	I	A	N	R	G	G	P	V	W	I	C
B	O	K	P	U	M	C	B	Y	Z	N	S	C	L	C	E	O	H	J	K
N	N	E	G	T	N	A	N	I	M	I	R	C	S	I	D	N	O	J	E
C	C	U	J	K	C	H	S	K	P	E	J	T	V	P	Y	C	T	I	Y

DISCRIMINANT	OPTIMIZATION	ITERATION	PARAMETER
PERMUTATION	VECTOR	HEXADECIMAL	QUEUE
SAMPLING	LOGARITHM	RECURSION	METHOD
TANGENT	TRIGONOMETRY	STACK	QUICKSORT
DERIVATIVE	RADIAN	EFFICIENCY	INHERITANCE

More Info:

Check the CEMC at Home webpage on Wednesday, June 17 for the solution to Can You Find the Terms?

CEMC at Home

Grade 11/12 - Tuesday, June 16, 2020
Can You Find the Terms? - Solution

C	V	O	W	I	Y	J	I	V	L	Q	V	S	M	L	R	B	R	U	O
O	N	S	N	W	R	O	A	Z	S	F	G	M	D	D	G	A	Q	D	U
J	U	Y	S	W	T	Y	G	T	F	E	E	L	H	E	S	R	F	L	H
S	M	X	Y	A	E	V	A	Y	V	M	Q	U	S	R	H	I	N	D	U
U	N	M	C	H	M	Q	N	E	R	Y	Z	O	Z	I	B	L	Q	J	O
H	O	N	N	F	O	P	C	G	X	S	L	H	H	V	E	A	T	R	C
U	I	O	E	U	N	T	L	D	X	O	B	E	Y	A	C	G	O	E	S
K	T	I	I	R	O	O	S	I	G	I	W	X	W	T	N	S	V	T	U
Z	A	S	C	R	G	S	I	A	N	P	M	A	T	I	A	B	Z	E	T
L	R	R	I	M	I	K	R	T	E	G	J	D	J	V	T	H	N	M	J
T	E	U	F	V	R	I	U	R	A	V	L	E	W	E	I	O	A	A	I
G	T	C	F	J	T	W	M	J	J	Z	U	C	A	J	R	M	R	R	T
D	I	E	E	H	G	U	R	K	Y	E	I	I	Q	J	E	M	J	A	L
G	O	R	M	U	T	A	Q	F	U	Q	T	M	O	V	H	R	F	P	S
Y	Z	H	H	A	W	L	C	Q	D	L	Q	A	I	J	N	U	T	K	T
H	L	U	T	R	O	S	K	C	I	U	Q	L	N	T	I	M	W	H	A
M	J	I	P	E	I	R	A	D	I	A	N	R	G	G	P	V	W	I	C
B	O	K	P	U	M	C	B	Y	Z	N	S	C	L	C	E	O	H	J	K
N	N	E	G	T	N	A	N	I	M	I	R	C	S	I	D	N	O	J	E
C	C	U	J	K	C	H	S	K	P	E	J	T	V	P	Y	C	T	I	Y

DISCRIMINANT	OPTIMIZATION	ITERATION	PARAMETER
PERMUTATION	VECTOR	HEXADECIMAL	QUEUE
SAMPLING	LOGARITHM	RECURSION	METHOD
TANGENT	TRIGONOMETRY	STACK	QUICKSORT
DERIVATIVE	RADIAN	EFFICIENCY	INHERITANCE

CEMC at Home

Grade 11/12 - Wednesday, June 17, 2020
Piecing it Together

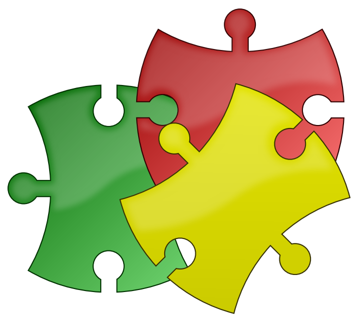
A positive integer is written on the back of each of three puzzle pieces. The numbers are not necessarily different but the sum of the three numbers is 14. Each puzzle piece is placed on a table so that the number cannot be seen. Alpha, Beta and Gamma each select one of the pieces, being careful not to let the other two see the number that is printed on the piece.

Alpha says, after looking at his puzzle piece, "I know that Beta and Gamma have different numbers."

Beta then says, "I already knew that all three numbers were different."

At this point, Gamma confidently exclaims, "I now know what all three of the numbers are!"

What were the numbers and who had which number?



More Info:

Check out the CEMC at Home webpage on Thursday, June 18 for a solution to Piecing it Together.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Wednesday, June 17, 2020
Piecing it Together - Solution

Problem:

A positive integer is written on the back of each of three puzzle pieces. The numbers are not necessarily different but the sum of the three numbers is 14. Each puzzle piece is placed on a table so that the number cannot be seen. Alpha, Beta and Gamma each select one of the pieces, being careful not to let the other two see the number that is printed on the piece.

Alpha says, after looking at his puzzle piece, “I know that Beta and Gamma have different numbers.”

Beta then says, “I already knew that all three numbers were different.”

At this point, Gamma confidently exclaims, “I now know what all three of the numbers are!”

What were the numbers and who had which number?

Solution:

We want to find the single solution to the problem $x + y + z = 14$ that satisfies the statements offered by Alpha, Beta and Gamma. It turns out that there are 78 different possible sums of three positive integers totalling 14. We could list all of the possible solutions and then proceed through the statements until we determine the required solution. Our approach will be far less exhausting. At the end of the solution, a justification of the existence of 78 possible positive integer solutions to the equation $x + y + z = 14$ will be provided.

The sum of three integers is 14, an even number. To generate an even sum, the three integers must all be even or one of the integers must be even and the other two integers must be odd.

Alpha says, after looking at his puzzle piece, “I know that Beta and Gamma have different numbers.” How can Alpha KNOW? If his number is even then Beta and Gamma could both have even numbers or both have odd numbers to generate the sum 14. For example, if Alpha had the number 6, Beta could have 6 and Gamma could have 2 or Beta could have 4 and Gamma could have 4. If his number was even, Alpha would not KNOW that the other two numbers were different. However, if Alpha had an odd number, then one of the others must have an odd number and the other must have an even number. In this case, Alpha would KNOW that Beta and Gamma have different numbers. Therefore, Alpha must have an odd number.

Beta then says, “I already knew that all three numbers were different.” Using the same logic as before, since Beta knows Alpha and Gamma have different numbers, Beta must have an odd number (and thus Gamma must have the even number). But how does Beta KNOW that all three numbers are different?

If Beta has a 1, 3 or 5, Alpha could have the same number. Beta would not know that all three numbers are different.

If Beta has a 7, 9, 11 or 13, Alpha could not have the same number in order for the three numbers to sum to 14. Furthermore, if Beta has a 7, Alpha must have a 5 or lower. If Beta has a 9, Alpha must have a 3 or lower. If Beta has an 11, Alpha must have a 1. Beta cannot have a 13 in order for the three numbers to sum to 14.

At this point our list of possible solutions has dropped from 78 to 6.

Beta	Alpha	Gamma
7	1	6
7	3	4
7	5	2
9	1	4
9	3	2
11	1	2

How does Gamma conclude, "I now know what all three of the numbers are!"?

If Gamma has a 4, Beta could have a 7 and Alpha could have a 3 or Beta could have a 9 and Alpha could have a 1. In this case, Gamma would not know what all three numbers are.

If Gamma has a 2, Beta could have a 7 and Alpha could have a 5 or Beta could have a 9 and Alpha could have a 3 or Beta could have an 11 and Alpha could have a 1. In this case, Gamma would not know what all three numbers are.

However, if Gamma has a 6, then Beta must have a 7 and Alpha must have a 1. This is the only possibility in which Gamma's statement is true.

Therefore, Alpha has a 1, Beta has a 7, and Gamma has a 6.

NOTE: It was mentioned at the beginning of the solution that there are 78 solutions to the equation $x+y+z=14$ where $x,\ y,\ z$ are positive integers. We can determine this by systematically counting the solutions:

x	y	z	# of possibilities
1	1	12	
1	2	11	
1	3	10	12
$\vdots$	$\vdots$	$\vdots$	
1	12	1	
2	1	11	
2	2	10	
2	3	9	11
$\vdots$	$\vdots$	$\vdots$	
2	11	1	
3	1	10	
3	2	9	
3	3	8	10
$\vdots$	$\vdots$	$\vdots$	
3	10	1	
$\vdots$	$\vdots$	$\vdots$	
10	1	3	
10	2	2	3
10	3	1	
11	1	2	
11	2	1	2
12	1	1	1

We can see that there are $12+11+10+9+8+7+6+5+4+3+2+1=78$ solutions to the equation $x+y+z=14$ where $x,\ y,\ z$ are positive integers.

CEMC at Home

Grade 11/12 - Thursday, June 18, 2020
Games and Puzzles

The CEMC has created lots of resources that we hope you have found interesting over the last few months. We also know that there are lots of online games and puzzles created by other organizations that make use of mathematics and logic. We've highlighted three examples below that you can explore for more mathematical fun!

Fraction Game from NCTM (https://www.nctm.org)

To make moves in this game, you need to use logic and number sense involving fractions.

Connect Three from NRICH (https://nrich.maths.org)

Use number sense to be the first to place three adjacent counters on the game board.

Kakuro Puzzles by Krazydad (https://krazydad.com)

A Kakuro puzzle is a kind of logic puzzle that is similar to a crossword puzzle. The objective of the puzzle is to insert a digit from 1 to 9 inclusive into each white cell such that the sum of the numbers in each "word" matches the clue associated with it and that no digit is duplicated in any "word".

You can find other interesting mathematics related games and puzzles online. Share your favourites using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Friday, June 19, 2020
Relay Day - Part 1

As part of the CEMC's Canadian Team Mathematics Contest, students participate in Math Relays. Just like a relay in track, you "pass the baton" from teammate to teammate in order to finish the race, but in the case of a Math Relay, the "baton" you pass is actually a number!

Read the following set of problems carefully.

Problem 1: The numbers $x+5$, 14, x, and 5 have an average of 9. What is the value of x?

Problem 2: *Replace N below with the number you receive.*
Each of the three lines having equations $x+Ny+8=0$, $5x-Ny+4=0$, and $3x-ky+1=0$ passes through the same point. What is the value of k?

Problem 3: *Replace N below with the number you receive.*
Quadrilateral $ABCD$ has vertices $A(0,3)$, $B(0,p)$, $C(N,10)$, and $D(N,0)$, where $p>3$ and $N>0$. The area of quadrilateral $ABCD$ is 50 square units. What is the value of p?

Notice that you can answer Problem 1 without any additional information.

In order to answer Problem 2, you first need to know the mystery value of N. The value of N used in Problem 2 will be the *answer* to Problem 1. (For example, if the answer you got for Problem 1 was 5 then you would start Problem 2 by replacing N with 5 in the problem statement.)

Similarly, you need the answer to Problem 2 to answer Problem 3. The value of N in Problem 3 is the *answer* that you got in Problem 2.

Now try the relay! You can use this tool to check your answers.

Follow-up Activity: Can you come up with your own Math Relay?
What do you have to think about when making up the three problems in the relay?
Can you just find three math problems and put them together to form a relay?

In Part 1 of this resource, you were asked to complete a relay on your own. But, of course, relays are meant to be completed in teams! In a team relay, three different people are in charge of answering the problems. Player 1 answers Problem 1 and passes their answer to Player 2; Player 2 takes Player 1's answer and uses it to answer Problem 2; Player 2 passes their answer to Player 3; and so on.

In Part 2 of this resource, you will find instructions on how to run a relay game for your friends and family. We will provide a relay for you to use, but you can also come up with your own!

CEMC at Home

Grade 4 to 12 - Friday, June 19, 2020
Relay Day - Part 2

Relay for Family and Friends

In Part 1 of this resource, you learned how to do a Math Relay. Now, why not try one out with family and friends!

You can put together a relay team and

- play just for fun, not racing any other team, or
- compete against another team in your household (if you have at least 6 people in total), or
- compete with a team from another family or household by
  - timing your team and comparing times with other teams to declare a winner, or
  - competing live using a video chat.

Here are the instructions for how to play.

Relay Instructions:

1. Decide on a team of three players for the relay. The team will be competing together.
2. Find someone to help administer the relay; let's call them the "referee".
3. Each teammate will be assigned a number: 1, 2, or 3. Player 1 will be assigned Problem 1, Player 2 will be assigned Problem 2, and Player 3 will be assigned Problem 3.
4. The three teammates should not see any of the relay problems in advance and should not talk to each other during the relay.
5. Right before the relay starts, the referee should hand out the correct relay problem to each of the players, with the problem statement face down (not visible).
6. The referee will then start the relay. At this time *all three players* can start working on their problems.

   Think about what Player 2 and Player 3 can do before they receive the value of N (the answer from the previous question passed to them by their teammate).
7. When Player 1 thinks they have the correct answer to Problem 1, they record their answer on the answer sheet and pass the sheet to Player 2. When Player 2 thinks they have the correct answer to Problem 2, they record their answer to the answer sheet and pass the sheet to Player 3. When Player 3 thinks they have the correct answer to Problem 3, they record their answer on the answer sheet and pass the sheet to the referee.

8. If all three answers passed to the referee are correct, then the relay is complete! If at least one answer is incorrect, then the referee passes the sheet back to Player 3.

9. At any time during the relay, the players on the team can pass the answer sheet back and forth between them, as long as they write nothing but their current answers on it and do not discuss anything. (For example, if Player 2 is sure that Player 1's answer must be incorrect, then Player 2 can pass the answer sheet back to Player 1, silently. This is a cue for Player 1 to check their work and try again.)

See the next page for a relay for family and friends! This includes instructions for the referee. You can also come up with your own relays to play. You can find many more relays from past CTMC contests on the CEMC's Past Contests webpage.

Sample answer sheets are provided below for you to use for your relays if you wish.

Answer Sheets:

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Relay For Three

Instructions for the Referee:

1. Multiple questions at different levels of difficulty are given for the different relay positions.
   - Assign one of the first three problems (marked "Problem 1") to Player 1.
   - Assign one of the next three problems (marked "Problem 2") to Player 2.
   - Assign one of the last three problems (marked "Problem 3") to Player 3.

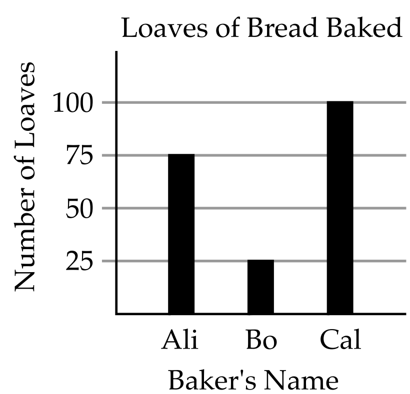
   Choose a problem so that each player is comfortable with the level of their question. The level of difficulty of each question is represented using the following symbols:

   - ● These questions should be accessible to most students in grade 4 or higher.
   - ■ These questions should be accessible to most students in grade 7 or higher.
   - ◆ These questions should be accessible to most students in grade 9 or higher.
2. Use this tool to find the answers for the relay problems in advance.

Relay Problems (to cut out):

Problem 1 ●

The graph shows the number of loaves of bread that three friends baked. How many loaves did Bo bake?

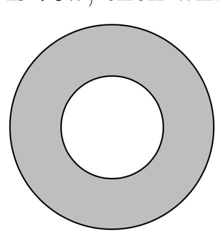


Problem 1 ■

An equilateral triangle has sides of length $x+4$, $y+11$, and 20. What is the value of $x+y$?

Problem 1 ◆

In the figure shown, two circles are drawn. If the radius of the larger circle is 10 and the area of the shaded region (in between the two circles) is 75π, then what is the *square* of the radius of the smaller circle?



Problem 2 ●

Replace N below with the number you receive.
Kwame writes the whole numbers in order from 1 to N (including 1 and N). How many times does Kwame write the digit '2'?

Problem 2 ■

Replace N below with the number you receive.
The total mass of three dogs is 43 kilograms. The largest dog has a mass of N kilograms, and the other two dogs have the same mass. What is the mass of each of the smaller dogs?

Problem 2 ◆

Replace N below with the number you receive.
The points $(6, 16)$, $(8, 22)$, and (x, N) lie on a straight line. Find the value of x.

Problem 3 ●

Replace N below with the number you receive.
You have some boxes of the same size and shape. If N oranges can fit in one box, how many oranges can fit in two boxes, in total?

Problem 3 ■

Replace N below with the number you receive.
One morning, a small farm sold 10 baskets of tomatoes, 2 baskets of peppers, and N baskets of zucchini. If the prices are as shown below, how much money, in dollars did the farm earn in total from these sales?

Basket of Tomatoes:	\$0.50
Basket of Peppers:	\$2.00
Basket of Zucchini:	\$1.00

Problem 3 ◆

Replace N with the number you receive.
Elise has N boxes, each containing x apples. She gives 12 apples to her sister. She then gives 20% of her remaining apples to her brother. After this, she has 120 apples left. What is the value of x?