CEMC at Home

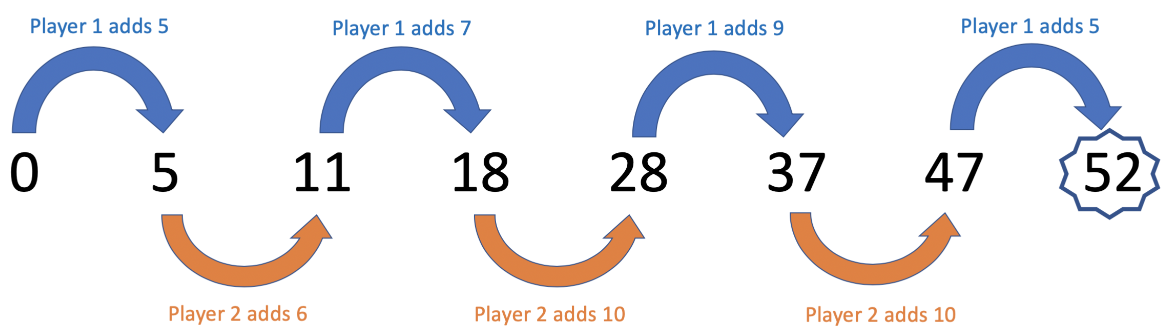
Grade 9/10 - Monday, March 23, 2020
Addition Magician

You Will Need:

- Two players
- A piece of paper and a pencil

How to Play:

1. Start with a total of 0 (on the paper).
2. The two players will alternate turns changing the total. Decide which player will go first.
3. On your turn, you can add 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 to the total.
   Numbers may be used more than once throughout the game.
4. The player who brings the total to 52 wins the game!



Play this game a number of times. Can you come up with a strategy that will allow you to win most of the time? What about every time? Is it better to go first or second or does this not matter?

Variations:

- How would your strategy change if the game was played to 55 instead of 52?
- What would your strategy be if the players are allowed to use the numbers from 1 to 15 but must play to 300 (instead of to 52)?
- What would your strategy be if the players are allowed to use the numbers from 1 to n, with $n > 1$, but must play to a total of T where T is some positive integer larger than $3n$?

More Info:

Check out the CEMC at Home webpage on Monday, March 30 for a discussion of a strategy for this game. We encourage you to discuss your ideas online using any forum you are comfortable with.

We sometimes put games on our math contests! Check out Question 2 on the 2003 Hypatia Contest for another game where we are looking for a strategy.

CEMC at Home
Grade 9/10 - Monday, March 30, 2020
Addition Magician - Solution

The Strategy

You likely noticed that the player that brings the total to 42, 43, 44, 45, 46, 47, 48, 49, 50, or 51 generally loses the game on the next turn. The next player can reach 52 by adding 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1, respectively, and so will win the game as long as they choose the correct number. Therefore, the player that brings the total to 41 is guaranteed to be able to bring the total to 52 on their next turn.

Using similar reasoning, the player that brings the total to 30 is guaranteed to be able to bring the total to 41 on their next turn. Also the player that brings the total to 19 is guaranteed to be able to bring the total to 30 on their next turn, and the player that brings the total to 8 is guaranteed to be able to bring the total to 19 on their next turn.

Putting all of this together, we see that there is a strategy that guarantees a win for the first player, regardless of what the second player does. (This is what is called a *winning strategy* for the game.) The first player starts by adding 8 to the total of 0. In the turns that follow, the first player will add whatever is needed to bring the totals to 19, 30, 41, and then 52. Our analysis above explains why this is always possible within the rules of the game.

Notice that the target numbers 8, 19, 30, 41, and 52 all differ by 11. We can describe the strategy more concisely as follows: Go first and start by adding 8. For all turns that follow, if the other player adds n, then you add $11 - n$.

The Variations

- In the first variation, the winning total is 55 which is a multiple of 11. A winning strategy in this variation is to go second and, on each turn, if the other player adds n, then add $11-n$, so that the total changes by 11 in total over the two turns. For example, if they add 4 then you add 7. This way the second player will bring the total to 11, 22, 33, 44, and then 55 to win.

- In the second variation, the player that brings the total to a number between 285 and 299 inclusive will generally lose the game since the next player can reach 300. Since the allowable numbers in this variation are 1 to 15, we focus on multiples of 16. Since 300 is 12 more than a multiple of 16, the winning strategy is to go first and start with 12. Then, if the other player adds n, you add $16 - n$, so that the total changes by 16 over the two turns. For example, if they add 7, you add 9. The first player can always bring the total to the next number that is 12 more than a multiple of 16, eventually reaching 300 to win.

- In the third variation, we need to consider different cases for T:

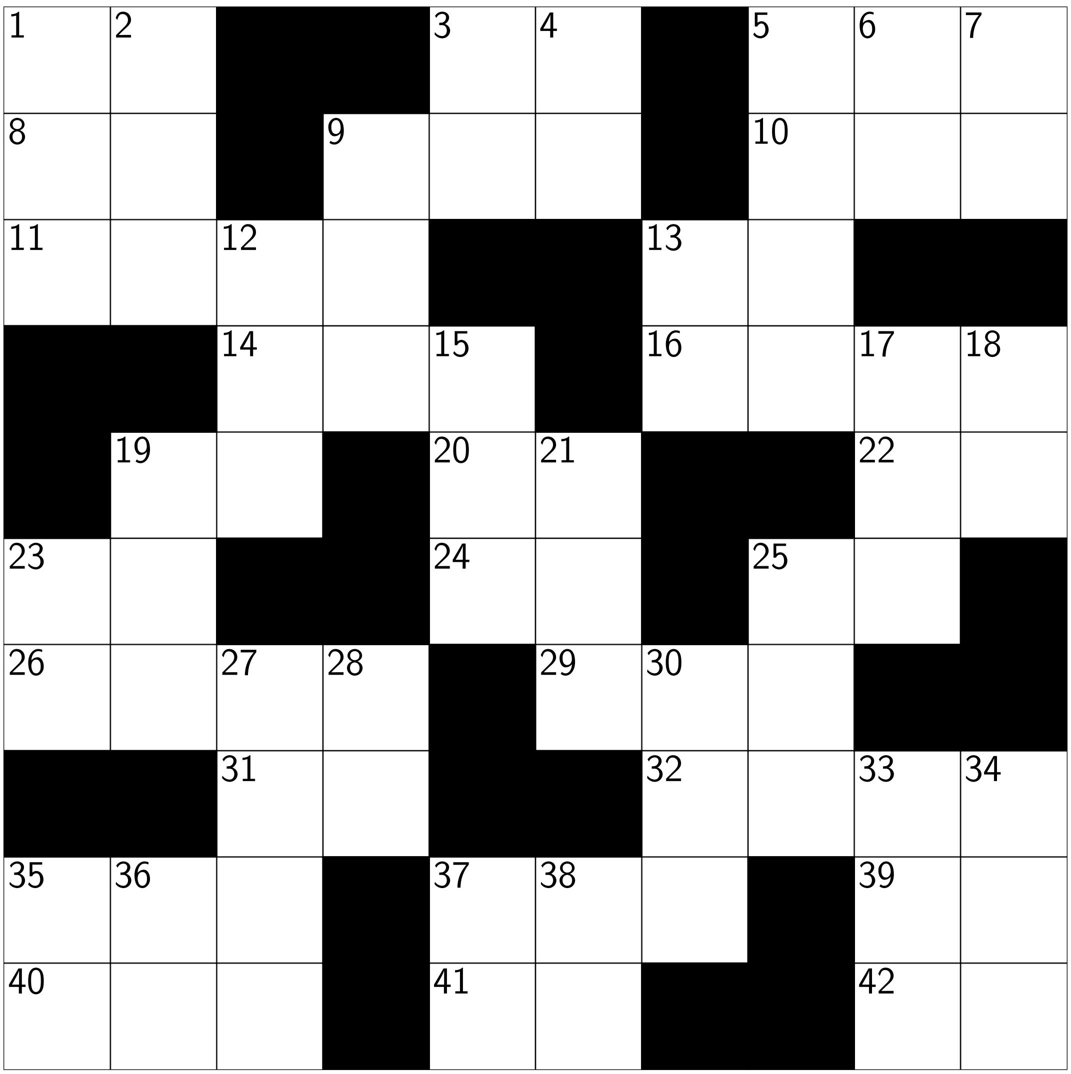
  If T is a multiple of $n + 1$, then go second. Whatever number the other player chooses, you choose the number that totals $n+1$ when summed with their chosen number. This means you will bring the total to each multiple of $n+1$, in turn, eventually reaching T. (The first variation above is an instance of this case.)

  If T is not a multiple of $n + 1$, then go first. Find the remainder when T is divided by $n + 1$ and start with this number. Whatever number the other player chooses, choose the number that totals $n + 1$ when summed with their chosen number. Eventually you will bring the total to T. (The second variation above is an instance of this case.)

CEMC at Home

Grade 9/10 - Tuesday, March 24, 2020
Crossnumber Puzzle

Use the clues on the next page to complete the crossnumber puzzle below. Each square of the grid will contain exactly one digit. Notice that some answers can be found using only the given clue, and some need the answers from other clues.
You may need to do a bit of research before you can figure out some of the clues!



More Info:

Check out the CEMC at Home webpage on Tuesday, March 31 for a solution to the Crossnumber Puzzle. We encourage you to spend some time discussing and investigating the references in this puzzle that are new to you.

Across

1. The sum of the squares of the first three primes.
3. The number of years the Grinch put up with the Whos' Christmas cheer.
5. A perfect cube.
8. With 31 ACROSS, a factor pair of 832.
9. 6!
10. A Mersenne prime.
11. The Hardy-Ramanujan number.
13. 25% of 300.
14. How much you spent if you received $4.17 in change from $10.
16. The prime factorization of 140.
19. A Fibonacci number.
20. A multiple of 11.
22. The smallest number in this grid.
23. The third side of a right triangle with hypotenuse 18 DOWN and other side 39 ACROSS.
24. A triangular number.
25. The number of bits in 5 bytes.
26. The number 9 in binary.
29. A palindrome.
31. The freezing point of water in degrees Fahrenheit.
32. MMDXIII.
35. 32 ACROSS - 16 ACROSS.
37. The balance after investing $100 at 3% simple interest for 8 years.
39. The same digit repeated.
40. The 11th, 12th, and 13th digits of pi.
41. ASCII value of lowercase b.
42. Atomic number of silver.

Down

1. Consecutive digits in decreasing order.
2. 1000 less than the year of Canada's Confederation.
3. The number of clues in this puzzle.
4. The number of edges in an icosahedron.
5. The digits of 32 ACROSS in reverse order.
6. The least common multiple of 6 and 7.
7. The number 55 in hexadecimal.
9. The last digit is the average of the first two digits.
12. The middle digit is the sum of the other two digits.
13. The smallest Achilles number.
15. The number of legs on a farm that has 24 chickens, 18 pigs, and 33 spiders.
17. The number of years in 5 centuries.
18. Sheldon Cooper's favourite number.
19. The sum of the interior angles of a triangle.
21. 9 ACROSS + 29 ACROSS.
23. The number of Mozart's last symphony.
25. The total value (in cents) of 9 quarters, 12 dimes, and 14 nickels.
27. Consecutive multiples of 3.
28. The number of sides in a dodecagon.
30. A perfect square.
33. Blaise Pascal's year of birth subtracted from Carl Gauss' year of birth.
34. Consecutive odd numbers.
35. The greatest common divisor of 13 ACROSS and 17 DOWN.
36. The number of minutes in 3480 seconds.
37. The sum of the digits of 11 ACROSS.
38. The number of days in 2 fortnights.

CEMC at Home

Grade 9/10 - Tuesday, March 24, 2020
Crossnumber Puzzle - Solution

3	8			5	3		3	4	3
2	6		7	2	0		1	2	7
1	7	2	9			7	5		
		5	8	3		2	2	5	7
	1	3		8	8			0	3
4	8			4	5		4	0	
1	0	0	1		1	3	1		
		3	2			2	5	1	3
2	5	6		1	2	4		5	5
5	8	9		9	8			4	7

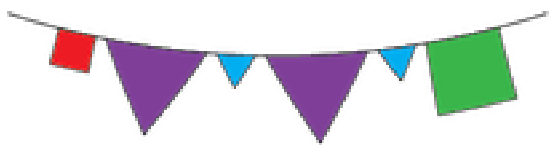
CEMC at Home

Grade 9/10 - Wednesday, March 25, 2020
Build A Banner

A computer program can be used to draw banners consisting of squares and triangles. The program makes use of the following five instructions:

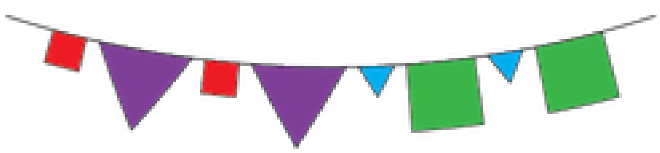
Instruction	Meaning
`S`	Draw a large square
`s`	Draw a small square
`T`	Draw a large triangle
`t`	Draw a small triangle
`N[I]`	Repeat the instructions, `I`, exactly `N` times

For example, the program `s 2[T t] S` draws the following banner:

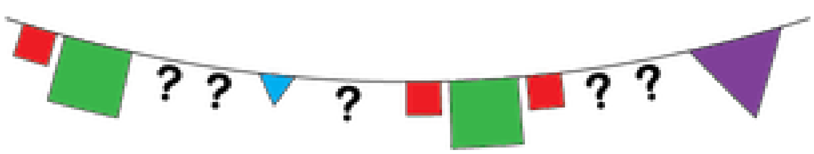


Questions:

1. Given the program `t 4[s] T 3[t S]`, draw the corresponding banner.
2. Create two different programs that will draw the following banner:

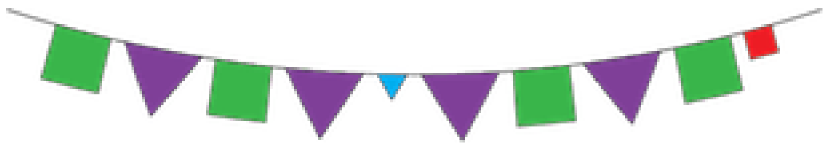


3. Given the program `2[2[s S] t T]`, draw the missing shapes in the following banner:

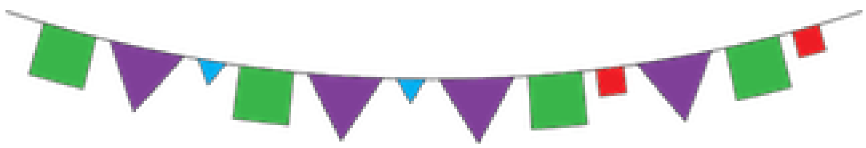


4. Given the incomplete program `?[2[?] t ?[s T ?]]`, complete the missing instructions in order to draw the following banner:

5. Suppose you want to draw the following banner:



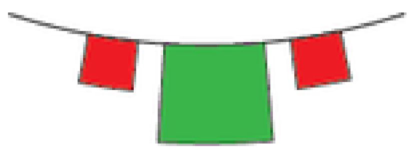
You create the program `2[S T t] 2[T S s]` which incorrectly draws this banner:



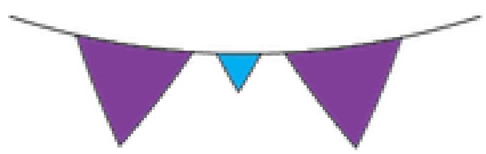
What are the mistakes in your program?

6. A new instruction named **if** is now available to you. The instruction `(a:b/c)` means that *if* the previous shape drawn was `a`, then the next shape drawn is `b`. *If* the previous shape drawn was **not** `a`, then the next shape drawn is `c`.

For example, the program `s (s:S/t) (t:T/s)` draws the following banner:



For each program in parts (a) to (f), decide whether or not it will draw the following banner:



(a) `2[T (t:T/t)]`
(b) `T (T:t/s) (t:T/S)`
(c) `T 2[(t:T/t)]`
(d) `t (t:T/s) (s:S/t)`
(e) `T (T:t/S) (S:s/T)`
(f) `3[(T:t/T)]`

7. Try creating your own new instructions. Perhaps add new shapes, or new capabilities such as chaining shapes vertically. Swap programs with a friend or family member and try to draw each other's banners.

More Info:

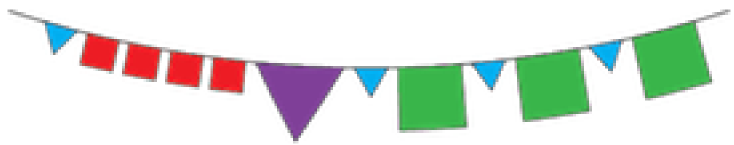
Check out the CEMC at Home webpage on Wednesday, April 1 for the solutions to these questions.

This task exercises your computational thinking muscles! For more information on how this task relates to computer science, check out Chain on the 2016 Beaver Computing Challenge.

CEMC at Home

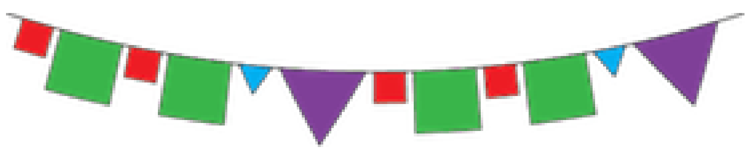
Grade 9/10 - Wednesday, March 25, 2020
Build a Banner - Solution

1. `t 4[s] T 3[t S]`



2. One possible program is `s T s T t S t S`.
   Another possible program is `2[s T] 2[t S]`.

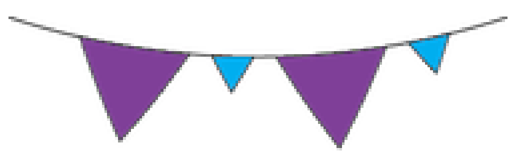
3. `2[2[s S] t T]`



4. `2[2[S] t 3[s T s]]`

5. The small triangle instruction, `t`, and the small square instruction, `s`, should be moved outside of their repeating blocks. The correct program is `2[S T] t 2[T S] s`.

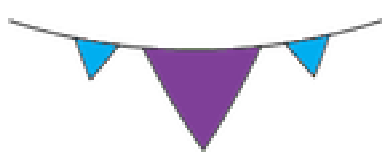
6. (a) No. The program draws this banner:



   (b) Yes

   (c) Yes

   (d) No. The program draws this banner:



   (e) Yes

   (f) No. The program is invalid and will not draw any banner. The first instruction is the new **if** instruction, but there is no previously drawn shape that it can use in order to make a decision about what to draw next.

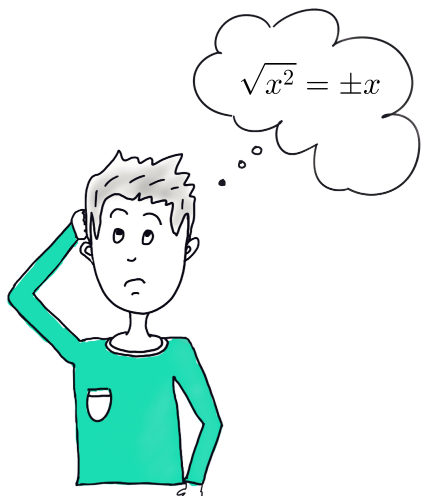
CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, March 26, 2020
What's that Total?

We know the following about the numbers a, b and c:

$$(a+b)^2 = 9,\ (b+c)^2 = 25,\ \text{and}\ (a+c)^2 = 81.$$

If $a+b+c \geq 1$, determine the **number** of possible values for $a+b+c$.

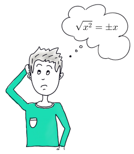


More Info:

Check the CEMC at Home webpage on Thursday, April 2 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 2.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem D and Solution

What's that Total?

Problem

We know the following about the numbers a, b and c:

$$(a+b)^2 = 9,\ (b+c)^2 = 25, \text{ and } (a+c)^2 = 81.$$

If $a+b+c \geq 1$, determine the **number** of possible values for $a+b+c$.

Solution

Since $(a+b)^2 = 9$, $a+b = \pm 3$. Since $(b+c)^2 = 25$, $b+c = \pm 5$. And since $(a+c)^2 = 81$, $a+c = \pm 9$.

Now $(a+b)+(b+c)+(a+c) = 2a+2b+2c = 2(a+b+c)$. This quantity is two times the value of the quantity we are looking for.

The following chart summarizes all possible combinations of values for $a+b$, $b+c$, and $a+c$ and the resulting values of $2a+2b+2c$ and $a+b+c$. The final column of the chart states a yes or no answer to whether the value of $a+b+c$ is ≥ 1.

$a+b$	$b+c$	$a+c$	$2a+2b+2c$	$a+b+c$	$a+b+c \geq 1$? (yes / no)
3	5	9	17	8.5	yes
3	5	−9	−1	−0.5	no
3	−5	9	7	3.5	yes
3	−5	−9	−11	−5.5	no
−3	5	9	11	5.5	yes
−3	5	−9	−7	−3.5	no
−3	−5	9	1	0.5	no
−3	−5	−9	−17	−8.5	no

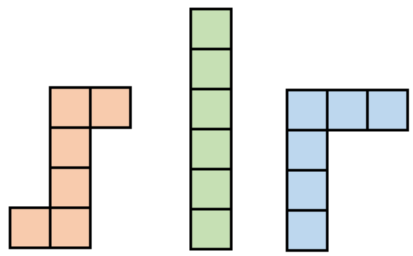
Therefore, there are three possible values of $a+b+c$ such that $a+b+c \geq 1$.

It should be noted that for each of the three possibilities, values for a, b, and c which produce each value can be determined but that was not the question asked.

CEMC at Home

Grade 9/10 - Friday, March 27, 2020
Hexominoes

A **hexomino** is a geometric shape composed of six equal-sized squares which are connected at one or more edges. Below are a few examples of hexominoes. Try drawing a few others yourself.

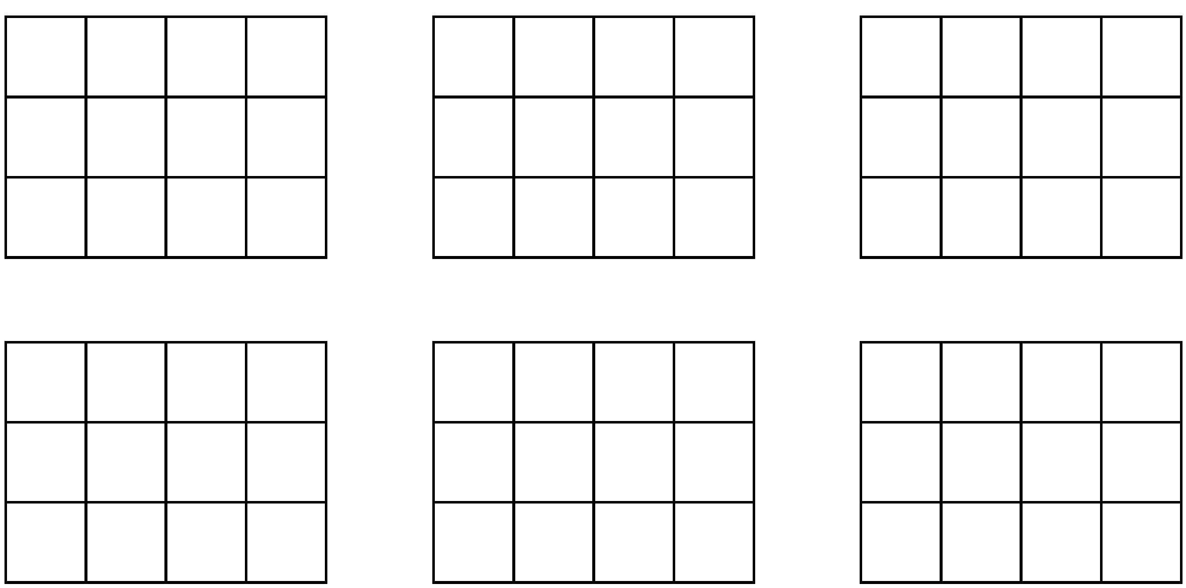


On the last page you will find 35 hexominoes drawn and numbered. Every other hexomino can be obtained by translating, rotating, or reflecting one of these 35 hexominoes, possibly using a combination of these transformations. In the following activities, you will be free to translate, rotate, and reflect the 35 shapes as needed to complete the tasks. The collection of shapes that we will be working with are sometimes called the 35 *free* hexominoes.

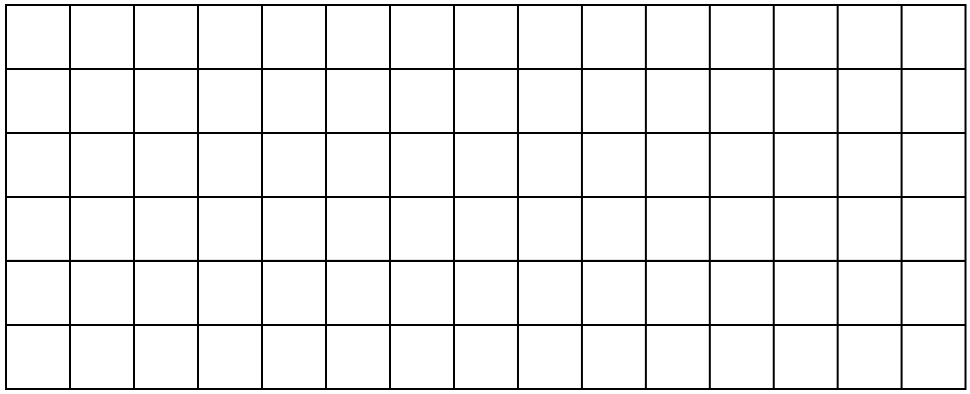
Questions:

1. Which of the 35 hexominoes represent the net of a cube? In other words, which hexominoes can be folded up into a cube? To help visualize this, you can print the hexominoes onto paper, cut them out, and fold them. Magnetic tiles would also work really well.

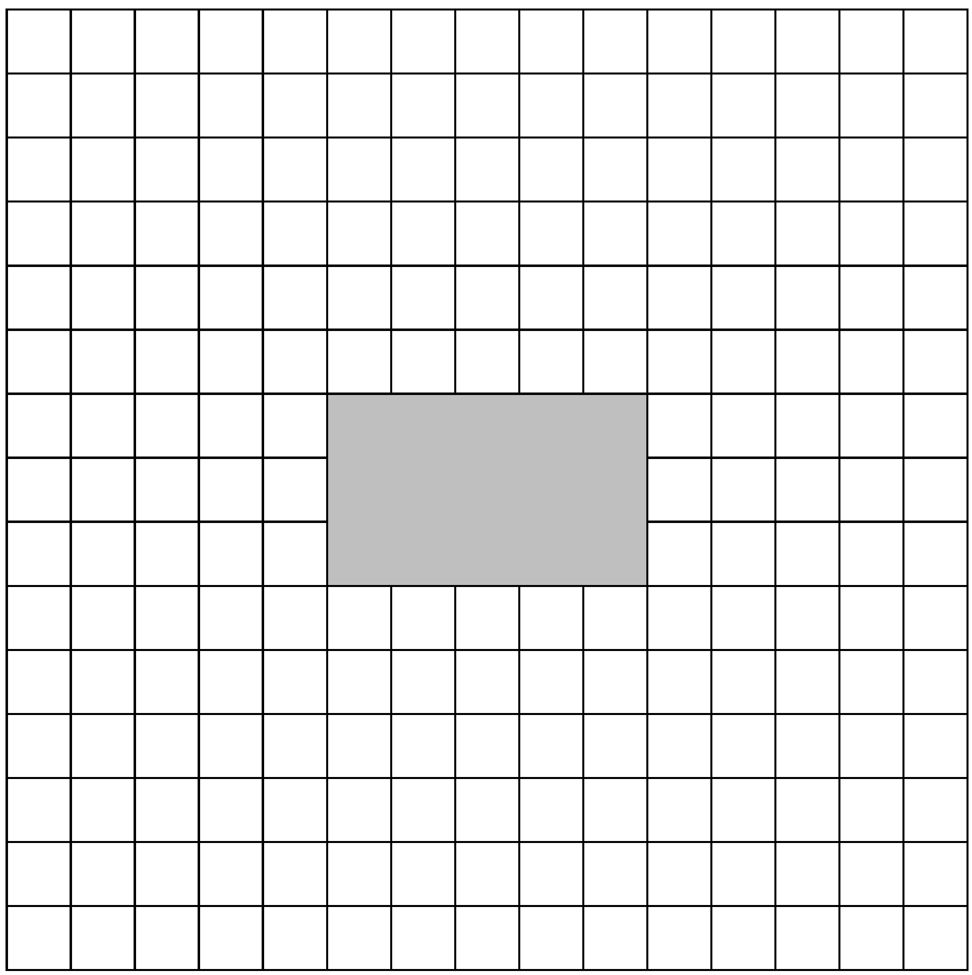
2. Cover a 3×4 rectangle using two copies of any single hexomino. How many different solutions can you come up with? (Remember you are free to translate, rotate, and/or reflect the shape.)

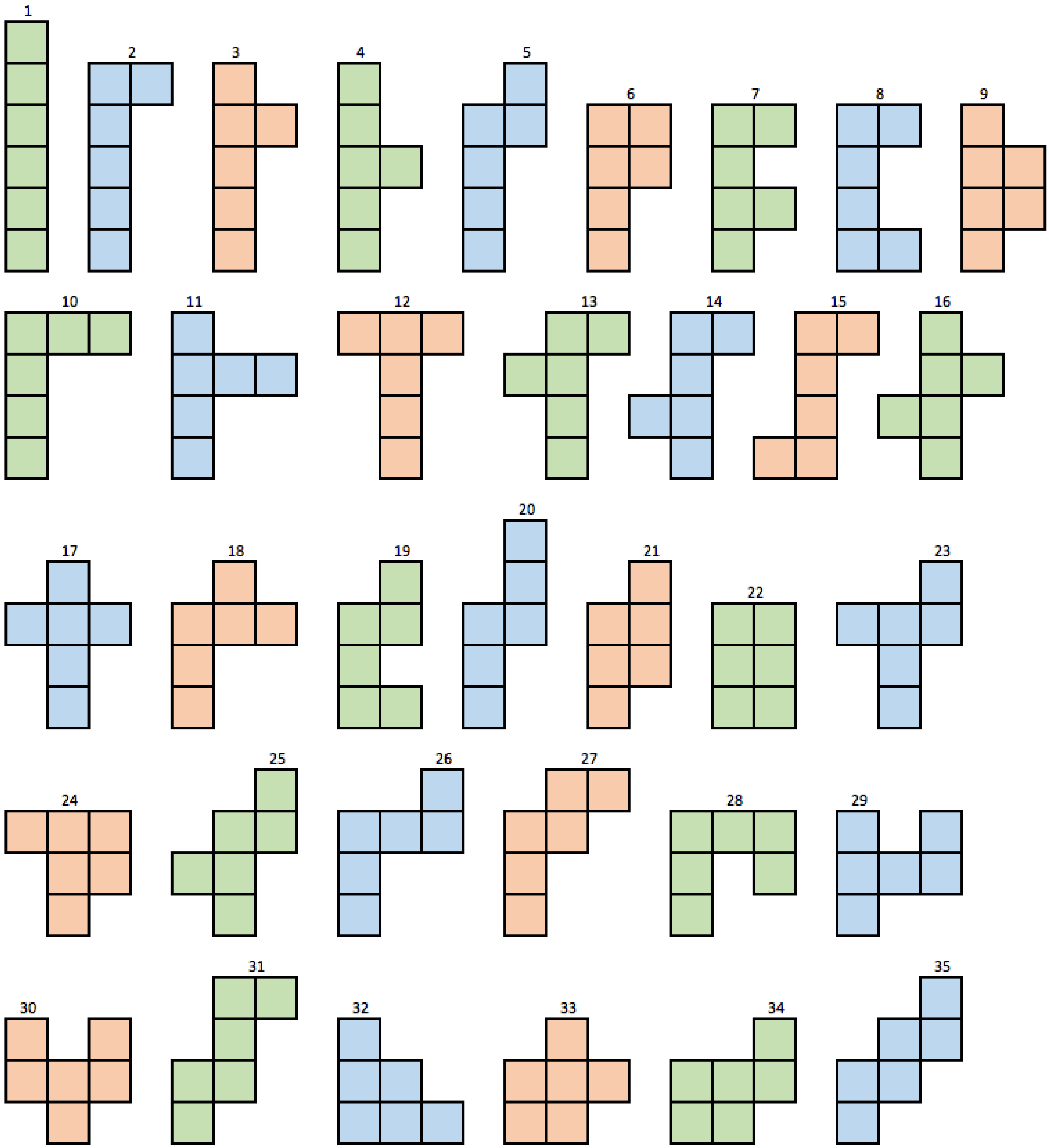


3. Cover the 6×15 rectangle below using any combination of hexominoes. Can you do so using each hexomino at most once?



4. Take a 15×15 square and cut out a 3×5 rectangle from the middle. Cover the remaining white squares using each of the 35 hexominoes exactly once.





More Info:

Check out the CEMC at Home webpage on Friday, April 3 for the solution to Hexominoes.

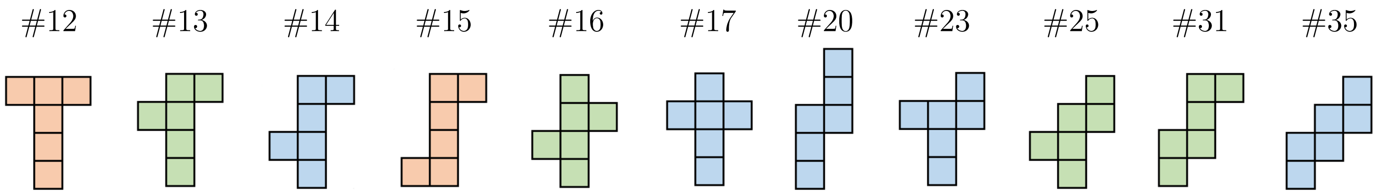
When four equal-sized squares are used instead of six, the geometric shapes are called **tetrominoes**. Tetrominoes are the building blocks of the original *Tetris* game.

CEMC at Home

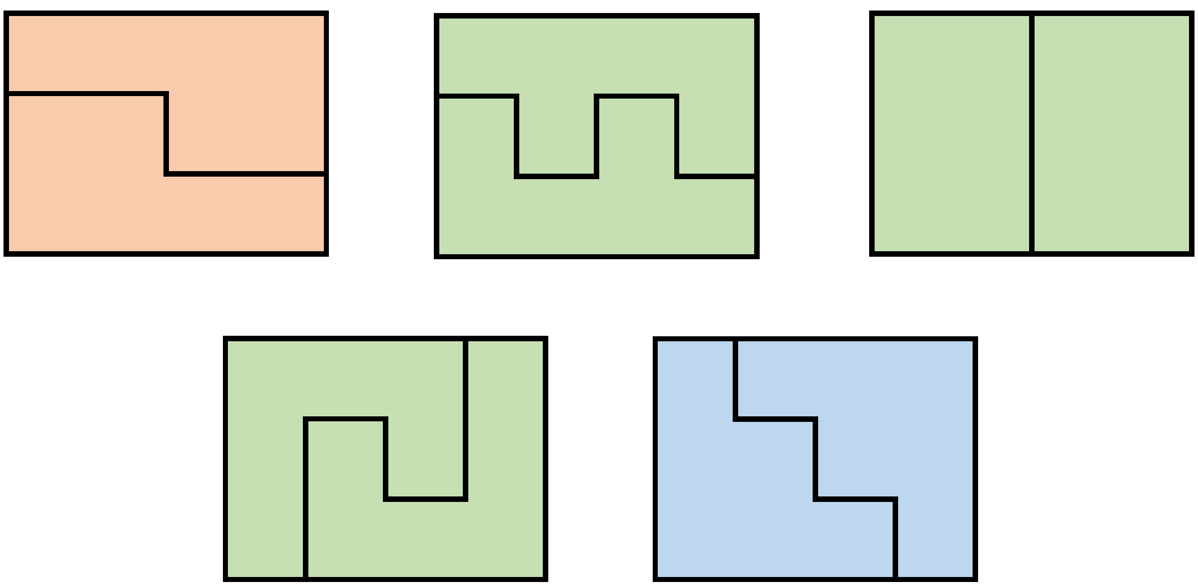
Grade 9/10 - Friday, March 27, 2020
Hexominoes - Solution

Answers:

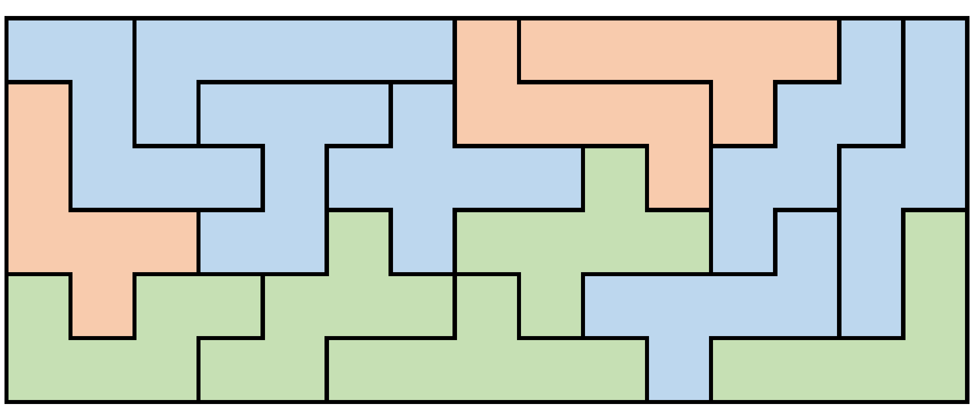
1. The following 11 hexominoes represent the net of a cube.



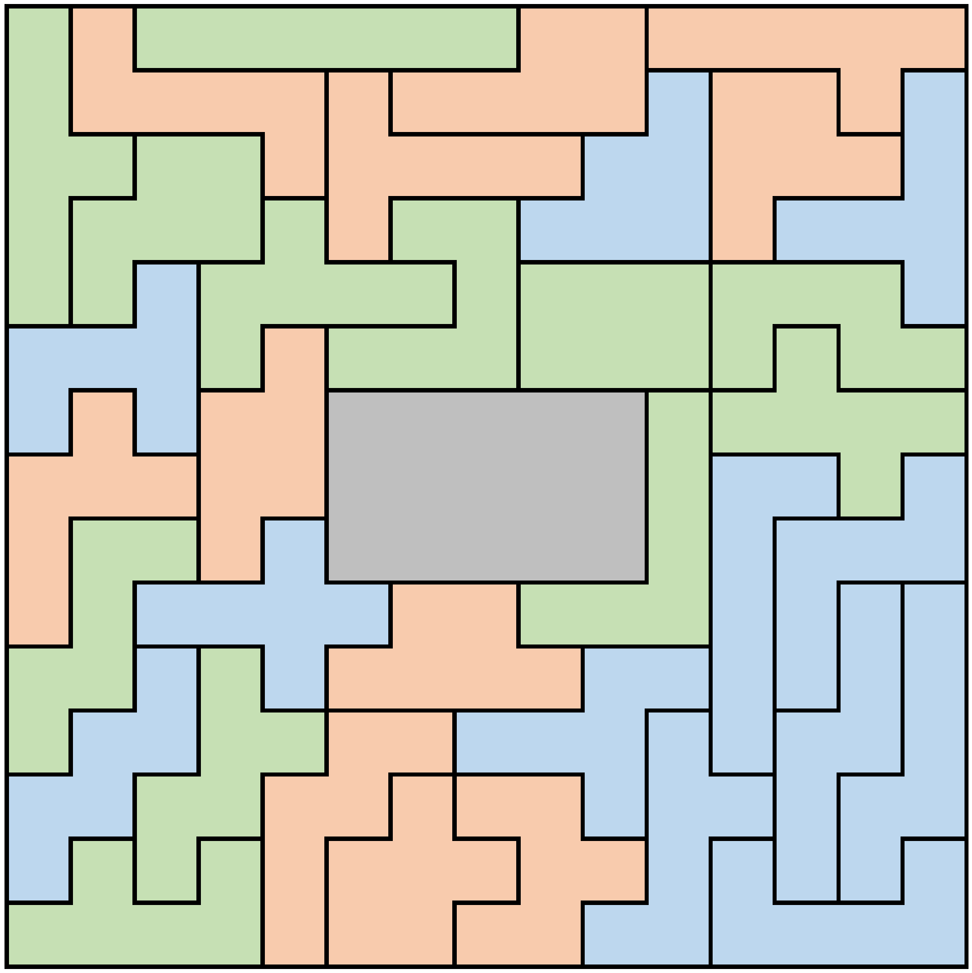
2. There are five different solutions using hexominoes #6, #7, #22, #28, and #32.



3. Here is one possible solution.



4. Here is one possible solution.



CEMC at Home

Grade 9/10 - Monday, March 30, 2020
Careful Clipping

You Will Need:

- Two players
- 10 paper clips
  (or other small objects)

How to Play:

1. Start with a pile of 10 paper clips.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On your turn, you can remove 1, 2 or 3 paper clips from the pile.
4. The player who removes the last paper clip, *loses*.

Play this game a number of times. Alternate which player goes first.
Can you determine a winning strategy* for this game?
Does the winning strategy depend on whether you are Player 1 or Player 2?

> * A *strategy* is a pre-determined set of rules that a player will use to play the game. The strategy dictates what the player will do for every possible situation in the game. It's a *winning strategy* if the strategy allows the player to always win, regardless of what the other player does.

Is there a connection between this game and the game we played on March 23 (Addition Magician)?

Variations:

A. Which player has a winning strategy if the game is won (instead of lost) by the player who removes the last paper clip? Describe this winning strategy.

B. Which player has a winning strategy if, in addition to variation A, players are instructed to instead take 1, 3 or 4 paper clips from the pile? Describe this winning strategy.

C. Which player has a winning strategy if, in addition to variations A and B, the pile starts with 14 paper clips? Describe this winning strategy.

More Info:

Check out the CEMC at Home webpage on Monday, April 6 for a solution to Careful Clipping.
We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 9/10 - Monday, March 30, 2020
Careful Clipping - Solution

The Strategy

Let the two players be Player 1 and Player 2.

You likely noticed that the player that brings the number of paper clips in the pile to 1 is guaranteed to win the game, and the player that brings the number of paper clips in the pile to 2, 3, or 4 generally loses the game. This is because the next player can bring the pile to 1 paper clip by removing 1, 2 or 3 paper clips, respectively.

Using similar reasoning, we can show that the player that brings the number of paper clips to 5 is guaranteed to be able to bring the number to 1 on their next turn, and the player that brings the number of paper clips to 9 is guaranteed to be able to bring the number to 5 on their next turn. This means that Player 1 has a winning strategy for this game and it goes as follows:

Start by removing 1 paper clip, reducing the total number of paper clips to 9. On your next turn, remove whatever number of paper clips are needed to bring the total to 5. On your turn after that, remove whatever number of paper clips are needed to bring the total to 1. (Our analysis above explains why each of these moves will be possible within the rules of the game.)

Notice that the "target numbers" (9, 5, and 1) all differ by 4. We can instead describe the strategy as follows: Go first and start by removing 1 paper clip. For all turns that follow, if the other player just removed n paper clips, then you remove $4 - n$ paper clips. (These two turns, combined, will reduce the number of paper clips by 4.)

The Variations

Variation A

The player that reduces the pile to 1, 2, or 3 paper clips will lose the game since the next player can remove all of the remaining paper clips. Therefore, you want to be the player that reduces the pile to 4 paper clips as you are guaranteed to be able to win the game on your next turn. Player 1 now has the following winning strategy: Start by removing 2 paper clips, reducing the pile to 8 paper clips. On your next turn, remove whatever number of paper clips are needed to bring the total to 4. On your turn after that, remove all remaining paper clips.

Variations B and C

In each of these variations, players can remove 1, 3, or 4 paper clips on their turn, and you *win* by removing the last paper clip from the pile. Player 1 has a winning strategy starting from 10 paper clips (Variation B) and Player 2 has a winning strategy starting from 14 paper clips (Variation C). We outline these strategies in the table on the next page by analyzing how to win the game starting with each of 1 through 14 paper clips, in turn. We give the first move(s) in each strategy and then give guidance on how to use earlier rows in the table to fill in the rest of the strategy.

Let the two players be Ally and Bri. In each game, Ally will go first.

Starting Pile	Winner	Reasoning
1	Player 1	Ally takes the clip and wins.
2	Player 2	Ally must take 1 clip. Bri takes the remaining clip and wins.
3	Player 1	Ally takes all 3 clips and wins.
4	Player 1	Ally takes all 4 clips and wins.
5	Player 1	Ally takes 3 clips, leaving 2 clips for Bri's turn. As above, if a pile has 2 clips, the second player will win. Since it is Bri's turn, the second player (starting from 2 clips) is Ally (Player 1).
6	Player 1	Ally takes 4 clips, leaving 2 clips for Bri's turn. As above, if a pile has 2 clips, the second player will win. Since it is Bri's turn, the second player (starting from 2 clips) is Ally (Player 1).
7	Player 2	Ally must take 1, 3, or 4 clips, leaving 6, 4, or 3 clips for Bri's turn. As above, if a pile has 6, 4, or 3 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).
8	Player 1	Ally takes 1 clip, leaving 7 clips for Bri's turn. As above, if a pile has 7 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
9	Player 2	Ally must take 1, 3, or 4 clips, leaving 8, 6, or 5 clips for Bri's turn. As above, if a pile has 8, 6, or 5 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).
10	Player 1	Ally takes 1 clip, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
11	Player 1	Ally takes 4 clips, leaving 7 clips for Bri's turn. As above, if a pile has 7 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
12	Player 1	Ally takes 3 clips, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
13	Player 1	Ally takes 4 clips, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
14	Player 2	Ally must take 1, 3, or 4 clips, leaving 13, 11, or 10 clips for Bri's turn. As above, if a pile has 13, 11, or 10 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).

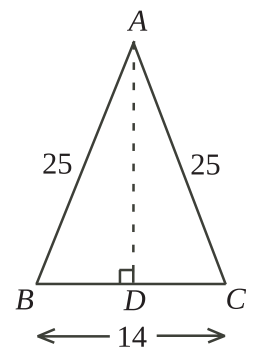
CEMC at Home

Grade 9/10 - Tuesday, March 31, 2020
Splitting Triangles

The CEMC offers many contests to inspire the next generation of mathematicians and computer scientists. Below is a favourite question from a past Fryer Contest (aimed at Grade 9 students). You can find a link to an additional question from a past Galois Contest (aimed at Grade 10 students) at the bottom of the page.

Fryer 2011 Contest, Question 2

In any isosceles triangle ABC with $AB = AC$, the altitude AD bisects the base BC so that $BD = DC$.



(a) (i) As shown in $\triangle ABC$, $AB = AC = 25$ and $BC = 14$. Determine the length of the altitude AD.

(ii) Determine the area of $\triangle ABC$.

(b) Triangle ABC from part (a) is cut along its altitude from A to D (Figure 1). Each of the two new triangles is then rotated $90°$ about point D until B meets C directly below D (Figure 2). This process creates a new triangle which is labelled PQR (Figure 3).

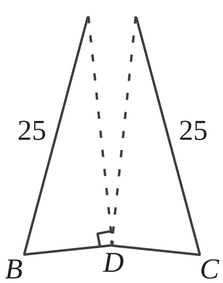


Figure 1

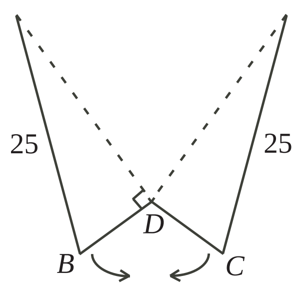


Figure 2

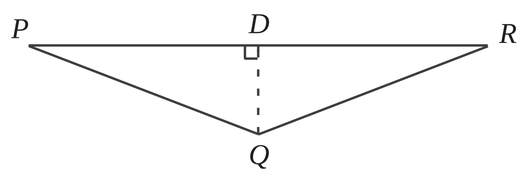
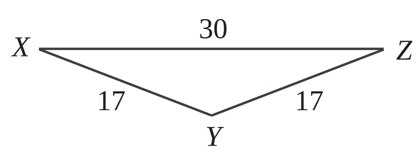


Figure 3

(i) In $\triangle PQR$, determine the length of the base PR.

(ii) Determine the area of $\triangle PQR$.

(c) There are two different isosceles triangles whose side lengths are integers and whose areas are 120. One of these two triangles, $\triangle XYZ$, is shown. Determine the lengths of the three sides of the second triangle.



More Info:

Check out the CEMC at Home webpage on Tuesday, April 7 for the solution to Splitting Triangles.

For an extra question from a past Galois Contest try Question 1 from the 2016 Galois Contest.

CEMC at Home

Grade 9/10 - Tuesday, March 31, 2020
Splitting Triangles - Solution

Fryer 2011 Contest, Question 2

(a) (i) Since $AB = AC$, then $\triangle ABC$ is isosceles. Therefore, the altitude AD bisects the base BC so that $BD = DC = \frac{14}{2} = 7$. Since $\angle ADB = 90^\circ$, then $\triangle ADB$ is right angled. By the Pythagorean Theorem, $25^2 = AD^2 + 7^2$ or $AD^2 = 25^2 - 7^2$ or $AD^2 = 625 - 49 = 576$, and so $AD = \sqrt{576} = 24$, since $AD > 0$.

(ii) The area of $\triangle ABC$ is $\frac{1}{2} \times BC \times AD$ or $\frac{1}{2} \times 14 \times 24 = 168$.

(b) (i) Through the process described, $\triangle ADB$ is rotated 90° counter-clockwise about D to become $\triangle PDQ$. Similarly, $\triangle ADC$ is rotated 90° clockwise about D to become $\triangle RDQ$. Through both rotations, the lengths of the sides of the original triangles remain unchanged. Thus, $PD = AD = 24$ and $RD = AD = 24$. Since P, D and R lie in a straight line, then base $PR = PD + RD = 24 + 24 = 48$.

(ii) When $\triangle ADC$ is rotated 90° clockwise about D, side DC becomes altitude DQ in $\triangle PQR$. Therefore, $DQ = DC = 7$. Thus, the area of $\triangle PQR$ is $\frac{1}{2} \times PR \times DQ$ or $\frac{1}{2} \times 48 \times 7 = 168$.

Note: The area of $\triangle PQR$ is equal to the area of $\triangle ABC$ from part (a)(ii). This is because $\triangle ABC$ is composed of $\triangle ADB$ and $\triangle ADC$, and $\triangle PQR$ is composed of rotated copies of these two right triangles.

(c) Since $XY = YZ$, then $\triangle XYZ$ is isosceles. Draw altitude YW from Y to W on XZ. Altitude YW bisects the base XZ so that $XW = WZ = \frac{30}{2} = 15$, as shown. Since $\angle YWX = 90^\circ$, then $\triangle YWX$ is right angled. By the Pythagorean Theorem, $17^2 = YW^2 + 15^2$ or $YW^2 = 17^2 - 15^2$ or $YW^2 = 289 - 225 = 64$, and so $YW = \sqrt{64} = 8$, since $YW > 0$. By reversing the process described in part (b), we rotate $\triangle XWY$ clockwise 90° about W and similarly rotate $\triangle ZWY$ counter-clockwise 90° about W. By the note at the end of the solution to part (b), the new isosceles triangle and the given isosceles triangle will have the same area. The new triangle formed has two equal sides of length 17 (since XY and ZY form these sides) and a third side having length twice that of YW or $2 \times 8 = 16$ (since the new base consists of two copies of YW).

CEMC at Home

Grade 9/10 - Wednesday, April 1, 2020
Where Am I?

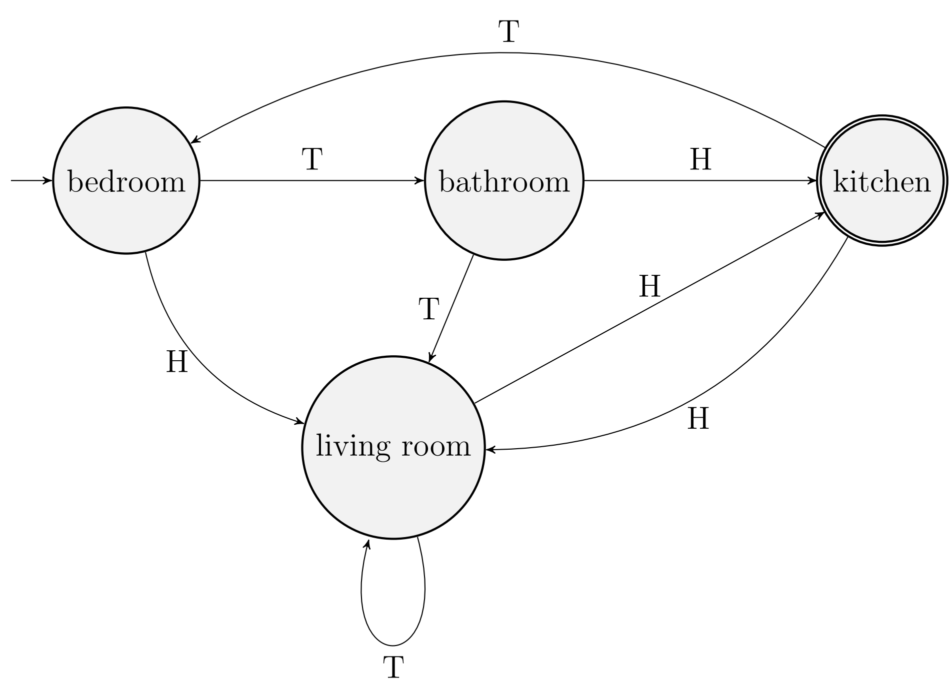
Let's play a game! This is a game without a *strategy* which makes it different from the other games we have played so far. All of our moves will be decided by tossing a coin. For this game you will need to label four different spaces as "bedroom", "bathroom", "kitchen", and "living room". The chosen spaces could be actual rooms, or four pieces of paper spread out around a single room, each having one of these labels. You will also need a coin.

Start in the bedroom (or standing on the paper labelled "bedroom") and toss the coin. Depending on the result of the coin toss, move according to the following rules:

- If you are in the bedroom and the coin lands "heads", move to the living room. If the coin lands "tails", move to the bathroom.
- If you are in the bathroom and the coin lands "heads", move to the kitchen. If the coin lands "tails", move to the living room.
- If you are in the kitchen and the coin lands "heads", move to the living room. If the coin lands "tails", move to the bedroom.
- If you are in the living room and the coin lands "heads", move to the kitchen. If the coin lands "tails", remain in the living room.

Continue the process of tossing the coin and moving from room to room until you have tossed the coin 10 times. The goal of this game is to end up in the kitchen. Which of the four rooms did you end up in?

The rules of this game can be illustrated using the following diagram.



This diagram is known as a *finite-state machine* (FSM). The circles (rooms) are the *states*. The arrows between circles are called *transitions* and they describe how to change states depending on input. The input in our game is a coin toss which resulted in either "heads" (`H`) or "tails" (`T`). The arrow coming from "nowhere" to a circle indicates the *start state*. In our game the start state is the bedroom. The double circle indicates an *accepting state*. An accepting state identifies a desired outcome. In our game, the desired outcome is the kitchen. Note that depending on your sequence of inputs (coin tosses) you may or may not have finished the game in the kitchen.

Finite-state machines are models. Using a finite-state machine to model a process allows you to analyze the process without having to actually implement the process.

Use the FSM model of our game to help you answer the following questions about our game.

Questions:

1. Biyu tosses the coin 6 times with the following results: `T H T H T H`. Which room does Biyu finish in?
2. Salmaan tosses the coin 10 times with the following results: `T T H T H H T T H T`. Which room does Salmaan finish in?
3. Leticia tosses the coin 7 times with the following results: `H H T T ? T T`. If Leticia finishes in the bathroom, what was the result of her 5th coin toss?
4. Pablo tossed the coin 12 times. His last coin toss landed "tails". Did Pablo finish in the kitchen? Yes, no, or maybe (depending on the actual sequence)?
5. Rashida tossed the coin 9 times. Her last coin landed "heads". Did Rashida finish in the kitchen? Yes, no, or maybe (depending on the actual sequence)?
6. Armando tossed the coin 3 times. Which room is it *not* possible for Armando to finish in?

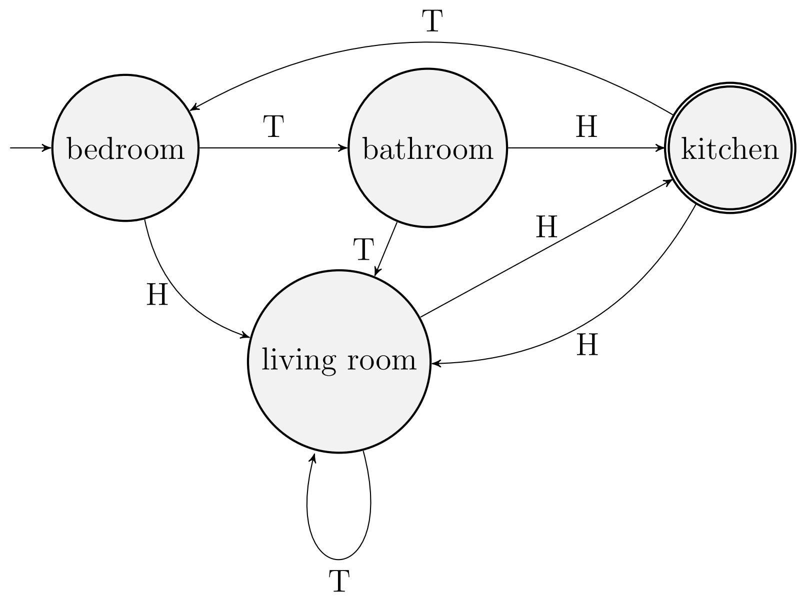
More Info:

Check out the CEMC at Home webpage on Wednesday, April 8 for the solution to Where Am I?

At a very abstract level, all computers are finite-state machines, moving from state to state depending on input received. To learn more about this topic, you can view videos of past Math Circles presentations such as the ones on Finite Automata recorded in Fall 2018.

CEMC at Home

Grade 9/10 - Wednesday, April 1, 2020
Where Am I? - Solution



Answers:

1.

START	T	H	T	H	T	H
bedroom	bathroom	kitchen	bedroom	living room	living room	kitchen

Biyu finishes in the kitchen.

2.

START	T	T	H	T	H	H	T	T	H	T
bed	bath	living	kitchen	bed	living	kitchen	bed	bath	kitchen	bed

Salmaan finishes in the bedroom.

3. After the first 4 coin tosses, Leticia must have ended up in the bathroom. Her 5th coin toss must have landed either "heads" or "tails". If it landed "heads", then she would have moved to the kitchen, and from the kitchen, her remaining 2 coin tosses would have taken her to the bathroom. If it landed "tails", then she would have moved to the living room, and her remaining 2 coin tosses would have kept her in the living room. Since we are told that Leticia finished in the bathroom, her 5th coin toss must have landed "heads".

4. Notice that the only way to arrive in the kitchen is for a coin toss to land "heads". Since Pablo's last coin toss landed on "tails", it is not possible for Pablo to finish in the kitchen.

5. It is not possible to tell which room Rashida finishes in without knowing the results of her other coin tosses. For example, suppose her first 8 coin tosses all land "heads". In this case, Rashida **will not** finish in the kitchen. However, if her first 8 coin tosses all land "tails", then Rashida **will** finish in the kitchen.

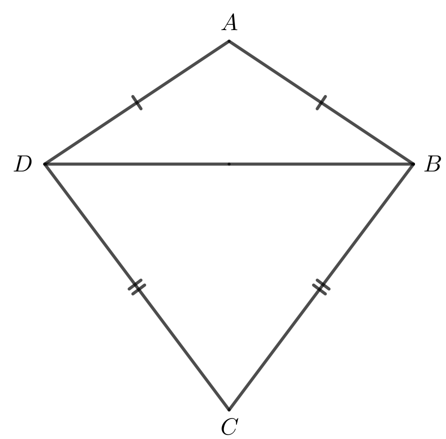
6. Armando's 3 coin tosses will result in one of 8 different combinations of "heads" and "tails". Trying each combination and tracking which room Armando finishes in reveals that for no combination does Armando finish in the bathroom.

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 2, 2020
Go Fly a Kite

Amanda wants to fly a kite. The kite is composed of two isosceles triangles, $\triangle ABD$ and $\triangle BCD$. The height of $\triangle BCD$ is 2 times the height of $\triangle ABD$, and the width of the kite, BD, is 1.5 times the height of the larger triangle.

If the area of the kite is 1800 cm^2, what is the perimeter of the kite?



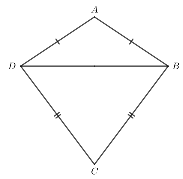
Did you know that in an isosceles triangle the altitude to the unequal side of the triangle bisects that unequal side?

More Info:

Check the CEMC at Home webpage on Thursday, April 9 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 9.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem D and Solution

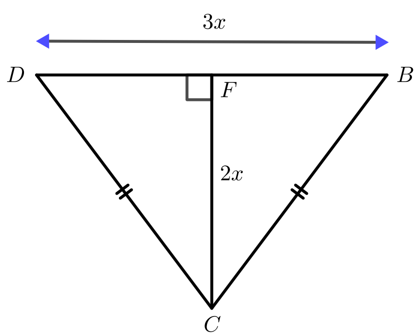
Go Fly a Kite

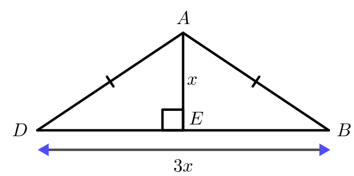
Problem

Amanda wants to fly a kite. The kite is composed of two isosceles triangles, $\triangle ABD$ and $\triangle BCD$. The height of $\triangle BCD$ is 2 times the height of $\triangle ABD$, and the width of the kite, BD, is 1.5 times the height of the larger triangle. If the area of the kite is 1800 cm^2, what is the perimeter of the kite?

Solution

Let the height of $\triangle ABD$ be $AE = x$. Therefore, the height of $\triangle BCD$ is $CF = 2x$. Also, the width of the kite is $BD = 3x$. Therefore, the base of each triangle is $3x$.





The area of $\triangle BCD = \dfrac{(3x)(2x)}{2} = 3x^2$ and the area of $\triangle ABD = \dfrac{(3x)(x)}{2} = \dfrac{3x^2}{2}$.
Also,

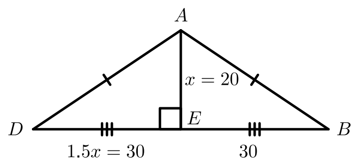
$$\begin{aligned}\text{area of kite } ABCD &= \text{area of } \triangle BCD + \text{area of } \triangle ABD\\ &= 3x^2 + \frac{3x^2}{2}\\ &= \frac{9x^2}{2}\end{aligned}$$

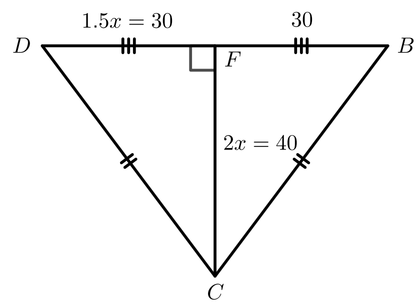
Therefore,

$$\begin{aligned}\frac{9x^2}{2} &= 1800\\ 9x^2 &= 3600\\ x^2 &= 400\\ x &= 20, \quad \text{since } x > 0\end{aligned}$$

Now, to find the perimeter of the kite, we need to find the lengths of the sides of the kite.

Since $\triangle ABD$ is isosceles, E will bisect BD and therefore
$DE = BE = 1.5x = 30$. This is shown below in the diagram to the left.
Similarily, $\triangle BCD$ is isosceles, F will bisect BD, and therefore
$DF = BF = 1.5x = 30$. This is shown below in the diagram to the right.





Using the Pythagorean Theorem in $\triangle AED$,

$$\begin{aligned} AD^2 &= 20^2 + 30^2 \\ &= 400 + 900 \\ &= 1300 \\ AD &= \sqrt{1300}, \text{ since } AD > 0. \end{aligned}$$

Also $AB = AD = \sqrt{1300}$ cm.

Similiarily in $\triangle DFC$,

$$\begin{aligned} DC^2 &= 30^2 + 40^2 \\ &= 2500 \\ DC &= 50, \text{ since } DC > 0. \end{aligned}$$

Also, $BC = DC = 50$ cm.

Now,

$$\begin{aligned} \text{the perimeter of the kite} &= \sqrt{1300} + \sqrt{1300} + 50 + 50 \\ &= 2\sqrt{1300} + 100 \\ &\approx 172.1 \end{aligned}$$

Therefore, the exact perimeter is $2\sqrt{1300} + 100$ cm or approximately 172.1 cm.

Note:

The expression $\sqrt{1300}$ can be simplified as follows:

$$\sqrt{1300} = \sqrt{100 \times 13} = \sqrt{100} \times \sqrt{13} = 10\sqrt{13}.$$

Therefore, the exact perimeter is

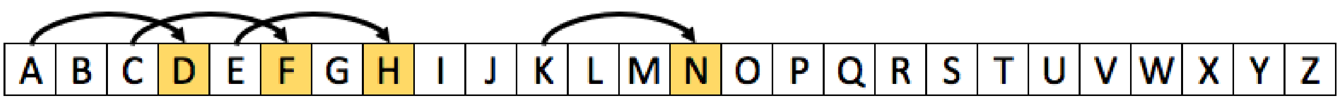
$$2\sqrt{1300} + 100 = 2(10\sqrt{13}) + 100 = 20\sqrt{13} + 100 \text{ cm}.$$

CEMC at Home

Grade 9/10 - Friday, April 3, 2020
Surprise Party

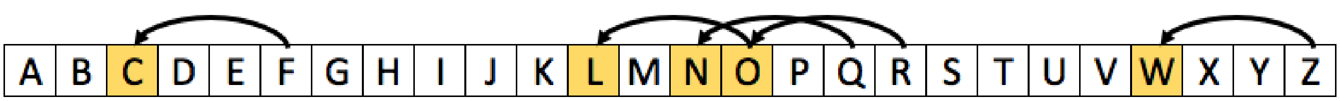
You are planning a surprise party for your friend, Eve. To prevent Eve from finding out about the details of the party, you and the other party planners have agreed to communicate in code. You have chosen to code your messages using a *substitution cipher* known as the *Caesar cipher*. A substitution cipher works by systematically replacing each letter (or symbol) in a message with a different letter (or symbol). A Caesar cipher involves "shifting" the alphabet.

In order to code messages using a Caesar cipher, your group first needs to choose an integer k from 1 to 25, inclusive. This integer k is called the *key* for the cipher, and determines by how many places the alphabet will be shifted. To *encrypt* a message (that is, to change the message from regular text to code) each letter in the message is replaced with the letter that appears k positions to the right in the alphabet. For example, to encrypt the message `C A K E` using a key of 3, the letter `C` is replaced with the letter `F`, which is 3 positions to the right, and the original message `C A K E` becomes the *encrypted* (or coded) message `F D N H`.



Note that if you cannot move k places to the right in the alphabet, then you wrap around to the beginning. For example, the letter 3 places to the right of `Y` is `B`.

To *decrypt* a message (that is, to change the code back to regular text) each letter in the coded message is replaced with the letter that appears k positions to the left, wrapping around if necessary. For example, to decrypt the message `F O R Z Q` using the same key of 3, the letter `F` is replaced with the letter `C`, and the coded message `F O R Z Q` can be revealed to be the message `C L O W N`.



For the questions below, consider making your own *Caesar Cipher Decoding Wheel* (see last page) to help you encrypt and decrypt. Alternatively, if you have some programming knowledge you can create a computer program that can encrypt and decrypt messages given some text and a key as input.

Questions:

1. Using a key of 6, encrypt the message `P A R T Y  S T A R T S  A T  S E V E N`.

2. Using a key of 24, decrypt the message `R F C  R F C K C  G Q  D Y L R Y Q W`.

3. The other party planners sent you the following message but the key got lost. Can you still decrypt the message? *Hint: What is the most commonly used letter in the English language?*

   `P F R P Y P  H T W W  M C T Y R  E S P  N L V P  L Y O  O P N Z C L E T Z Y D`

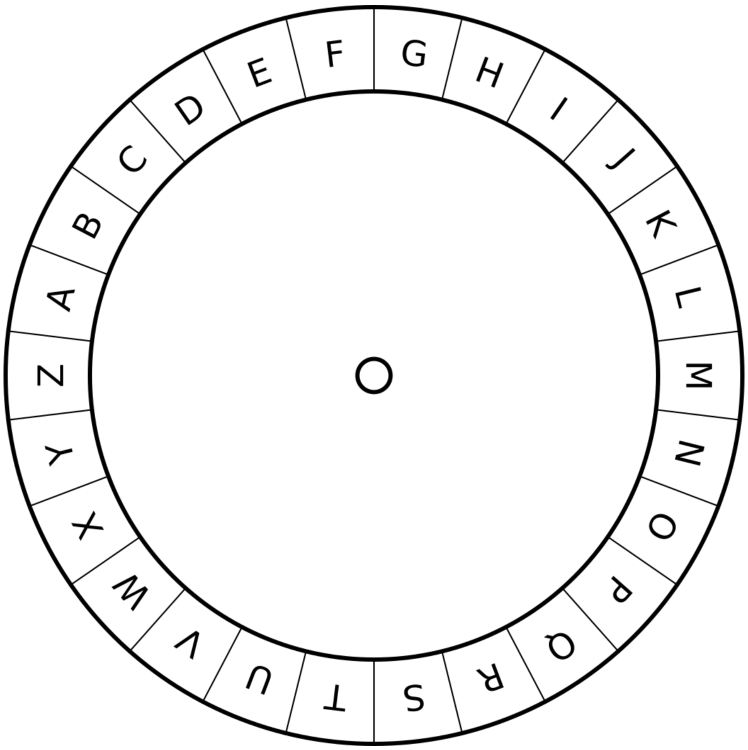
More Info:

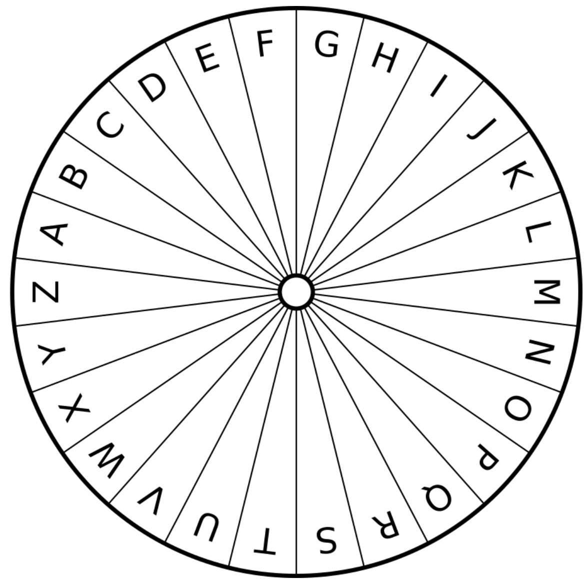
Check out the CEMC at Home webpage on Thursday, April 9 for the solution to Surprise Party.

For a slightly more challenging substitution cipher, check out the Vatsyayana Encryption Scheme.

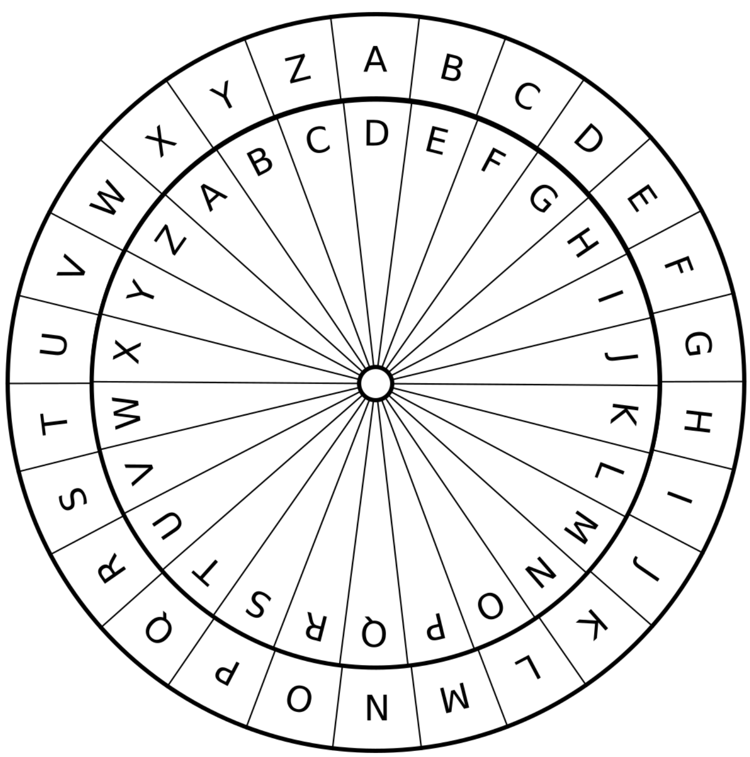
Caesar Cipher Decoding Wheel

Print and cut out the following two circles. Place the smaller circle on top of the larger circle and attach them through the middle using a paper fastener (brad).





Rotate the circles so that the `A`'s are aligned. Then set your key by rotating the inner circle **counter clockwise**. In the diagram below the key is set to 3.



You are now ready to encrypt and decrypt! To encrypt, replace each letter on the outer circle with the corresponding letter on the inner circle. To decrypt, replace each letter on the inner circle with the corresponding letter on the outer circle.

CEMC at Home

Grade 9/10 - Friday, April 3, 2020
Surprise Party - Solution

Answers:

1. The encrypted message is `V G X Z E  Y Z G X Z Y  G Z  Y K B K T`.

2. The decrypted message is `T H E  T H E M E  I S  F A N T A S Y`.

3. One way to decode a message that is encrypted using a Caesar Cipher, when the key is unknown, is to try all possible keys until one key produces a message that makes sense. There are only 25 possible keys, so this wouldn't take too long.

   A more clever way is to take advantage of letter frequencies in the English language. The most common letter in the English language is `E`. The most common letter in the encrypted message is `P`. This means that a good guess might be that the letter `E` has been shifted to the letter `P`. This would make the key equal to 11. Using a key of 11, the decrypted message is:

   `E U G E N E  W I L L  B R I N G  T H E  C A K E  A N D  D E C O R A T I O N S`

Note: *An attempt to break a substitution cipher by using knowledge of commonly used letters or phrases in a language, as we did above, is an example of what is called frequency analysis. For frequency analysis to be as reliable as possible, we want to study as much text, encrypted using the same cipher, as we can. If we have only a short message to work with, then it is very possible that the letter* `E` *will not be the most frequently occurring letter in the original message (and we will be tricked). If we have a very long message, or a very large quantity of messages, chances are good that within a few tries we will have found the right match for the letter* `E`.

CEMC at Home

Grade 9/10 - Monday, April 6, 2020

Sprouts

You Will Need:

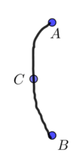
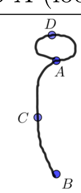
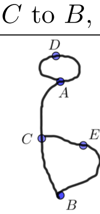
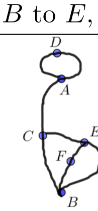
- Two players
- A piece of paper and a pencil

How to Play:

1. Start with two or three dots on the page, reasonably spaced out.
2. Players alternate turns. Decide which player will go first.
3. On your turn, do the following (if possible, according to the restrictions given in 4):

   Draw a curve joining two existing dots and add a dot to the newly drawn curve.
   Note that this curve can be drawn between two different dots, or in the form of a loop from one dot back to itself.
4. Here are the restrictions on the moves performed in 3:
   - You cannot draw a curve if it will result in a dot having more than three curve segments coming in or out of the dot. In particular, you cannot draw a loop on a dot that already has more than one curve segment coming in or out.
   - You cannot draw a curve if it will have to cross an existing curve.
   - The added dot cannot be placed on top of an existing dot.
5. The last person to successfully draw a new curve according to the rules wins the game!

An example of a complete game starting with 2 dots:

Start	Player 1 Joins A to B, adds C	Player 2 Joins A to A (loop), adds D	Player 1 Joins C to B, adds E	Player 2 Joins B to E, adds F
				

Notice that, after these four turns, Player 1 cannot draw a new curve. Player 1 cannot draw a curve from A since there are already three curve segments coming in or out of A (with two from the loop). This is the same for dots B, C, and E. Player 1 cannot join D to F since the curve would have to cross an existing curve, and cannot draw a loop on D or F as they each already have two curve segments coming in or out. Therefore, Player 2 wins!

Play the game a number of times starting with 2 dots. Keep track of the total number of turns it takes for each game to be won. Is there a certain number of turns after which the game is guaranteed to have ended?

Play the game a number of times starting with 3 dots. Is there a certain number of turns after which the game is guaranteed to have ended? How does this answer compare to your answer for the game starting with 2 dots?

More Info: Check the CEMC at Home webpage on Tuesday, April 14 for a discussion of Sprouts.

CEMC at Home

Grade 9/10 - Monday, April 6, 2020
Sprouts - Solution

Sprouts starting with 2 dots

While playing games of Sprouts starting with 2 dots, you may have noticed that each game ended after at most 5 turns. Did you also notice that each game included at least 4 turns?

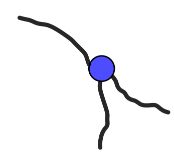
Sprouts starting with 3 dots

While playing games of Sprouts starting with 3 dots, you may have noticed that each game ended after at most 8 turns. Did you also notice that each game included at least 6 turns?

We encourage you to think about each of these observations and see if you can explain why this happened. We provide a discussion below to explain why any game starting with 3 dots must end after at most 8 turns.

A game of Sprouts starting with 3 dots must end after at most 8 turns

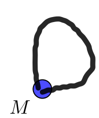
First, we note that the game is over as soon as no dots in the game can be part of a newly drawn curve. Remember that all dots in the game can have at most three curve segments attached to them. We will think of each dot as having three "slots" that can be filled. Curve segments can be attached to a dot M in a few different ways as shown below.



Curve drawn from M to another dot

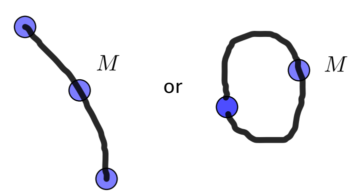
This curve takes up one slot on dot M

Loop drawn at M



This loop takes up two slots on dot M

M added to a newly drawn curve



This curve/loop takes up two slots on dot M

The game starts with three dots and no curves. Since there are three dots in total, each having three slots, the game starts with nine available slots.

After one turn is complete, one of two things has happened:

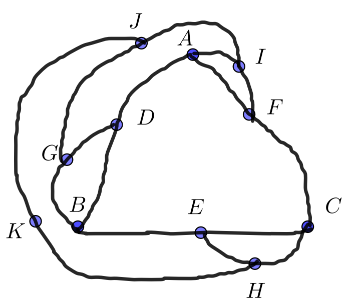
- a loop was added to one of the three dots, and a fourth dot was added to this loop, or
- a curve was drawn between two of the three dots, and a fourth dot was added to this curve.

In either case, after the first turn, there will be eight available slots remaining. Here we explain why:

Drawing a loop on a dot fills two of the three available slots on that particular dot, reducing us to $9-2=7$ slots available for the three original dots. However, a fourth dot is also added to this loop. Since two slots of this new dot are already taken, there is exactly one slot open on this new dot. This means there are $7+1=8$ slots available among the four dots now in the game. Notice that the situation is similar if a curve is drawn from one dot to another dot: we fill two slots (one on each dot at the ends of this curve), but gain one new slot from the fourth dot that is added.

In a similar way, we can argue that for *each turn that follows*, there is a net total loss of one slot per turn. After turn 2 there are 7 slots left, after turn 3 there are 6 slots left, and so on. If the game makes it to turn 8, then there can be only 1 slot left after turn 8 is complete. Since there must be at least 2 slots available in order for a new curve to be drawn, we can be sure that there is no legal move to make on turn 9. Therefore, we see that any game starting with 3 dots must end after at most 8 turns.

Did one of your games last exactly 8 turns? Note that our discussion above does not argue that a game can actually make it all the way to 8 turns, just that it is impossible for a game to make it to 9 turns. Below is an example of a game that lasted exactly 8 turns, which shows that 8 is the *maximum* number of turns attainable in a game starting with 3 dots.



Move	Endpoints	Added Point
1	A and B	D
2	B and C	E
3	A and C	F
4	B and D	G
5	E and C	H
6	A and F	I
7	G and I	J
8	H and J	K

Now try the following on your own:

- Explain why a game of Sprouts starting with 3 dots must last for at least 6 turns.
- Determine if a game of Sprouts starting with 3 dots can end in exactly 6 turns.

The game of Sprouts was invented by mathematicians John H. Conway (who died recently) and Michael S. Paterson at Cambridge University.

CEMC at Home

Grade 9/10 - Tuesday, April 7, 2020
Sum Code

In this puzzle, every letter of the alphabet represents a different integer from 1 to 26. Your task is to figure out which number is assigned to each letter. To get you started, you are given that H = 20 and N = 17. Use the algebraic equations to crack the code and figure out the remaining assignments.

A	B	C	D	E	F	G	H	I	J	K	L	M
							20					
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
17												

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ~~17~~ 18 19 ~~20~~ 21 22 23 24 25 26

Algebraic Equations

$E = D \times D$

$B = E - D$

$H = E + D$

$B = T \times D$

$H = D \times C$

$J = C - T$

$V = C \times C$

$Y \times Y = P + I$

$Y + M = P - Y$

$P = V + 1$

$R = F - R$

$S = R - J$

$A = K + L$

$U = K \times T$

$Z = O + W - K$

$O = W + C$

$X = T \times C$

$Q = G - N + U$

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for a solution to Sum Code.

CEMC at Home

Grade 9/10 - Tuesday, April 7, 2020
Sum Code - Solution

Answers

A	B	C	D	E	F	G	H	I	J	K	L	M
8	12	5	4	16	22	19	20	10	2	7	1	14
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
17	18	26	23	11	9	3	21	25	13	15	6	24

Explanation

Since we are told that H = 20, and there are two equations involving the letter H, a good place to start is with these equations.

E = D × D	V = C × C	A = K + L
B = E − D	Y × Y = P + I	U = K × T
H = E + D	Y + M = P − Y	Z = O + W − K
B = T × D	P = V + 1	O = W + C
H = D × C	R = F − R	X = T × C
J = C − T	S = R − J	Q = G − N + U

The equation H = D × C tells us that D and C are a factor pair of 20. This means they could be 2 and 10 (in some order) or 4 and 5 (in some order). Note that they cannot be 1 and 20. (Why not?)

The equations E = D × D and V = C × C tell us more about this factor pair. If the factor pair is 2 and 10, then E and V are 4 and 100 (in some order). This is not possible since the numbers in this code only range from 1 to 26. Therefore, it must be the case that the factor pair D and C are 4 and 5 (in some order) which means that E and V are 16 and 25 (in some order).

Suppose D = 5 and C = 4. Then E = 25 and V = 16. Using the equation H = E + D we get that H = 25 + 5 = 30 which we know must be false. Since this is not the correct order of the factor pair, we know we must have D = 4 and C = 5. In this case, we get E = 16 and V = 25. We confirm with equation H = E + D that we get H = 20 as expected.

We now know for certain the values of D, C, E and V. By substituting these values into all of the relevant equations above, we can also determine the values for B, T, J, P, and X.

To proceed further, consider the equation Y × Y = P + I. This tells us that P + I is a perfect square. What does this tell us about possible values for I and Y? What does this information, combined with the equation Y + M = P − Y, reveal about the value of M?

By substituting values we already know into equations, and combining equations that contain common letters, we can proceed to crack the rest of the code, as indicated in the answer key above.

CEMC at Home

Grade 9/10 - Wednesday, April 8, 2020

Buying Local

Last week five people (Charlie, Manuel, Priya, Sal, and Tina) shopped at a local farmers' market. Each person went on a different day (Monday through Friday), bought a different item (carrots, blueberries, tomatoes, apples, or potatoes), and spent a different amount of money ($1.50, $2.00, $2.50, $3.50, or $3.75).

Use the clues below to determine who went on each day, what they bought, and for how much.

1. Sal went to the market two days before Priya.
2. Manuel spent $3.75 at the market the day before someone bought tomatoes.
3. Charlie paid $2.50 for carrots the day after someone spent $3.50.
4. The person who went on Wednesday bought apples.
5. Someone bought potatoes for $2.00 on Monday

You may find the following table useful in organizing your solution.

	Monday	Tuesday	Wednesday	Thursday	Friday	Carrots	Blueberries	Tomatoes	Apples	Potatoes	$1.50	$2.00	$2.50	$3.50	$3.75
Charlie															
Manuel															
Priya															
Sal															
Tina															
$1.50															
$2.00															
$2.50															
$3.50															
$3.75															
Carrots															
Blueberries															
Tomatoes															
Apples															
Potatoes															

More Info:

Check out the CEMC at Home webpage on Wednesday, April 15 for a solution to Buying Local.

This type of puzzle is known as a logic puzzle and we sometimes include them in Problem of the Week. Here is one called Winter Carnival Event.

CEMC at Home
Grade 9/10 - Wednesday, April 8, 2020
Buying Local - Solution

Answer

- Charlie bought carrots for $2.50 on Friday
- Manuel bought apples for $3.75 on Wednesday
- Priya bought tomatoes for $3.50 on Thursday
- Sal bought blueberries for $1.50 on Tuesday
- Tina bought potatoes for $2.00 on Monday

Explanation

There are many different ways to arrive at the answers above. You may have used the chart provided with the problem to keep track of matches that were confirmed or deemed impossible while examining and combining the different clues. Below we present an explanation in words only. It may be helpful to follow along by filling out the chart given with the problem as you read.

The apples were purchased on Wednesday (clue 4). The potatoes were purchased for $2.00 on Monday (clue 5). The carrots were not purchased on Tuesday because then the potatoes would cost $3.50 instead of $2.00 (clue 3). So the carrots were purchased for $2.50 on either Thursday or Friday. The tomatoes were not purchased on Tuesday because then the potatoes would cost $3.75 instead of $2.00 (clues 2 and 5). The tomatoes were not purchased on Friday because if they were, then the carrots were purchased on Thursday, and the carrots would cost $3.75 instead of $2.50 (clues 2 and 3).

So the potatoes were purchased for $2.00 on Monday, the blueberries were purchased on Tuesday, the apples were purchased on Wednesday, the tomatoes were purchased on Thursday, and the carrots were purchased for $2.50 on Friday.

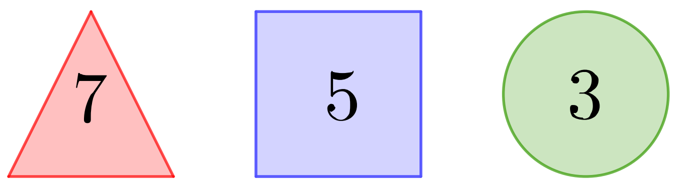
Manuel spent $3.75 on Wednesday purchasing apples (clue 2). Charlie spent $2.50 on Friday purchasing carrots, and someone spent $3.50 on Thursday purchasing tomatoes (clue 3). Since the potatoes cost $2.00 (clue 5) someone spent $1.50 on Tuesday purchasing blueberries. Priya, Sal, and Tina went to the market on Monday, Tuesday, and Thursday in some order. Since Sal went two days before Priya (clue 1), then it must be the case that Sal went on Tuesday, Priya went on Thursday, and Tina went on Monday.

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 9, 2020
Tokens Taken

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.



You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

More Info:

Check the CEMC at Home webpage on Thursday, April 16 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 16.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

7 5 3

Problem of the Week
Problem D and Solution
Tokens Taken

Problem

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

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You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

Solution

Solution 1

There are 22 different numbers which can be chosen from the green bag, 15 different numbers which can be chosen from the red bag, and 10 different numbers which can be chosen from the blue bag. So there are a total of $22 \times 15 \times 10 = 3300$ different combinations of numbers which can be produced by selecting one token from each bag.

To count the number of possibilities for a 5 to appear on at least two of the tokens, we will consider cases.

1. Each of the selected tokens has a 5 on it.

   This can only occur in 1 way.

2. A 5 appears on the green token and on the red token but not on the blue token.

   There are 9 choices for the blue token excluding the 5. A 5 can appear on the green token and on the red token but not on the blue token in 9 ways.

3. A 5 appears on the green token and on the blue token but not on the red token.

   There are 14 choices for the red token excluding the 5. A 5 can appear on the green token and on the blue token but not on the red token in 14 ways.

4. A 5 appears on the red token and on the blue token but not on the green token.

   There are 21 choices for the green token excluding the 5. A 5 can appear on the red token and on the blue token but not on the green token in 21 ways.

Summing the results from each of the cases, the total number of ways for a 5 to appear on at least two of the tokens is $1 + 9 + 14 + 21 = 45$. The probability of 5 appearing on at least two of the tokens is $\frac{45}{3300} = \frac{3}{220}$.

Solution 2

This solution uses a known result from probability theory. If the probability of event A occurring is a, the probability of event B occurring is b, the probability of event C occurring is c, and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.

The probability of a specific number being selected from the green bag is $\frac{1}{22}$ and the probability of any specific number not being selected from the green bag is $\frac{21}{22}$.

The probability of a specific number being selected from the red bag is $\frac{1}{15}$ and the probability of any specific number not being selected from the red bag is $\frac{14}{15}$.

The probability of a specific number being selected from the blue bag is $\frac{1}{10}$ and the probability of any specific number not being selected from the blue bag is $\frac{9}{10}$.

In the following we will use $P(p, q, r)$ to mean the probability of p being selected from the green bag, q being selected from the red bag, and r being selected from the blue bag. So, $P(5, 5, \text{ not } 5)$ means that we want the probability of a 5 being selected from the green bag, a 5 being selected from the red bag, and anything but a 5 being selected from the blue bag.

$$
\begin{aligned}
& \text{Probability of 5 being selected from at least two of the bags} \\
= \; & \text{Probability of 5 from each bag} + \text{Probability of 5 from exactly 2 bags} \\
= \; & \text{P(5, 5, 5)} + \text{P(5, 5, not 5)} + \text{P(5, not 5, 5)} + \text{P(not 5, 5, 5)} \\
= \; & \frac{1}{22} \times \frac{1}{15} \times \frac{1}{10} + \frac{1}{22} \times \frac{1}{15} \times \frac{9}{10} + \frac{1}{22} \times \frac{14}{15} \times \frac{1}{10} + \frac{21}{22} \times \frac{1}{15} \times \frac{1}{10} \\
= \; & \frac{1}{3300} + \frac{9}{3300} + \frac{14}{3300} + \frac{21}{3300} \\
= \; & \frac{45}{3300} \\
= \; & \frac{3}{220}
\end{aligned}
$$

The probability of 5 appearing on at least two of the tokens is $\dfrac{3}{220}$.

CEMC at Home

Grade 9/10 - Tuesday, April 14, 2020
Some Loose Change

In the little-known country of Galbrathia, there are only two types of money: a 5 $\mathbb{J}$ coin and a 9 $\mathbb{J}$ bill. Using this money, you can make some amounts, but not others. For example, you can make 19 $\mathbb{J}$ using two 5 $\mathbb{J}$ coins and one 9 $\mathbb{J}$ bill, but there is no way to make 17 $\mathbb{J}$ using these coins and bills.

Our goal is to determine the largest amount of money that cannot be made using these bills and coins.

1. Show how you can make 23 $\mathbb{J}$ using these bills and coins.

2. Explain why you cannot make 17 $\mathbb{J}$ using these bills and coins.

3. The table below shows one way to make each value from 40 $\mathbb{J}$ to 44 $\mathbb{J}$, inclusive:

Amount (in $\mathbb{J}$)	# of 5 $\mathbb{J}$ coins	# of 9 $\mathbb{J}$ bills
40	8	0
41	1	4
42	3	3
43	5	2
44	7	1

   Explain how we could use this information to make each value from 45 $\mathbb{J}$ to 49 $\mathbb{J}$, inclusive.

   To start, think about how the information in the table about 40 $\mathbb{J}$ *can help you make* 45 $\mathbb{J}$.

4. Explain how the information in 3. allows us to say that we can make *every* amount that is at least 40 $\mathbb{J}$.

   To start, think about how you might make the values from 50 $\mathbb{J}$ *to* 54 $\mathbb{J}$.

5. What is the largest amount N $\mathbb{J}$ that cannot be made using these bills and coins?

   To answer this question fully, you will need to do some work to figure out what you think the value of the positive integer N is. Next, you'll need to argue that N $\mathbb{J}$ cannot in fact be made. Finally, you'll need to argue that every value from $(N+1)$ $\mathbb{J}$ onward can be made. You'll likely want to "work backwards" using what you learned from 3. and 4..

Extensions:

A. There are three ways to make 90 $\mathbb{J}$ using these bills and coins: using 18 coins, using 9 coins and 5 bills, or using 10 bills. (To move from one way to the next way, we have traded 9 coins for 5 bills, since these have equal value.) How many ways are there to make 900 $\mathbb{J}$ using these bills and coins?

B. In the neighbouring country of Pnoll, stamps are issued in amounts of 7 $\mathbb{K}$, 9 $\mathbb{K}$, and 11 $\mathbb{K}$. What is the largest amount of postage that *cannot* be made using these stamps?

More Info:

Check out the CEMC at Home webpage on Tuesday, April 21 for a solution to Some Loose Change.

CEMC at Home

Grade 9/10 - Tuesday, April 14, 2020
Some Loose Change - Solution

1. Since $23\ \mathbb{J} = 2 \times 9\ \mathbb{J} + 1 \times 5\ \mathbb{J}$, then we can can use 2 bills and 1 coin.

2. Since 2×9 is greater than 17, then to make 17 $\mathbb{J}$, we must use either zero bills or one bill.
If zero 9 $\mathbb{J}$ bills are used, then the entire 17 $\mathbb{J}$ must be made using 5 $\mathbb{J}$ coins.
Since 17 is not divisible by 5, this is not possible.
If one 9 $\mathbb{J}$ bill is used, then the remaining 8 $\mathbb{J}$ must be made using 5 $\mathbb{J}$ coins.
Since 8 is not divisible by 5, this is not possible.
Therefore, it is not possible to make 17 $\mathbb{J}$ using these bills and coins.

3. We start with the combinations of coins and bills that make up each of 40 $\mathbb{J}$ through 44 $\mathbb{J}$, inclusive.
If we add one 5 $\mathbb{J}$ coin to each of these combinations, we add 5 $\mathbb{J}$ to each total to get totals from 45 $\mathbb{J}$ through 49 $\mathbb{J}$, inclusive.
We can summarize these in a table:

Amount (in $\mathbb{J}$)	# of 5 $\mathbb{J}$ coins	# of 9 $\mathbb{J}$ bills
40	8	0
41	1	4
42	3	3
43	5	2
44	7	1
45	9	0
46	2	4
47	4	3
48	6	2
49	8	1

4. Adding one 5 $\mathbb{J}$ coin to each of the combinations that make five consecutive amounts (like 40 $\mathbb{J}$ to 44 $\mathbb{J}$) allows us to make the next five consecutive amounts (in this case, 45 $\mathbb{J}$ to 49 $\mathbb{J}$).
Because we can continue to do this indefinitely, we can make any amount that is at least 40 $\mathbb{J}$.
Given any such amount, can you describe a quick way to determine a number of bills and coins that can be used to make this amount?

5. The table below shows that we can make each amount from 32 $\mathbb{J}$ to 36 $\mathbb{J}$, inclusive:

Amount (in $\mathbb{J}$)	# of 5 $\mathbb{J}$ coins	# of 9 $\mathbb{J}$ bills
32	1	3
33	3	2
34	5	1
35	7	0
36	0	4

Repeating the argument from 4., this table shows that we can in fact make every amount that is at least 32 $\mathbb{J}$.

It is impossible to make 31 $\mathbb{J}$ using only these coins and bills. (Can you modify the argument from 2. to show this?)
Since we cannot make 31 $\mathbb{J}$ and we can make every amount that is at least 32 $\mathbb{J}$, then 31 $\mathbb{J}$ is the largest amount that we cannot make using these bills and coins.

Extensions:

A. We can make 900 $\mathbb{J}$ using 100 bills and 0 coins.
Since 5 bills and 9 coins have the same value, we can trade 5 bills for 9 coins.
This means that we can make 900 $\mathbb{J}$ using 95 bills and 9 coins.
How many times can you repeat this process before no further trades are possible?

B. Try working with just 7 $\mathbb{K}$ and 9 $\mathbb{K}$ stamps first and work through similar steps to 3., 4., and 5. from earlier.
Once you know the largest amount that cannot be made with just 7 $\mathbb{K}$ and 9 $\mathbb{K}$ stamps, try introducing 11 $\mathbb{K}$ stamps to see what amounts that you previously couldn't make can now be made.

CEMC at Home

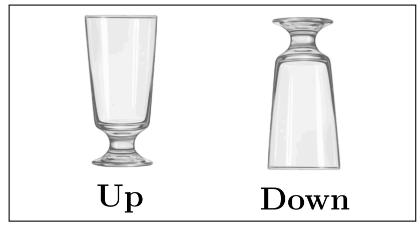
Grade 9/10 - Wednesday, April 15, 2020
Flipping Glasses

In this activity, we will flip glasses according to a set of rules and try to achieve certain targets.

You Will Need: Three empty glasses.

Rules:

- On each turn, ***you must flip two adjacent glasses.***
- Flipping a glass that is facing up changes it to facing down.
- Flipping a glass that is facing down changes it to facing up.



Activities:

1. Line up your three glasses as shown in the picture to the right. Is it possible to flip the glasses, according to the rules, so that all three glasses are facing up? If so, how many turns does it take?

2. There are exactly eight different starting positions for the three glasses.

   (a) Two possible starting positions are shown below. Complete the rest of the table.

   (b) From which of these eight starting positions is it possible to flip the glasses, according to the rules, so that all three glasses are eventually facing up?

   (c) From part (b), what do you notice about the original number of glasses facing down in the scenarios where you *can* flip all three glasses so that they are eventually facing up?

Extension: Suppose you now have eight glasses lined up as shown. Is it possible to flip the glasses, according to the rules, so that all eight glasses are eventually facing up?

More Info:

Check out the CEMC at Home webpage on Wednesday, April 22 for a solution to Flipping Glasses.

You can model your solutions using a *finite state machine* as shown with the *Glasses* question on the 2012 Beaver Computing Challenge and the CEMC at Home resource *Where Am I?* from April 1.

CEMC at Home

Grade 9/10 - Wednesday, April 15, 2020
Flipping Glasses - Solution

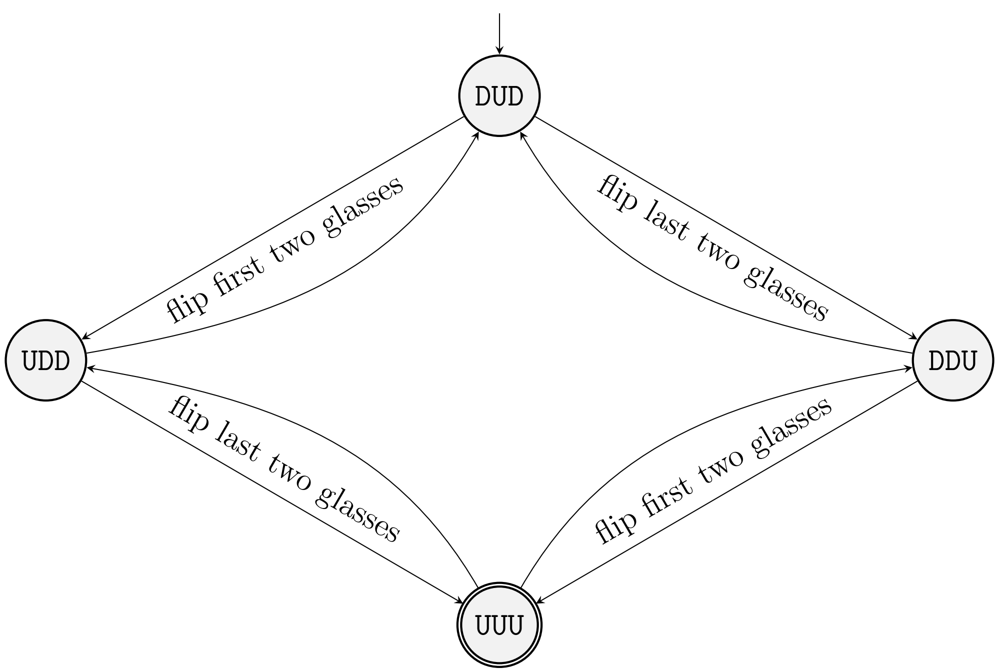
1. It is possible to flip the three glasses in two turns so that all three glasses are facing up.

   Here are images describing one possible solution:

Start **Turn 1** **Turn 2**

   To visualize the transformation of the glasses after each turn (i.e. facing up or down), we can use a *finite state machine*:

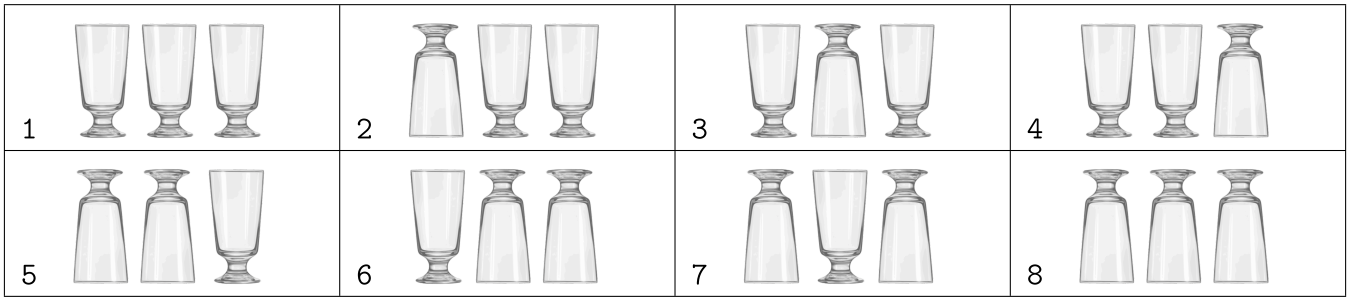
   - Each circle represents a possible *state* of the three glasses, where `D` represents a glass facing down and `U` represents a glass facing up.
   - The *starting state* is the circle with the arrow "out of nowhere" pointing to it (`DUD`).
   - To *transition* between states, follow an arrow by performing the action describing it.
   - The desired state is the circle with the double border around it (*final state*).



From the diagram above, we see that there are two different ways to arrive at the desired state (`UUU`) in exactly two turns (or two transitions from the starting state). One of these two solutions was outlined earlier (using images).

Note: If we flip the first and the third glass from the starting state, this would also result in the desired state of `UUU`. However, this is not a valid transition since these glasses are *not* adjacent.

2. (a) Here are the eight different starting positions for the three glasses.



(b) You are able to flip all three glasses so that they are facing up only if you start with position `1`, `5`, `6`, or `7`. Starting position `1` already has all three glasses facing up, so we don't have to do anything. In question 1, we showed that the glasses starting in position `7` can be flipped so that all three glasses are facing up. Positions `5` and `6` can be flipped to `UUU` in one turn. Another way to think about this is to notice that states `1`, `5`, `6`, and `7` are exactly the states in the finite state machine from Question 1. By following arrows, it is possible to move between any two of these four states. In particular, it is possible to get from any of states `5`, `6`, and `7` to state `1`.

You may have also figured out that it is impossible to reach `UUU` from the starting states `2`, `3`, `4`, or `8`. An explanation of this will be provided in part (c).

(c) In each of the starting positions for which it is possible to flip the glasses so that all three are facing up, there are either 0 or 2 glasses facing down.

On each turn, two adjacent glasses are flipped causing one of the following three outcomes:

* If the two glasses were facing down, the number of glasses facing up increases by two.
* If the two glasses were facing up, the number of glasses facing up decreases by two.
* If one glass was facing up and the other was facing down, the number of glasses facing up does not change.

Therefore, on each turn, the number of glasses facing down either increases by two, decreases by two, or stays the same. A consequence of this is that if the number of glasses facing down at the beginning is odd, then the number of glasses facing down will always be odd, and hence can never be 0.

In starting positions `2`, `3`, `4`, and `8`, an odd number of glasses are facing down. This means that no sequence of valid flips can change any of them to the state `UUU` (which has 0 glasses facing down). Put differently, a state must have an even number of glasses facing down for there to be a chance to change it to `UUU`. In part (b), we we confirmed that it is actually possible to do this for all states starting with an even number of glasses facing down.

Extension: It is not possible to have all eight glasses facing up. We can generalize our reasoning from 2(c) beyond three glasses. The observation at the end of part (c) is true for any number of glasses. That is, if there are an odd number of glasses facing down at the beginning, then there will always be an odd number of glasses facing down. Since we start with an odd number of glasses (five) facing down, the number of glasses facing down will always be odd, and hence will never be 0. At best, we can reach a state where 1 glass is facing down and 7 glasses are facing up. Can you convince yourself that it is indeed possible to reach a state where exactly one glass is facing down?

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 16, 2020
Mind Your Ps and Qs and Rs

Three digit numbers PQR, QQP and PQQ are formed using the single digits P, Q and R such that:

$$PQR + 2 = PQQ$$

AND

$$PQR \times 2 = QQP$$

Determine all possible combinations of values of P, Q and R that satisfy the two equations.

More Info:

Check the CEMC at Home webpage on Thursday, April 23 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 23.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

PQR + 2 = PQQ

Problem of the Week

Problem D and Solution

Mind Your Ps and Qs and Rs

PQR × 2 = QQP

Problem

The letters P, Q, and R represent single digits. Determine all possible combinations of values of P, Q and R, given that: $PQR + 2 = PQQ$ and $PQR \times 2 = QQP$

Solution

From $PQR + 2 = PQQ$, we note that the number PQQ is 2 more than the number PQR and that the first two digits of each number are the same. From this, $Q = R + 2$ follows. We can prove this using place value. The number $PQR = 100 \times P + 10 \times Q + R = 100P + 10Q + R$. The number $PQQ = 100 \times P + 10 \times Q + Q = 100P + 10Q + Q$.
Then,

$$\begin{aligned}(100P + 10Q + R) + 2 &= 100P + 10Q + Q\\ 100P + 10Q + R + 2 &= 100P + 11Q\\ R + 2 &= Q, \text{ as above.}\end{aligned}$$

From this expression we can obtain a restriction on the possible integer values of R. R must be an integer from 0 to 7, inclusive. If $R = 8$, then $R = Q + 2 = 10$ and Q is not a single digit.

From here, we will show two possible solutions.

<u>Solution 1</u>

Representing $PQR \times 2 = QQP$ using place value,

$$\begin{aligned}(100P + 10Q + R) \times 2 &= 100Q + 10Q + P\\ 200P + 20Q + 2R &= 110Q + P\\ 199P &= 90Q - 2R\\ \text{Substituting } R + 2 \text{ for } Q: \quad 199P &= 90(R + 2) - 2R\\ 199P &= 90R + 180 - 2R\\ P &= \frac{88R + 180}{199}\end{aligned}$$

We are looking for an integer value of R from 0 to 7 such that $88R + 180$ is a multiple of 199. The only value of R that produces a multiple of 199 when substituted into $88R + 180$ is $R = 7$. When $R = 7$, $P = \frac{88R+180}{199} = \frac{88(7)+180}{199} = 4$ and $Q = R + 2 = 9$.

Therefore, the only values that satisfy the system of equations are $P = 4$, $Q = 9$ and $R = 7$.

We can verify this result:

$$PQR + 2 = 497 + 2 = 499 = PQQ \text{ and}$$

$$PQR \times 2 = 497 \times 2 = 994 = QQP.$$

Solution 2

If R must be an integer from 0 to 7, inclusive, then Q must be an integer from 2 to 9, inclusive. From the second equation, we also know that P would be the units digit of $2R$.
We can now use a table to determine all possible values for P, Q and R

Q	R	$2R$	P	PQR	QQP	Does $2 \times PQR = QQP$?
2	0	0	0	020	220	No
3	1	2	2	231	332	No
4	2	4	4	442	444	No
5	3	6	6	653	556	No
6	4	8	8	864	668	No
7	5	10	0	075	770	No
8	6	12	2	286	882	No
9	7	14	4	497	994	Yes

We will now need to verify that the final row also works for $PQR + 2 = PQQ$.

$$PQR + 2 = 497 + 2 = 499 = PQQ.$$

Therefore, the only values that satisfy the system of equations are $P = 4$, $Q = 9$ and $R = 7$.

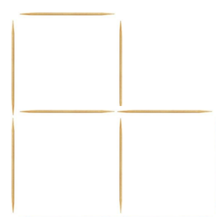
CEMC at Home

Grade 9/10 - Friday, April 17, 2020
Toothpick Challenges

In the following toothpick puzzles, you will be presented with some toothpicks as well as some instructions. Your task is to arrange, rearrange, or remove toothpicks in order to produce the desired result.

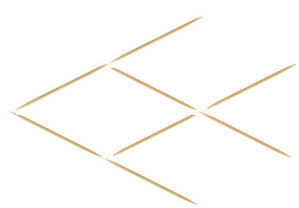
Example

Starting with 10 toothpicks arranged into 3 squares as shown, move exactly 2 toothpicks to create a figure that has exactly 2 squares. (The solution is on the second page.)



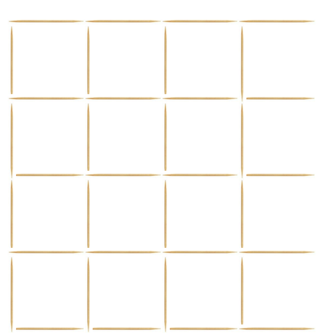
Reversing Fish

Starting with 8 toothpicks arranged into a fish facing left as shown, move exactly 3 toothpicks in order to produce an identical fish, but now facing right.



Many Squares to No Squares

Start with 40 toothpicks arranged into a 4 by 4 grid as shown. Notice that this grid contains many squares of various sizes. Remove exactly 9 toothpicks so that no square, of any size, remains.



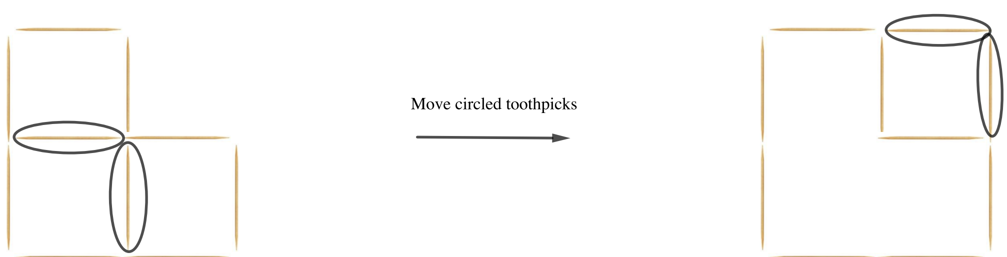
Can you count the total number of squares in this figure? Hint: There are more than 17.

Triangles

Using exactly 6 toothpicks, arrange the toothpicks to create exactly 4 equilateral triangles. Each triangle will have side lengths equal to the length of one toothpick.

Hint: You will not be able to lay your solution flat on a table.

Solution for Example



We now have two squares: a large 2 by 2 square, and a small 1 by 1 square.

More Info:

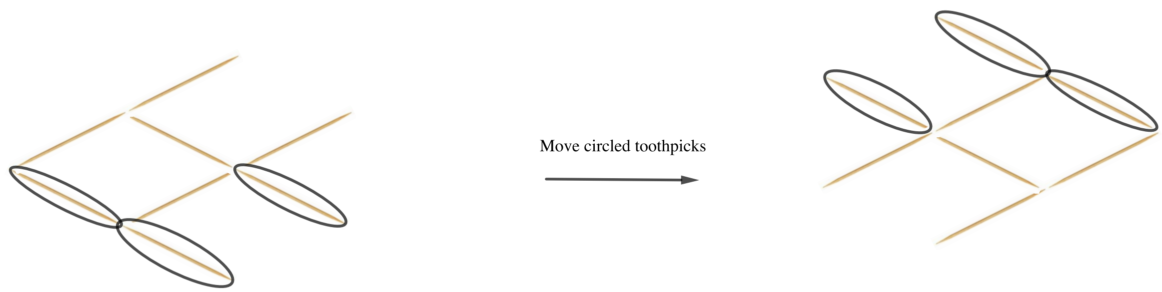
Check out the CEMC at Home webpage on Friday, April 24 for a solution to Toothpick Challenges.

CEMC at Home

Grade 9/10 - Friday, April 17, 2020
Toothpick Challenges - Solutions

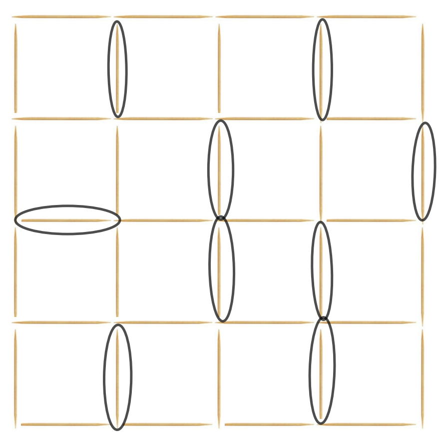
Reversing Fish

Move the circled toothpicks to their corresponding places noted.



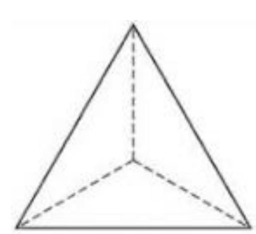
Many Squares to No Squares

Remove the circled toothpicks.



Triangles

Make a triangular based pyramid. (A regular tetrahedron.)



CEMC at Home

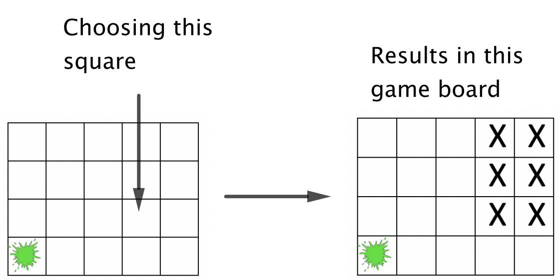
Grade 9/10 - Monday, April 20, 2020
Splat

You Will Need:

- Two players
- A Splat game board which is a rectangular grid with the bottom left corner marked with a Splat

How to Play:

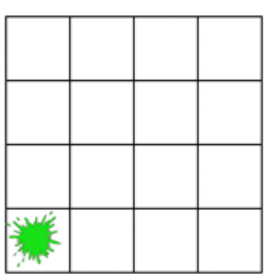
1. Start with a Splat game board.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On each turn, the current player must select a square and removes that square and all of the squares above and it and to the right of it (see example below).



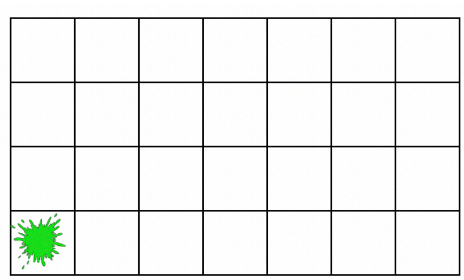
Any square that is removed (marked with an X*) cannot be chosen on any turns that follow.*

4. The player who is forced to remove the square with the Splat loses the game.

Play this game a number of times using the game boards below. Alternate which player goes first. For each of the sample game boards (Game Board 1 and Game Board 2), is there a strategy that guarantees one of the players (Player 1 or Player 2) a win every time?



Game Board 1



Game Board 2

More Info:

Check out the CEMC at Home webpage on Monday, April 27 for a discussion of a strategy for Splat. We encourage you to discuss your ideas online using any forum you are comfortable with.

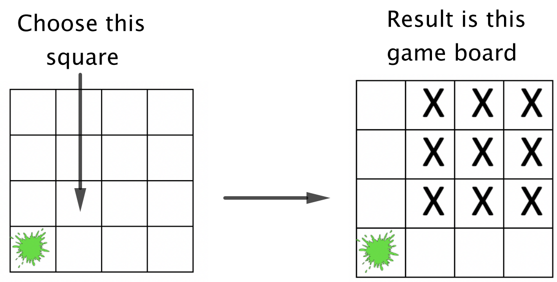
This game is often called *Chomp*.

CEMC at Home

Grade 9/10 - Monday, April 20, 2020
Splat - Solution

Game Board 1

Player 1 has a winning strategy for this game board. Player 1 should choose the square one over and one up from the square with the Splat as shown in the diagram below.



From here, Player 1 will mimic whatever Player 2 does in the following way:

- If Player 2 chooses a square from the remaining column, then Player 1 will choose the corresponding square in the remaining row.
- If Player 2 chooses a square from the remaining row, then Player 1 will choose the corresponding square in the remaining column.

For example, if Player 2 chooses the square marked as P2 (in Diagram 1 below), then Player 1 should chose the square marked as P1. The resulting game board is shown in Diagram 2 below.

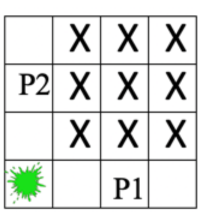


Diagram 1

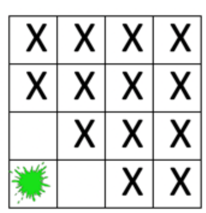


Diagram 2

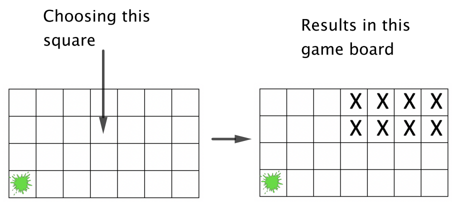
If Player 1 keeps mimicking Player 2 in this way, eventually the square with the Splat will be the only square left after Player 1 makes a move. Then Player 2 must choose that square and lose the game.

Note: This solution is valid for any game board with an equal number of rows and columns (except for a 1 by 1 game board). To start, Player 1 should always choose the square one over and one up from the Splat. Then Player 1 mimics Player 2 as described above.

The strategy for Game Board 2 is on the next page.

Game Board 2

It turns out that Player 1 has a winning strategy for this game board but it is not easy to find or to describe. In one strategy, Player 1 can start with the following play.



From here, the strategy depends on what move Player 2 chooses to make. Player 1 will then have to react to this move and there are many different ways this can unfold. We will not present the strategy here but we encourage you to think about it and look into it further on your own. (Remember that this game is often called *Chomp*.)

It is actually possible to argue that Player 1 must have a winning strategy without actually finding one! An argument would involve explaining the following two facts:

1. Exactly one player (Player 1 or Player 2) must have a winning strategy for this game board.
2. It is not possible for Player 2 to have a winning strategy for this game board.

For point 1., it turns out that since there is no way for the game to end in a tie, we can argue that at least one player must have a winning strategy (although this argument is not as easy to explain carefully as you might think). By the definition of a winning strategy, it is not possible for both players to have a winning strategy for the same game board.

To argue point 2. we can use what is sometimes called a *strategy-stealing argument*. The idea here is as follows: If Player 2 had a winning strategy, then Player 1 could use that strategy before Player 2 has a chance to use it. The name "strategy stealing argument" is slightly misleading. A winning strategy must work regardless of the moves made by the other player and so cannot be "stolen" (at least not without a mistake!). That Player 1 can "steal" the winning strategy simply means that Player 1 must have had the winning strategy in the first place.

Let's assume that Player 2 has a winning strategy. This means that no matter what first move Player 1 makes, Player 2 is guaranteed to win the game following this strategy.

In particular, this means that if Player 1 starts the game by selecting the top right hand square in the grid, then Player 2 must have a winning strategy from this point. This means Player 2 has a move in response to Player 1 removing the top right hand square that leaves Player 1 with a game board from which they cannot possibly win.

No matter what the game board looks like after Player 2 responds, Player 1 could have made their first move in a way that leaves this exact same game board! (Can you see why?) In this way, Player 1 can "steal" the strategy by making their first move to leave Player 2 with this game board from which they cannot win.

This argument may take a few times through to understand, but it achieves something quite remarkable. It explains that Player 1 has a winning strategy, but gives no hint whatsoever as to what the strategy is. In fact, it doesn't even tell us what the first move should be!

The above explanation that Player 2 cannot have a winning strategy involves a type of argument called a proof by contradiction.

CEMC at Home

Grade 9/10 - Tuesday, April 21, 2020
Some Counting

Consider the integers from 10 to 30, inclusive.

If you wrote down all of these integers, you would use a total of twelve 2s: one 2 for the units (ones) digit of the integer 12, then ten 2s for the tens digit in each of the integers from 20 to 29, and an additional 2 for the units digit of 22.

If you were asked how many of the integers from 10 to 30 contain the digit 2, you would answer eleven, not twelve. While the number 22 contains two copies of the digit 2, we only count this number once.

Solve the following problems that involve counting digits and numbers in a longer list of integers.

Problem 1: Consider the integers from 300 to 600, inclusive.

(a) If you wrote down all of the integers from 300 to 600, inclusive, how many times would you write the digit 4?

(b) How many of the integers from 300 to 600, inclusive, contain the digit 4?

Problem 2: If you wrote down all of the integers from 300 to 600, inclusive, what is the sum of all of the digits that you would write?

Discussion for Problem 2: To answer part (a) above, you need to count how many times you would write the digit 4 when writing the integers from 300 to 600. Since there would be 301 integers in this list, actually writing them all down is not the best approach.

We have a similar issue in Problem 2. One way to solve this problem would be to write out the 301 integers from 300 to 600, inclusive, and then add the resulting 903 digits together. However, there are nicer ways to solve this problem. Do you see how your work in Problem 1 may help you in answering this question?

If you need help getting started, you can always return to our first example, the list of integers from 10 to 30, which can be written down quickly:

$$10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30$$

What do you notice when calculating the sum of the 42 digits in this list of integers? You might find it helpful to consider the 21 units digits and the 21 tens digits separately.

More Info:

Check out the CEMC at Home webpage on Tuesday, April 28 for a solution to Some Counting.

CEMC at Home

Grade 9/10 - Tuesday, April 21, 2020
Some Counting - Solution

Problem 1: Consider the integers from 300 to 600, inclusive.

(a) If you wrote down all of the integers from 300 to 600, inclusive, how many times would you write the digit 4?

(b) How many of the integers from 300 to 600, inclusive, contain the digit 4?

Solution to Problem 1:

(a) There are a number of ways to approach this question, but we will use an organized count.

First, consider the hundreds digits of the 301 integers from 300 to 600. None of the integers from 300 to 399 have a 4 in the hundreds position. The 100 integers from 400 to 499 each have a 4 in the hundreds position. None of the integers from 500 to 600 have a 4 in the hundreds position. So there are a total of 100 4s appearing in the hundreds position in the list of integers from 300 to 600.

Next, consider the tens digits. Only the integers from 340 to 349, 440 to 449, and 540 to 549 have a 4 in the tens position. So there are $3 \times 10 = 30$ 4s appearing in the tens position in the list of integers from 300 to 600.

Finally, consider the units (ones) digits. In the integers from 300 to 399 there are 10 integers with a 4 in the units position. These integers are as follows:

$$304, 314, 324, 334, 344, 354, 364, 374, 384, 394$$

Similarly, there are 10 integers from 400 to 499 with a 4 in the units position and 10 integers from 500 to 600 with a 4 in the units position. So there are $3 \times 10 = 30$ 4s appearing in the units position in the list of integers from 300 to 600.

Combining the counts for the three different positions, when writing the integers from 300 to 600 the digit 4 will be written a total of $100 + 30 + 30 = 160$ times.

(b) In our solution to part (a), we determined that there are 100 integers from 300 to 600 with a 4 in the hundreds position, 30 integers with a 4 in the tens position, and 30 integers with a 4 in the units position, but we note that $100 + 30 + 30 = 160$ is not a correct count of the number of integers that contain a 4. Some of these integers have more than one digit equal to 4 and so were counted more than once in the total of 160.

We now need to determine how many integers were counted more than once, and exactly how many times they were counted.

How many of the integers have a 4 in both the hundreds and the tens positions? These integers will begin with 44, and thus are from 440 to 449, and there are 10 of them. These 10 integers have been counted at least twice.

How many of the integers have a 4 in both the hundreds and the units positions? These integers will begin with a 4 and end with a 4. These integers have the form 4X4, where the X can be any of the ten digits, and there are 10 of them. These 10 integers have been counted at least twice.

How many of the integers have a 4 in both the tens and the units positions? These integers have the form X44, where X can be 3, 4 or 5. Thus there are 3 of these integers. These 3 integers have been counted at least twice.

There is only one integer with a 4 in all three positions, 444, and it was counted three times because it belongs to each of the three groups we just examined. All other integers from these groups were counted exactly two times because they each belong to two of the three groups.

To count the number of integers that contain at least one digit 4 we do the following calculation. Start with the total of 160 and subtract the number of integers that were counted at least twice (due to having a digit 4 in at least two positions). This results in the following:

$$160 - 10 - 10 - 3 = 137$$

But there is one last thing to consider before we obtain the final answer. The integer 444 was counted (or included) three times and then removed (or excluded) three times in our calculation above. (Do you see why?) This means it is not included in the count of 137. Adding the integer 444 back into our count we get a final answer of

$$160 - (10 + 10 + 3) + 1 = 138$$

Therefore, there are 138 integers between 300 and 600 that contain the digit 4.

Note: In this solution, the method of *inclusion-exclusion* has been demonstrated.

Problem 2: If you wrote down all of the integers from 300 to 600, inclusive, what is the sum of all of the digits that you would write?

Solution to Problem 2:

In the solution to this problem, it is again necessary to be organized in your approach. We give two possible ways to calculate this sum. The second approach uses our work from Problem 1(a).

Approach 1:

First, consider the units digits of the integers from 300 to 399. In each set of ten consecutive integers in this range, each of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ appears once (in the units position) and their sum is 45. There are ten distinct sets of 10 consecutive integers in the range from 300 to 399, and so the sum of the units digits in this range is $10 \times 45 = 450$. Similarly, in both the ranges from 400 to 499 and 500 to 599, the sum of the units digits will be 450. Thus, from 300 to 599, the sum of the units digits will be $3 \times 450 = 1350$.

Now consider the tens digits of the integers from 300 to 399. We note that each of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ appears 10 times in the tens position in this list of 100 integers. For example, the digit 0 appears as the tens digit in the integers from 300 to 309. The sum of these tens digits is $10 \times 45 = 450$. As with the units digits, the same number of each digit occurs again in the ranges 400 to 499 and 500 to 599. Thus, from 300 to 599, the sum of the tens digits will be $3 \times 450 = 1350$.

In the range 300 to 599 there are 100 integers with a 3 in the hundreds position, 100 integers with a 4 in the hundreds position, and 100 integers with a 5 in the hundreds position. The sum of these 300 digits is $100 \times 3 + 100 \times 4 + 100 \times 5 = 100 \times (3 + 4 + 5) = 100 \times 12 = 1200$.

Note that the only digits we have not yet included in our sum are the digits of the integer 600. Therefore, the sum of all digits used in writing down the integers from 300 to 600, inclusive, is

$$1350 + 1350 + 1200 + 6 + 0 + 0 = 3906$$

Approach 2:

You may have noticed that you can use your work from Problem 1(a) to calculate the sum in Problem 2. We outline how to do this here.

Recall that when writing the integers from 300 to 600, inclusive, the digit 4 will be written a total of 160 times. A similar argument can be used to show that the digits 3 and 5 will also appear a total of 160 times. Convince yourself that the digits 3 and 5 will appear exactly as often as the digit 4 in each of the three positions: hundreds, tens, and units.

The counts for the remaining digits are a bit different because of the particular range of integers. Notice that the digits 1, 2, 7, 8 and 9 do not appear as hundreds digits, and the digit 6 appears only once as a hundreds digit (in 600).

Using the work done in Problem 1(a), we can see that the digit 9 will appear $30 + 30 = 60$ times. This is because the digit 9 appears exactly as often as the digit 4 in the units position and in the tens position, but does not appear at all in the hundreds position. (Remember that the digit 4 appears 30 times in the tens position and 30 times in the units position.) Similar reasoning shows that the digits 1, 2, 7, and 8 also appear 60 times each, and that the digit 6 appears 61 times. (Remember that the digit 6 appears once in the hundreds position.)

When we add up all of the digits, each digit 4 contributes 4 to the sum, and so the 160 4s contribute a total of $160 \times 4 = 640$ to the sum of the digits. Dealing with the other digits in a similar way, we can calculate the sum of all of the digits as follows:

$$60 \times 1 + 60 \times 2 + 160 \times 3 + 160 \times 4 + 160 \times 5 + 61 \times 6 + 60 \times 7 + 60 \times 8 + 60 \times 9 = 3906$$

CEMC at Home

Grade 9/10 - Wednesday, April 22, 2020
Sunken Treasure

Video
Watch the following presentation on *The Knapsack Problem*, a problem involving optimization:

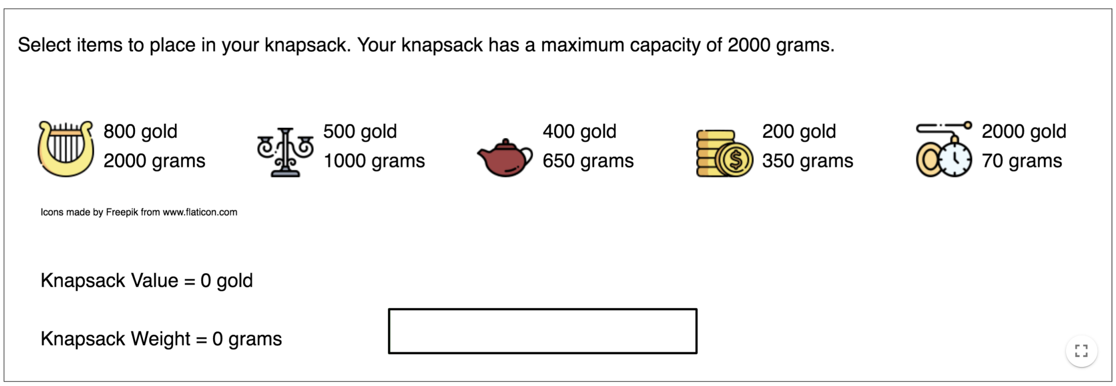
https://youtu.be/qihjUx8Qakk

The two scenarios from the presentation, along with the questions, are included for your reference.

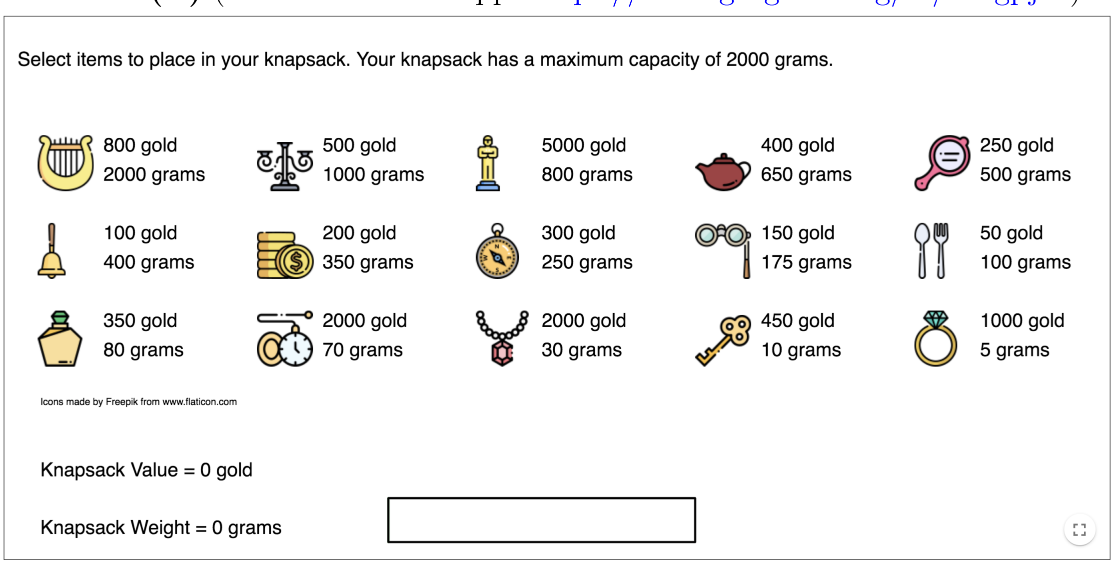
Questions:

1. What is the maximum knapsack value you can achieve?
2. Which subset of items achieves this maximum value?
3. What was your process?
4. How do you know for sure that your solution is correct?

Sunken Treasure (A) (with Interactive App: https://www.geogebra.org/m/hvbnqhzg)



Sunken Treasure (B) (with Interactive App: https://www.geogebra.org/m/dnrgpjcd)



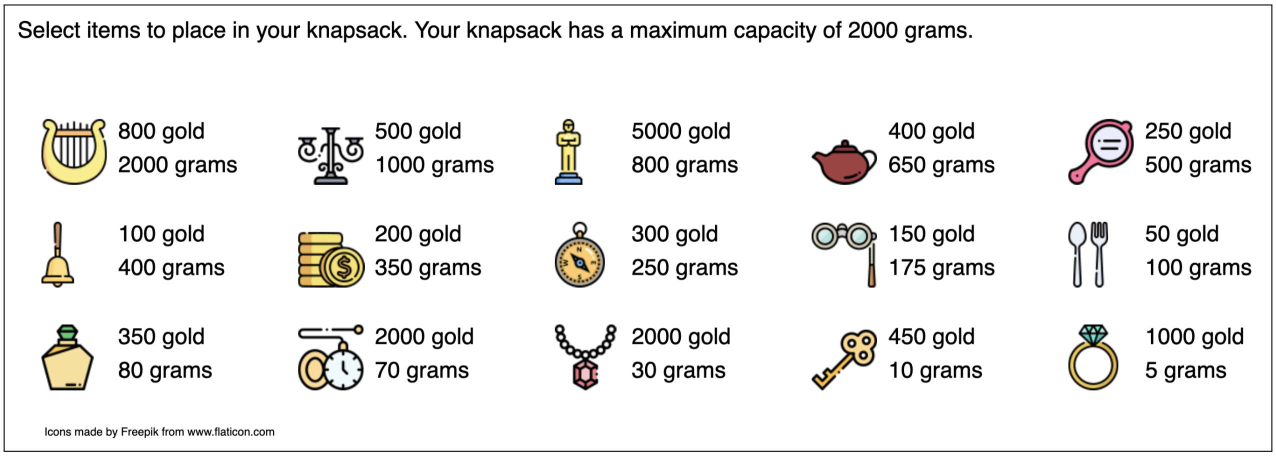
More Info:

Check out the CEMC at Home webpage on Wednesday, April 29 for a solution to Sunken Treasure.

CEMC at Home

Grade 9/10 - Wednesday, April 22, 2020
Sunken Treasure - Solution

Recall the set up for the activity Sunken Treasure (B):

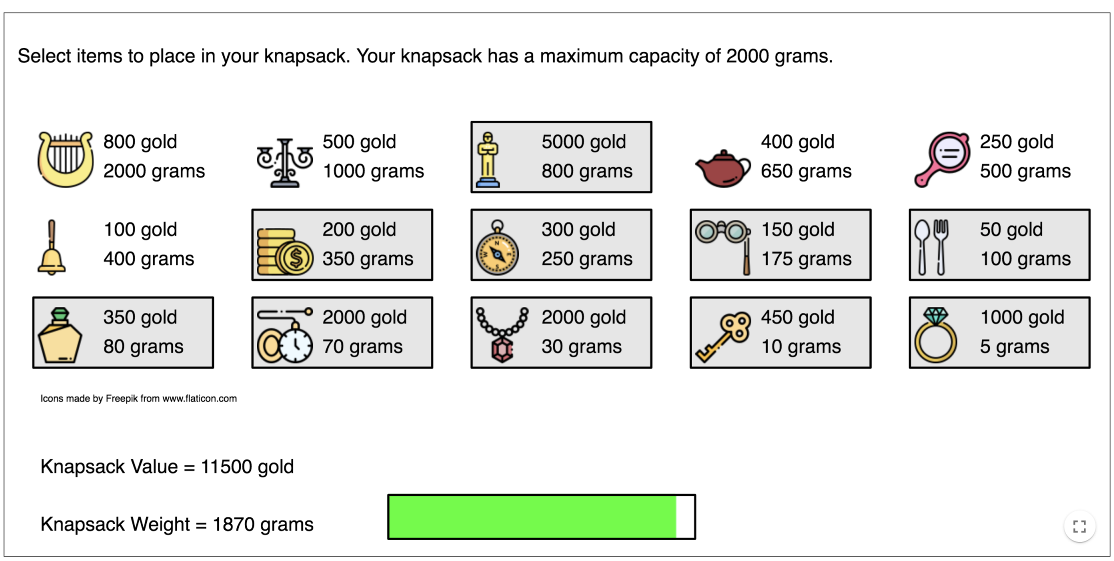


What is the maximum knapsack value you can achieve?

The maximum knapsack value you can achieve is 11 550 gold, although this value may have been difficult for you to obtain.

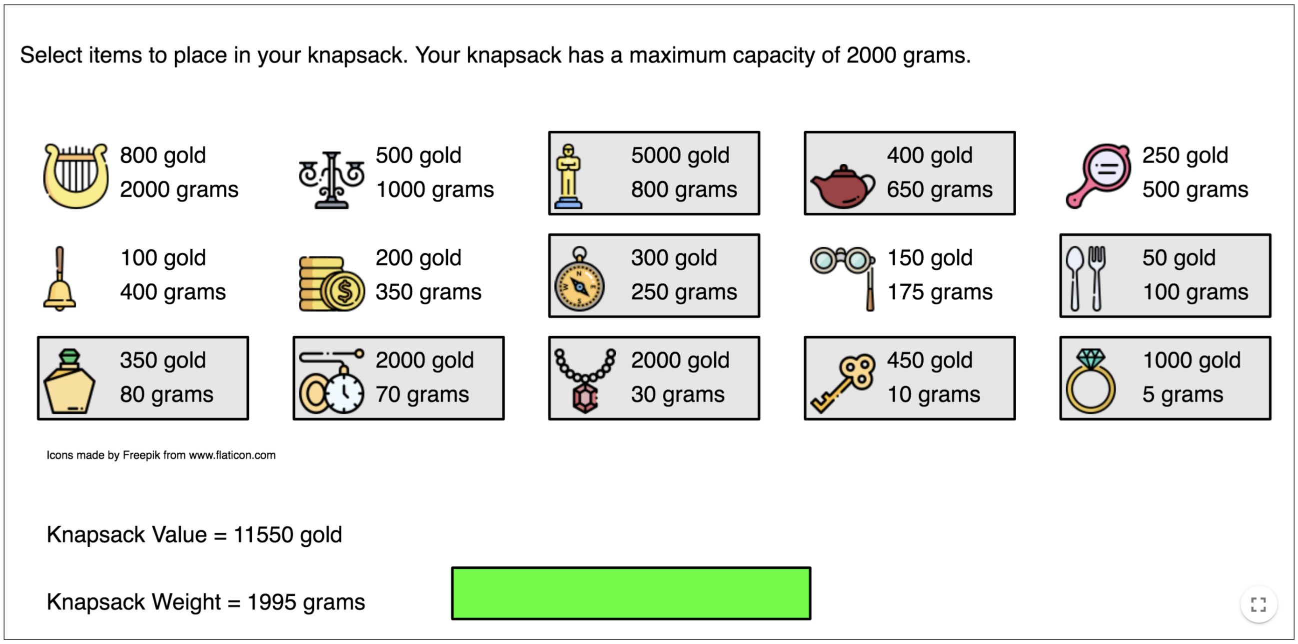
If you tried to fill your knapsack by choosing items with the greatest rate of *gold per gram*, then you most likely achieved a value of 11 500.

If you calculate each item's rate of gold per gram, sort the items from largest to smallest rate, and then place items in your bag in this same order as long as they fit in your knapsack, then you will end up with the following subset of the available items:



This algorithm, where you choose the best item at each step in the hope of getting the best outcome overall, is known as a *greedy algorithm*. Greedy algorithms do not look at the bigger picture and do not plan ahead.

If we step back and look at the bigger picture, we can see that if we remove the coins and the opera glasses and instead take the teapot, we can increase the knapsack's value by 50 gold as shown below.



It turns out that this solution is optimal, but how do we know for sure that this is the case? We could check the value attained by every possible subset of items, and make sure our value is at least as high as all of the rest. This approach is often called a *brute force algorithm*.

How long would this approach take? When forming a subset, each item is either in the subset or not. So there are 2 possibilities for each of the 15 available items: in or out. This means there are $2^{15} = 32\,768$ subsets we would have to check.

Of course, there are many subsets that we can eliminate immediately, or at least very quickly, because it is clear they will not produce an optimal result.

Checking all of these subsets is not feasible by hand, but suppose a computer can check 1 million subsets every second. In this case, it would take a computer less than 1 second to check all of the subsets and find the optimal solution. The problem is, this approach does not scale well. If the number of available items is increased from 15 to 100, then there would be 2^{100} subsets to check which would take a computer approximately 40 quadrillion years to consider.

Using brute force to find the optimal solution is often not feasible, but unfortunately the best known algorithms to solve this type of problem are also not always feasible. This explains why the knapsack problem is a hard problem in general. For hard problems, it is often good enough to find a solution which is *close to the optimal solution*.

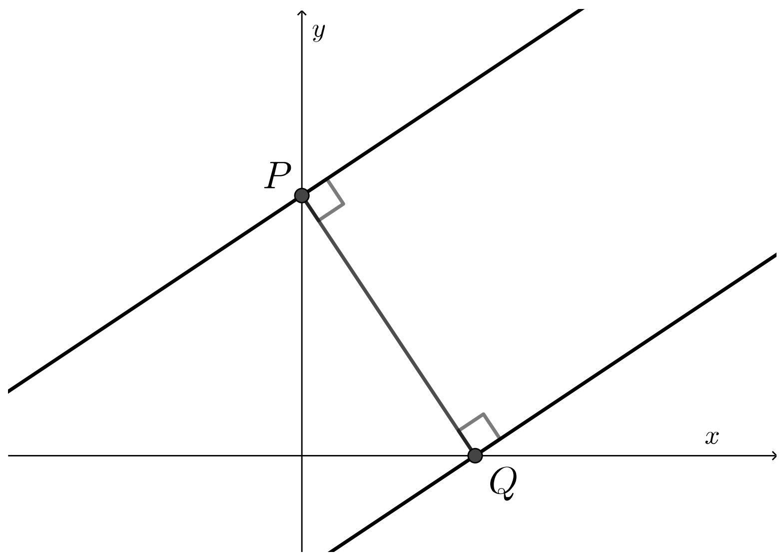
Can you find a way to reason that the value given above is optimal without the help of a computer? If you have some programming experience, can you write a computer program that can check this for you?

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 23, 2020
Crossing Points in General

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines.

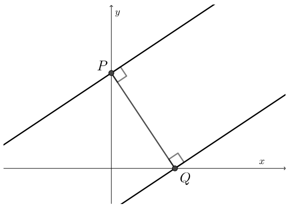
If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

More Info:

Check the CEMC at Home webpage on Thursday, April 30 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 30.



Problem of the Week

Problem D and Solution

Crossing Points in General

Problem

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines. If the line through P has equation $y = mx + k$, then determine the y-intercept of line through Q in terms of m and k.

Solution

For ease of reference, we will call the first line l_1 and the second line l_2.

Let b represent the y-intercept of l_2.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P is the point $(0, k)$.

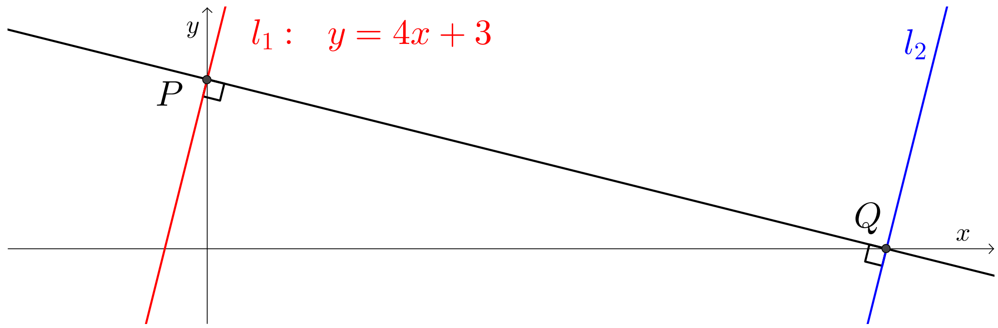
Since PQ is perpendicular to both lines, it follows that l_1 is parallel to l_2. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, $\text{slope}(PQ) = -\frac{1}{m}$. Since k is the y-intercept of the perpendicular segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

To find the x-intercept of $y = -\frac{1}{m}x + k$, set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of $y = -\frac{1}{m}x + k$ is mk and the coordinates of Q are $(mk, 0)$.

We can now find the y-intercept of l_2 since we know $Q(mk, 0)$ is on l_2 and the slope of l_2 is m. Substituting into the slope-intercept form of the line, $y = mx + b$, we obtain $0 = (m)(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2, the line through Q. A full solution to this problem is provided on the next page.



Let l_1 represent the line $y = 4x + 3$. Let l_2 represent the second line, the line through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P is the point (0,3).

Since $PQ \perp l_1$ and $PQ \perp l_2$, it follows that $l_1 \parallel l_2$. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, slope$(PQ) = -\frac{1}{4}$. Since 3 is the y-intercept of PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-intercepts of the line through perpendicular PQ and the line l_2 are the same since both lines intersect at Q on the x-axis. To find this x-intercept, set $y = 0$ in $y = -\frac{1}{4}x + 3$. Then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result, $x = 12$, follows. The x-intercept of the line through perpendicular PQ and the line l_2 is 12 and point Q is $(12, 0)$.

We can now find equation of l_2 since $Q(12, 0)$ is on l_2 and the slope of l_2 is 4. Substituting $x = 12$, $y = 0$ and $m = 4$ into $y = mx + b$, we obtain $0 = (4)(12) + b$ which simplifies to $b = -48$. The equation of l_2 is $y = 4x - 48$ and the y-intercept is -48.

This is the same result we obtained from the general solution on the previous page.

CEMC at Home

Grade 9/10 - Friday, April 24, 2020
Colouring Fun

You Will Need:

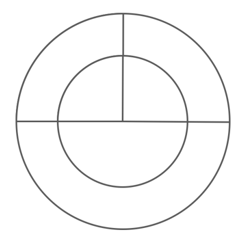
- Pieces of paper
- Pencil crayons or markers for colouring

Colouring Figures

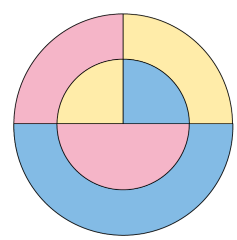
In the following activities, we will be colouring figures using different colours. Here we outline the properties of a *valid colouring* for the purposes of today's activities:

- All regions in the figure are coloured.
- Any pair of neighbouring regions in the figure (that is, regions sharing an edge or border) are coloured with two different colours.
- Regions that meet only at a single point can be coloured with the same colour.

Consider the example shown to the right. Since there are six different regions in the top figure shown, one easy way to produce a valid colouring of the figure is to use six different colours and colour each region using one of these colours. However, the figure can be coloured according to the rules using fewer than six colours. Can you figure out how many colours are actually *needed* to colour this figure?

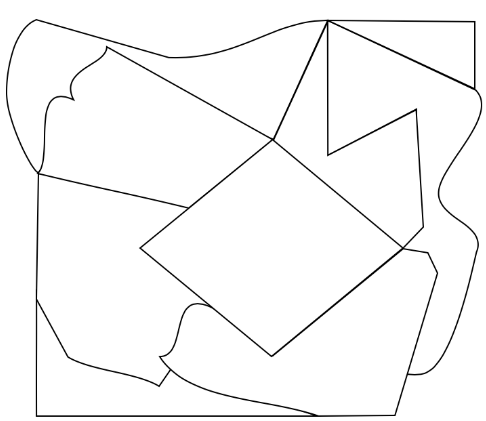


It turns out that there are valid colourings of this figure that only use three different colours, but no valid colouring that uses fewer than three colours. This means that the *minimum* number of colours needed to colour this figure is 3. The bottom image to the right shows one possible valid colouring using three colours. Can you explain why this figure cannot be coloured using only two colours?

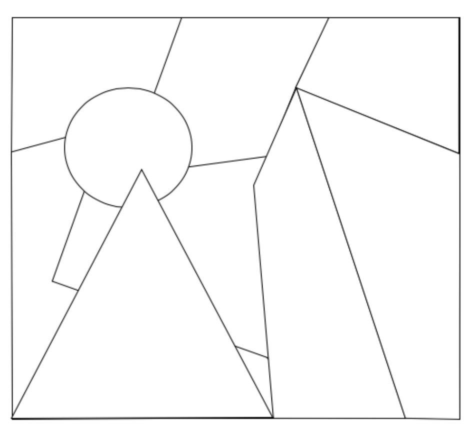


Activity 1: Find a valid colouring for each of the figures shown below that uses the fewest colours possible. Can you explain why the figure cannot be coloured using fewer colours than what you have?

(a)

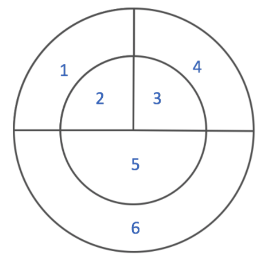


(b)



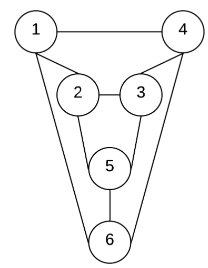
Modelling Figures Using Graphs

As figures become more complex, it can become difficult to see how to colour them using the fewest colours possible. To help with this, we can translate all of the necessary information needed for colouring from the original figure into a simpler figure called a *graph*. In other words, we *model* the figure using a graph. We can then colour the graph instead of the figure, and then translate this colouring back to the figure. Let's investigate this idea by revisiting the circle figure from earlier.



To model a figure using a graph for the purposes of colouring, follow these steps:

i. Label each region of the figure with a unique positive integer.

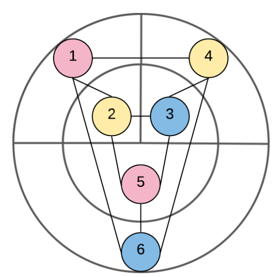


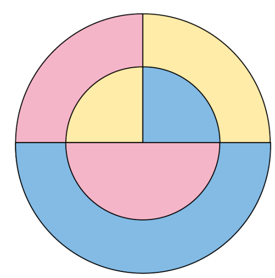
ii. Model the picture from i. using a *graph*. Graphs consist of points (called *vertices*) and lines (called *edges*). Create a point (vertex) in the graph for each region of the picture. We will actually use a circle (as shown) as we would like to include the region number with the point. Next, connect various pairs of vertices using lines (edges). Two vertices should be connected by an edge exactly if they represent neighbouring regions in the original figure.

For example, since regions 1 and 2 share a border in the figure, there is an edge between vertices 1 and 2 in the graph, and since regions 4 and 5 do not share a border in the figure, there is no edge between vertices 4 and 5 in the graph.

iii. Colour all of the vertices in the graph so that no adjacent vertices (vertices connected by an edge) share the same colour.

iv. Transfer the colours from the vertices of your graph to the corresponding regions in the original figure. This must correspond to a valid colouring of the figure. (Can you see why?)





Since the original goal was to colour figures using the fewest colours possible, the goal when modelling these problems with graphs is to colour graphs using the fewest colours possible.

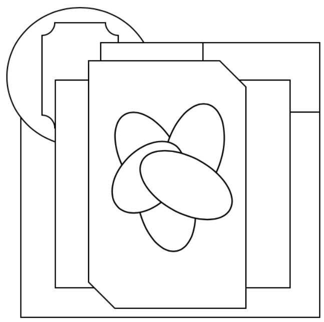
For example, here is the method that was used to colour the graph above: Start with vertex 1, and assign it a colour (pink). Next, identify all of the vertices that are adjacent to this vertex, which in this example are 2, 4 and 6. If possible, colour all of these vertices with the same second colour (yellow). If this is not possible, colour as many as you can with the second colour. In this example, we colour 2 and 4 yellow, but not 6 as it is connected to 4. We now know we need a third colour (blue). It turns out that we can colour all remaining vertices with these three colours.

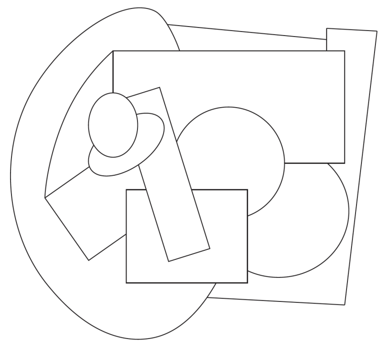
We can also argue that three colours is the best we can do. Notice that we have no choice but to use three different colours for the three vertices 2, 3, and 5. You need two different colours for 2 and 3, but neither of these colours can then be used for 5.

Activity 2:

(a) Consider the two figures shown below. For each figure, do the following:

- Model the figure using a graph as outlined in i. and ii. on the previous page.
- Find a colouring of the graph as outlined in iii. on the previous page that uses the fewest colours possible.
- Explain why the graph cannot be coloured according to the guidelines using fewer colours than what you have.





If you complete these steps, you will have determined the minimum number of colours needed for a valid colouring of each figure. If you would like, you can now colour the original figures as outlined by your graph colourings!

(b) Can you create a two-dimensional figure that requires five colours in order to achieve a valid colouring? Spend some time thinking about whether or not you think this is possible.

Graph Colouring in Action!

Many real-world problems can be translated into graph colouring problems. These problems often involve resources (colours) and conflicts (two regions that cannot be coloured the same), and you are tasked with assigning resources in an optimal way (using the fewest colours possible) while ensuring that no conflicts arise. This type of problem often arises when attempting to make schedules and timetables, and large scale versions are famously difficult to solve!

Check out the problems Timetabling and Aircraft Scheduling from past Beaver Computing Challenges. Read the story for the problem and try to figure out how to model this problem as a graph colouring problem. Ask yourself the following questions:

- What should the vertices represent?
- How do you decide whether or not an edge should be drawn between a particular pair of vertices?
- What do the colours represent?
- Does finding a valid colouring of your graph which uses the fewest colours possible provide a solution to the problem?

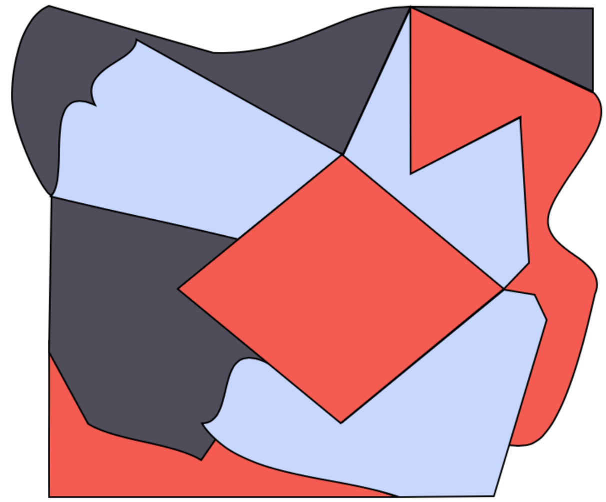
More Info:

Check out the CEMC at Home webpage on Friday, May 1 for a solution to Colouring Fun.

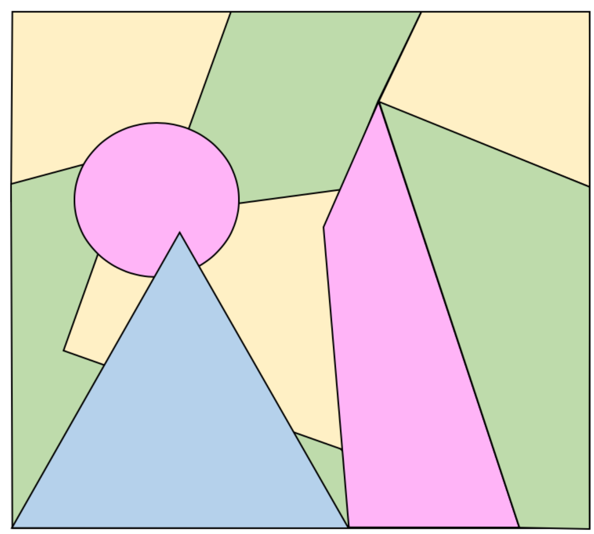
CEMC at Home

Grade 9/10 - Friday, April 24, 2020
Colouring Fun - Solution

1(a) The colouring to the right shows one way this figure can be coloured using three colours. Is there a valid colouring that uses fewer colours? Consider the region in the bottom left corner that is coloured red. This region has exactly two neighbours and these two neighbours are also neighbours of each other. Thus, in a valid colouring, these three regions must all be given different colours. That is, at least three colours are required. Since at least three colours are required and we have found a colouring that uses exactly three colours, the minimum number of colours required to colour this figure is three.



1(b) The colouring to the right shows one way this figure can be coloured using four colours. Is there a valid colouring that uses fewer colours? Consider the following four regions: the circle coloured pink along with the green region to its left, the yellow region to its bottom-left, and the blue region below it. Notice that every pair of regions in this group of four are neighbours. This means that all four of these regions must be given a different colour in a valid colouring. That is, at least four colours are required.
Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this figure is four.



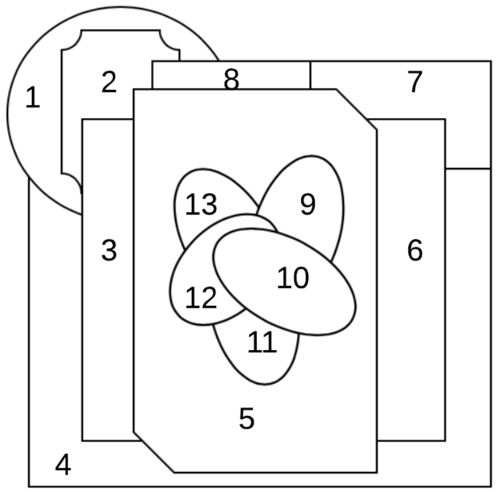
Graph colouring is the assignment of labels, in this case colours, to each vertex of a graph, G, such that no adjacent vertices (i.e. connected by an edge) share the same colour. The goal is to colour G with the minimum number of colours possible. This minimum number is known as the *chromatic number* of G.

There are many applications of graph colouring in the real world. Here are a few examples:

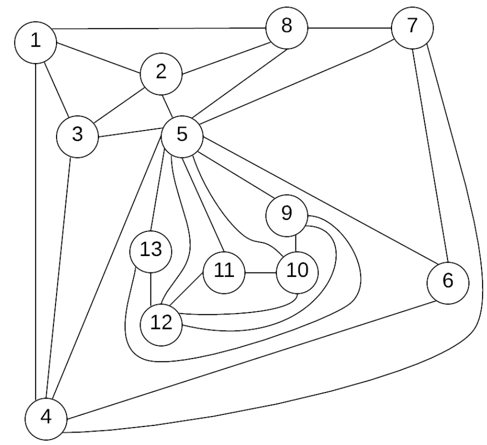
- Scheduling: classes, exams, meetings, sports, flights
- Creating and solving Sudoku puzzles
- Internet bandwidth allocation

2(a) We first work through the activity for the first figure.

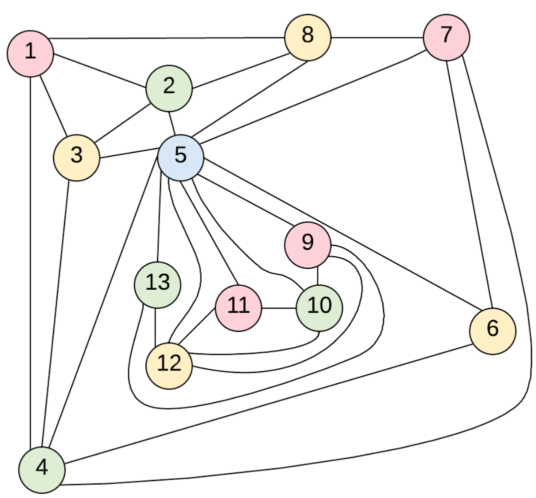
i. Label the regions



ii. Translate into a graph

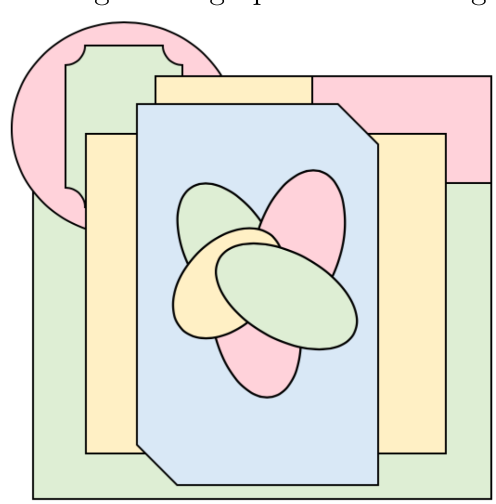


iii. Colour the graph



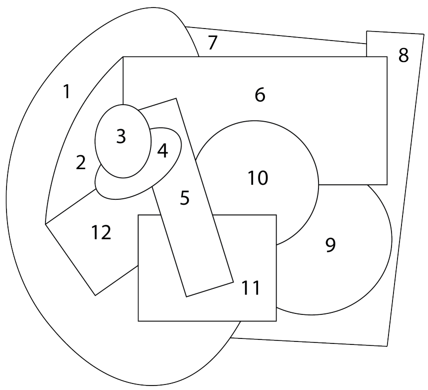
The colouring to the left shows one way this graph can be coloured using four colours. Is there a valid colouring that uses fewer colours? Notice that the group of vertices labelled 5, 10, 11, and 12 has the property that all pairs of vertices from this group are adjacent. Therefore, vertices 5, 10, 11, and 12 all require a different colour in a valid colouring, so at least four colours are required. Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this graph is four. That is, the *chromatic number* of this graph is 4.

Finally, we use translate the colouring of the graph to a colouring of the figure:

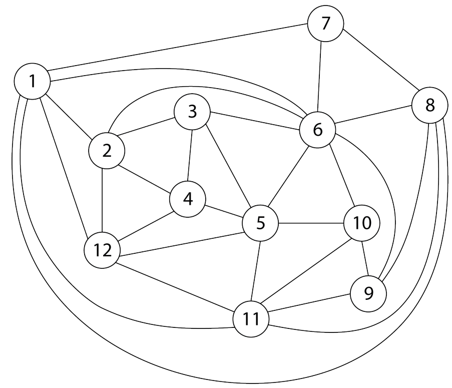


Next we work through the activity for the second figure.

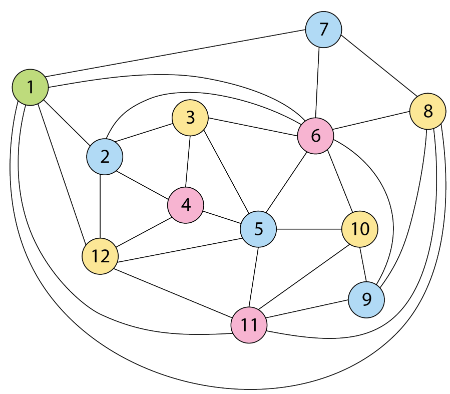
i. Label the regions



ii. Translate into a graph



iii. Colour the graph

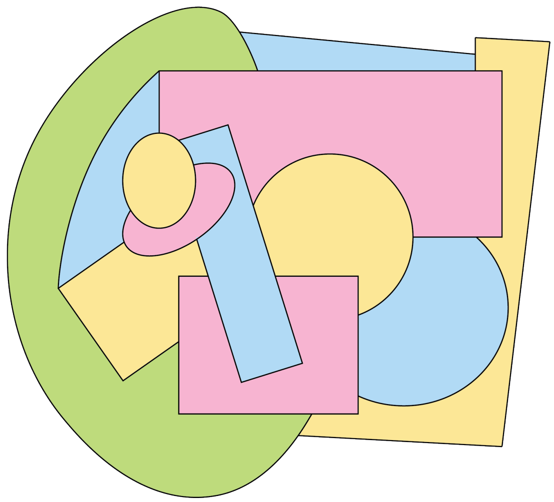


The colouring above shows one way this graph can be coloured using four colours. Is there a valid colouring that uses fewer colours? You can spend a bit of time trying to find a group of four vertices for which each pair of vertices is adjacent, but how do you proceed if you cannot find a group like this? (*Note that vertices 1, 6, 7, and 8 satisfy this property, but we will proceed as though we have not spotted these.*)

Let's try to colour this graph using only three colours: pink, yellow, and blue. Consider the vertex labelled 4 and assign this vertex a colour, say pink. (Note that exactly what colour we choose is not important here.) The vertex 4 is adjacent to the vertices labelled 2, 3, 5, and 12 and so these vertices cannot be coloured pink. Looking at the connections between these four vertices, convince yourself that since only two colours remain, vertices 2 and 5 must be coloured with the same colour, say blue, and vertices 3 and 12 must be coloured with the other colour, yellow. Now consider the vertex numbered 11, which is adjacent to vertex 5 (coloured blue) and vertex 12 (coloured yellow), and so must be coloured pink. Finally, consider the vertex numbered 1. This vertex is adjacent to vertex 2 (coloured blue), vertex 12 (coloured yellow), and vertex 11 (coloured pink). In order to produce a valid colouring, vertex 1 must be assigned a colour, but that colour cannot be pink, yellow, or blue! This means we cannot possibly finish our colouring without adding a fourth colour.

Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this graph is four. That is, the *chromatic number* of this graph is 4.

Finally, we use translate the colouring of the graph to a colouring of the figure:



2(b) It turns out that, as hard as you try, you will not be able to create a two-dimensional figure that requires five colours in order to achieve a valid colouring. This result follows from a famous theorem in graph theory known as the *Four Colour Theorem*. The Four Colour Theorem states that any two-dimensional map, like the figures we have been colouring in this activity, always has a valid colouring with four colours. That is, it can always be coloured with four or fewer colours.

The proof of the Four Colour Theorem is not trivial. In fact, this theorem was first conjectured in 1852, but was not proven until the 1970s, and the proof required the aid of a computer!

CEMC at Home

Grade 9/10 - Monday, April 27, 2020
Race Track

You Will Need:

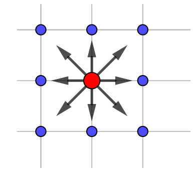
- One to four players
- A Race Track on grid paper
  A Race Track is provided for you on the second to last page. You are also given a blank grid on the last page where you can create your own track!
- A different coloured pen or pencil for each player.
  Since you are likely to play this game multiple times, you may want to place the Race Track inside a sheet protector and then use dry erase markers to play instead. If you do not have a sheet protector, try using clear tape to create an erasable surface for the track.

How to Play:

1. Start with a Race Track.
2. Players take turns. Decide which player will go first, second, and so on.
3. To start the game, each player must place their "car" at a different place on the starting line. Players can do so, one at a time, based on the chosen order of the players.
   Placing your "car" on the starting line actually means drawing a dot on top of one of the grid points lying on the starting line. Each player needs to place their car on a different grid point. You can place your car on the boundary of the track.
4. On each turn, the current player will move their car according to the allowed moves in the game.
   Moving your car means placing a new dot at a new grid point on the track. See below for a description of the rules allowed in the game.
5. The winner is the first player to complete a lap, that is, the first player whose car crosses the finish line.

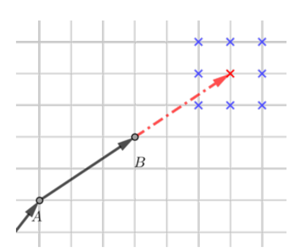
Allowed Moves

All moves must be from one grid point to another grid point. Each grid point has eight neighbouring grid points as shown to the right.



From the starting line, each player's first move must be moving their car to one of the eight neighbours of their starting position.

For all subsequent moves, players must move their car the same distance in the same direction as their previous move, or to one of the eight neighbours of that final position. For example, if arrow AB represents the player's previous move as shown to the right, then on this player's next turn, they can move their car either to the spot marked with a red $\times$, or to any of the eight neighbours of the point with a red $\times$, each marked with a blue $\times$.

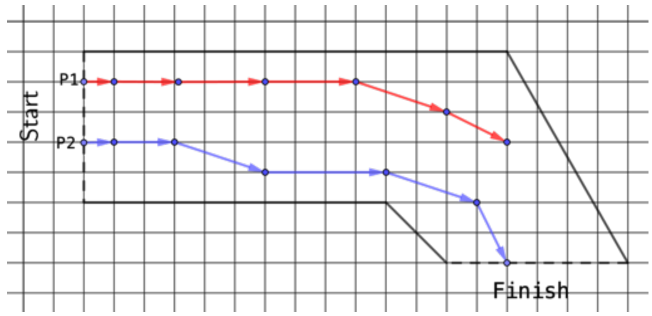


Notice that a move from B to the red $\times$ is represented by an arrow that is the same length and in the same direction as the arrow from A to B.

A car cannot be moved to a grid point where another car is already located.

Here is an example of a two player game on a simple Race Track.

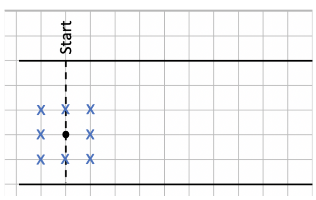
Player 1 (P1) goes first.
Player 2 (P2) wins in 6 moves.

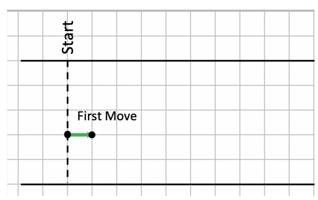


Here is an explanation of Player 2's first four moves in the sample game above.

First Move

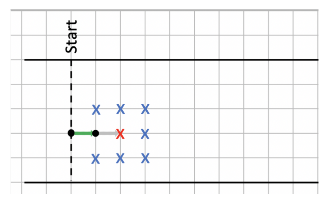
P2 can move their car to any of the eight locations marked with an ×. P2 chooses to move one grid point to the right. Note that P2 could move backward, but this may not be the best choice if P2 is hoping to complete a lap quickly. (Moving backwards to a finish line does not count!)

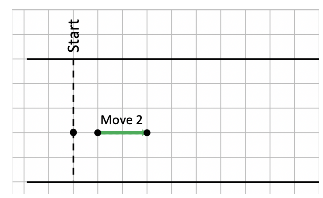




Second Move

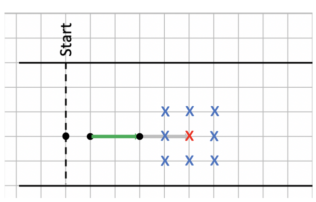
Since P2's previous move was one grid point to the right, we place the red × one grid point to the right of P2's current position. P2 can move to this × or any of the eight locations surrounding it (including P2's current position). P2 moves two grid points to the right.

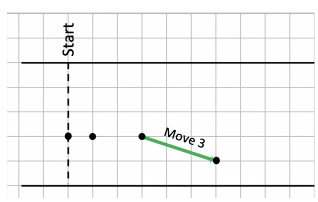




Third Move

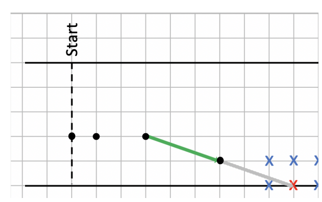
Since P2's previous move was two grid points to the right, we place the red × two grid points to the right of P2's current position. P2 can move to this × or any of the eight locations surrounding it. P2 moves three grid points to the right and one grid point down.

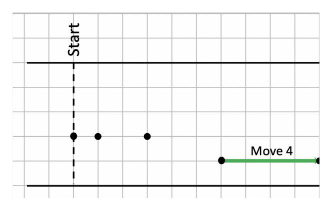




Fourth Move

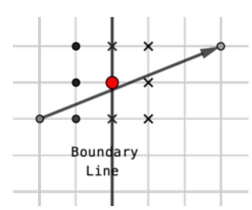
Since P2's previous move was three to the right and one down, we place the red × three to the right and one down from P2's current position. P2 can move to this × or to *some* of the eight locations surrounding it. P2 moves four grid points to the right.





Dealing with the Boundary

During the game, there may be a time when a player has no choice but to move their car onto or through a boundary line of the Race Track on their turn (as shown to the right). If this happens, then the player places their car at the grid point nearest to where their move touched the boundary (as shown by the red dot). On this player's next turn, they move their car to one of the eight neighbours of their current place (red dot) that *lies inside the track* (shown with black dots).

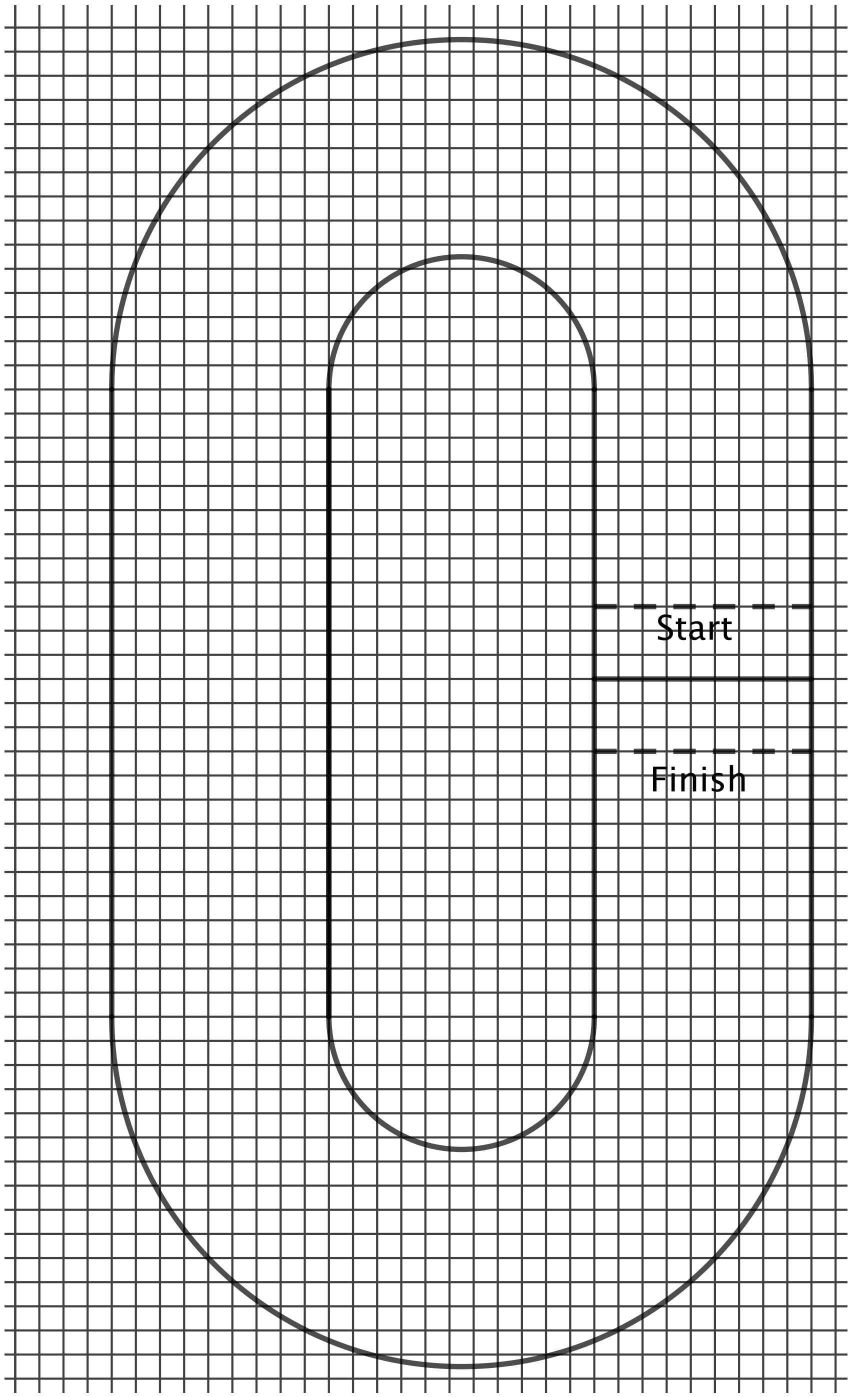


Let's Play!

Play this game a number of times using the track given on the next page. Alternate which player goes first. Were you able to figure out how to avoid hitting a boundary of the Race Track?

More Info: A *vector* is defined as a quantity which has both a magnitude and a direction. In Race Track, each move can be represented by a vector. To learn more about vectors see this Math Circles lesson.

Sample Race Track



Make Your Own Race Track!

You can use your own grid paper or the grid below. Add some sharp corners for an extra challenge!

CEMC at Home

Grade 9/10 - Tuesday, April 28, 2020
More Counting

In some areas of mathematics, we study things called *permutations*. A permutation of a collection of objects is an arrangement of the objects in some order.

For example, consider the integers 1, 2, and 3. There are six different ways to arrange these three objects, in some order, and so there are six permutations of these objects. The permutations are given below:

$$(1,\ 2,\ 3) \qquad (1,\ 3,\ 2) \qquad (3,\ 1,\ 2) \qquad (3,\ 2,\ 1) \qquad (2,\ 1,\ 3) \qquad (2,\ 3,\ 1)$$

In the questions below, we will work with permutations of consecutive integers, and we will think about a particular type of permutation which we will call a VALROBSAR permutation.

> A permutation will be called a **VALROBSAR** permutation if *no* integer in the permutation has two neighbours that both are less than it.
>
> *Two integers in the permutation are neighbours if they appear directly beside each other.*

From our example above, the permutations (1, 3, 2) and (2, 3, 1) are *not* VALROBSAR permutations. This is because, in each of these permutations, the integer 3 has a smaller integer immediately to its left and immediately to its right. That is, the integer 3 has two neighbours that are both less than 3.

The other four permutations shown above *are* VALROBSAR permutations. For example, let's look at the permutation (3, 1, 2). The integer 3 has only one neighbour and so does not have two neighbours less than 3. The integer 2 also has only one neighbour and so does not have two neighbours less than 2. The integer 1 has two neighbours but they are both greater than 1. As a second example, let's look at (3, 2, 1). The integers 3 and 1 each have only one neighbour and so do not have two neighbours less than themselves, and the integer 2 has two neighbours but only one of them is less than 2. You should work through the remaining two permutations on your own to verify that they are indeed VALROBSAR permutations.

Problems:

1. List all permutations of the integers 1, 2, 3, and 4.
2. How many of the permutations of the integers 1, 2, 3, and 4 are VALROBSAR permutations?
3. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, and 5?
4. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, 5, and 6?

Extension: Can you see a pattern forming based on your work in problems 1. to 4.? Suppose that n is a positive integer satisfying $n \geq 2$ and consider the permutations of the integers $1, 2, 3, 4, \ldots, n$. What can you say about the number of VALROBSAR permutations of these integers?

More Info:

Check out the CEMC at Home webpage on Tuesday, May 5 for a solution to More Counting.

CEMC at Home

GRADE 9/10 - Tuesday, April 28, 2020
More Counting - Solution

A permutation will be called a **VALROBSAR** permutation if *no* integer in the permutation has two neighbours that both are less than it.

Two integers in the permutation are neighbours if they appear directly beside each other.

Problems:

1. List all permutations of the integers 1, 2, 3, and 4.
2. How many of the permutations of the integers 1, 2, 3, and 4 are VALROBSAR permutations?
3. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, and 5?
4. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, 5, and 6?

Solutions:

1. There are 24 permutations of the four integers:

1 2 3 4	1 2 4 3	1 3 2 4	1 3 4 2	1 4 2 3	1 4 3 2
2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1
3 1 2 4	3 1 4 2	3 2 1 4	3 2 4 1	3 4 1 2	3 4 2 1
4 1 2 3	4 1 3 2	4 2 1 3	4 2 3 1	4 3 1 2	4 3 2 1

2. There are 8 VALROBSAR permutations. They are the red permutations in the table below:

1 2 3 4	1 2 4 3	1 3 2 4	1 3 4 2	1 4 2 3	1 4 3 2
2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1
3 1 2 4	3 1 4 2	**3 2 1 4**	3 2 4 1	3 4 1 2	3 4 2 1
4 1 2 3	4 1 3 2	**4 2 1 3**	4 2 3 1	**4 3 1 2**	**4 3 2 1**

3. Looking at the solution to 2., you might have noticed that all of the VALROBSAR permutations have the number 4 at one of the two ends. It will be helpful to think about why this must be true. In order to be a VALROBSAR permutation, every number must have at most one neighbour that is smaller than it. Since 4 is the largest among 1, 2, 3, and 4, if a permutation has 4 in one of the two middle positions, then 4 is guaranteed to have two neighbours that are smaller than it. Hence, a permutation cannot be a VALROBSAR permutation unless 4 is at one of the ends. It is worth noting that there are permutations with a 4 at the end that fail to be VALROBSAR permutations, for example, 4132.

   We now focus our attention on the VALROBSAR permutations of 1, 2, 3, 4, and 5. By the same reasoning as in the previous paragraph, a VALROBSAR permutation of these integers must have 5 at one of the ends. The next key observation is that if we "remove" this 5 from the end, what remains will be a VALROBSAR permutation of 1, 2, 3, and 4. This is because removing a number on the end of a permutation does not introduce any new neighbouring pairs, and so cannot cause a failure of the VALROBSAR condition.

   This means all of the VALROBSAR permutations of 1, 2, 3, 4, and 5 take the form $5abcd$ or $abcd5$ where $abcd$ is a VALROBSAR permutation of 1, 2, 3, and 4. On the other hand, if we take a VALROBSAR permutation of 1, 2, 3, and 4 and place a 5 on either end, what results is a VALROBSAR permutation of 1, 2, 3, 4, and 5. To see this, suppose $abcd$ is a VALROBSAR permutation of 1, 2, 3, and 4 and consider the permutation $5abcd$. The neighbours of b, c, and d in $5abcd$ are the same as they are in the permutation $abcd$. Since we are assuming $abcd$ is a VALROBSAR permutation, each of a, b, and c has at most one neighbour smaller than it in $abcd$, and hence, has at most one neighbour smaller than it in $5abcd$. We know that a is equal to one of 1, 2, 3, and 4, so $a < 5$, which means a has at most one neighbour smaller than it in $5abcd$ (namely, b *could* be smaller than a). The number 5 is at the end of the permutation, so it cannot possibly cause a failure of the VALROBSAR condition.

   We are now able to quickly count the number of VALROBSAR permutations of 1, 2, 3, 4, and 5. Using the discussion above, we get all of *these* VALROBSAR permutations by taking a VALROBSAR permutation of 1, 2, 3, and 4 and placing a 5 on one of the two ends. There are 8 VALROBSAR permutations of 1, 2, 3, and 4, so this gives $2 \times 8 = 16$ VALROBSAR permutations of 1, 2, 3, 4, and 5. Furthermore, each of these 16 VALROBSAR permutations must be different. *Can you see why?*

   The VALROBSAR permutations of 1, 2, 3, 4, and 5 are given below:

   1 2 3 4 5   2 1 3 4 5   3 1 2 4 5   3 2 1 4 5

   4 1 2 3 5   4 2 1 3 5   4 3 1 2 5   4 3 2 1 5

   5 1 2 3 4   5 2 1 3 4   5 3 1 2 4   5 3 2 1 4

   5 4 1 2 3   5 4 2 1 3   5 4 3 1 2   5 4 3 2 1

   Note: There are more direct ways of counting these permutations without building on the permutations of 1, 2, 3, and 4. (An idea of this form will be discussed in the Extension on the last page.) The method presented above doesn't just give us an easy to count the "next order" of VALROBSAR permutations, but also gives an easy way to list them (based on the "previous list"). When using permutations in mathematics, sometimes we are interested in only the count, and sometimes we are interested in the actual list of permutations. It is often helpful to think about building them in "stages" like we have done here.

4. Similar to the argument in the previous solution, a VALROBSAR permutation of 1, 2, 3, 4, 5, and 6 must have the 6 at one of the ends and what remains after removing the 6 must be a VALROBSAR permutation of 1, 2, 3, 4, and 5. Furthermore, we get a VALROBSAR permutation of 1, 2, 3, 4, 5, and 6 by taking any VALROBSAR permutation of 1, 2, 3, 4, and 5 and adding a 6 to either end of the permutation. For every choice of a VALROBSAR permutation of 1, 2, 3, 4, and 5 and choice of which side to add the 6, we get a VALROBSAR permutation of 1, 2, 3, 4, 5, and 6. There are 16 VALROBSAR permutations of 1, 2, 3, 4, and 5, so this means there are $2 \times 16 = 32$ VALROBSAR permutations of 1, 2, 3, 4, 5, and 6.

Extension: Can you see a pattern forming based on your work in problems 1. to 4.? Suppose that n is a positive integer satisfying $n \geq 2$ and consider the permutations of the integers $1, 2, 3, 4, \ldots, n$. What can you say about the number of VALROBSAR permutations of these integers?

Discussion:

To recap, we found the following counts of the VALROBSAR permutations:

- $n = 4$: $2^3 = 8$ VALROBSAR permutations
- $n = 5$: $2^4 = 16$ VALROBSAR permutations
- $n = 6$: $2^5 = 32$ VALROBSAR permutations

You might guess from this pattern that the number of VALROBSAR permutations of $1, 2, \ldots, n$ is 2^{n-1}, in general. In fact, the arguments in 3. and 4. actually showed why the number of VALROBSAR permutations seemed to double at each stage, and these arguments can be used to justify this formula.

Here is a more direct way to count the number of VALROBSAR permutations of $1, 2, \ldots, n$. Let's build such a permutation by placing each integer in turn and keeping track of how many choices we have at each stage. (This argument could have been used in 3. and 4. as well.)

First, consider n, the largest integer. No matter where you place it in the permutation it will be larger than its neighbours, and so it must have only one neighbour. The integer n must be placed at an end of the permutation, either first or last, which means you have 2 choices.

$$n \_ \_ \_ \ldots \_ \qquad \text{or} \qquad \_ \_ \_ \ldots \_ n$$

Once you place n, you have $(n-1)$ places left for the remaining integers $1, 2, \ldots, n-1$. As with n above, you have 2 choices for where to place the integer $n-1$: beside n or at the other end of the permutation.

For example, if we placed n as in the left image shown above, then the two leftmost images below show the choices for where to place $n-1$. *Can you see why $n-1$ must be placed like this in order to form a VALROBSAR permutation?* In the other case above, the choices are shown below on the right.

$$n\ (n-1) \_ \_ \ldots \_ \quad \text{or} \quad n \_ \_ \_ \ldots (n-1) \quad \text{or} \quad (n-1) \_ \_ \ldots \_ n \quad \text{or} \quad \_ \_ \_ \ldots (n-1)\ n$$

After each placement of the largest remaining integer, you then have 2 choices for where to place the next largest integer. This continues until you have placed the integer 2, and then the integer 1 must go in the only remaining place.

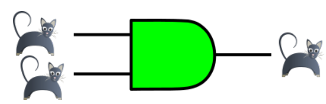
Thus, for each of the first $n-1$ integers, $n, (n-1), (n-2), (n-3), \ldots, 2$, you have 2 choices of where each is placed, and then the 1 goes in the last open place. Thus, the number of VALROBSAR permutations of the integers $1, 2, \ldots, n$ is 2^{n-1}.

CEMC at Home

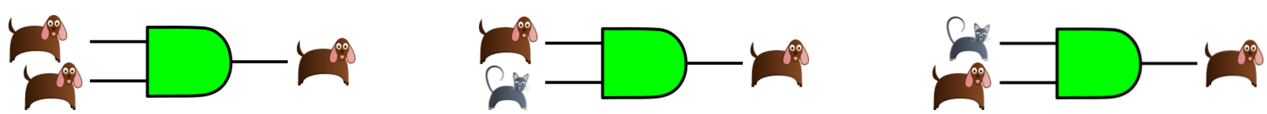
Grade 9/10 - Wednesday, April 29, 2020
Magic Carpets

In a magic castle there are magic carpets and an unlimited supply of toy cats and dogs. The magic carpets behave as follows:

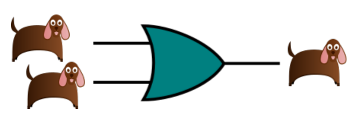
* If two cats walk on a **green** magic carpet, exactly one cat walks off.



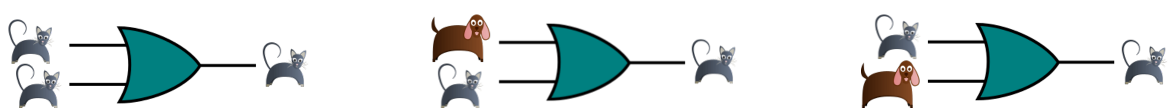
* If any other pair of animals walk on a **green** magic carpet, exactly one dog walks off.



* If two dogs walk on a **teal** magic carpet, exactly one dog walks off.



* If any other pair of animals walk on a **teal** magic carpet, exactly one cat walks off.

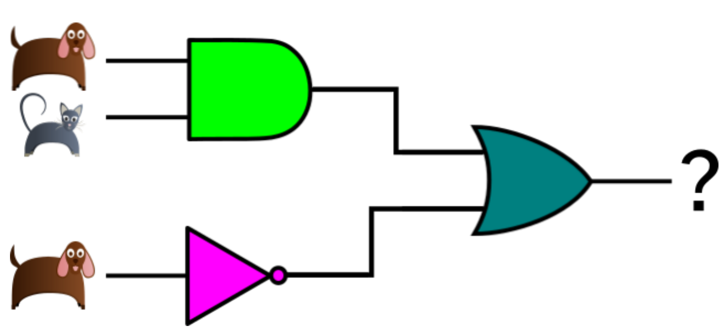


* If a cat walks on a **pink** magic carpet, a dog walks off. If a dog walks on a **pink** magic carpet, a cat walks off.

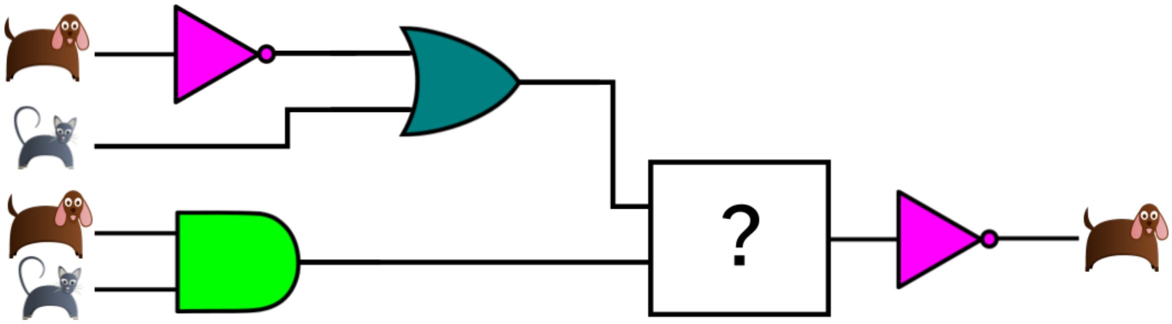
Questions:

Note that in the diagrams in the questions that follow, a line between two carpets means that the animal that walks off the left carpet is the same animal that then walks on the next carpet to the right.

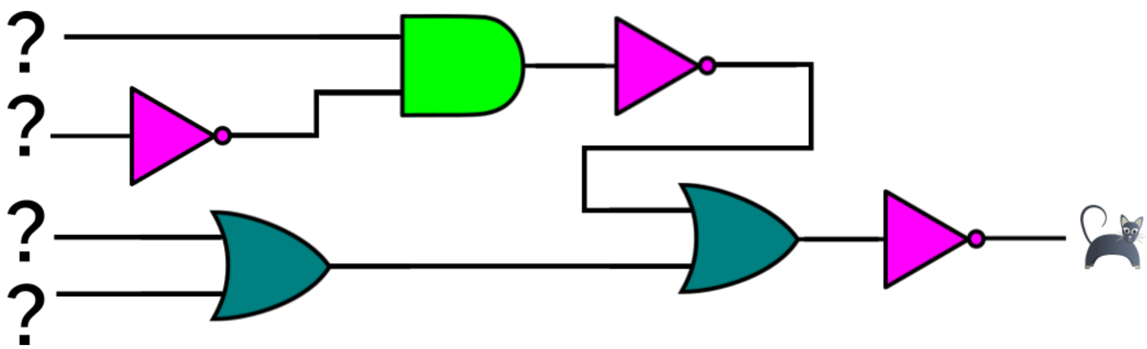
1. On the first floor of the castle, the magic carpets are arranged as shown. If cats and dogs walk on the carpets as indicated, which animal will walk off the rightmost carpet?



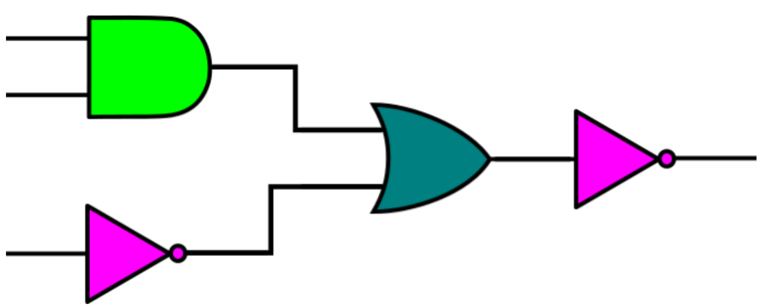
2. On the second floor of the castle, the magic carpets are arranged as shown. If cats and dogs walk on the carpets as indicated, and a dog walks off the rightmost carpet, identify the colour of the missing magic carpet.



3. On the third floor of the castle, the magic carpets are arranged as shown. If a cat walks off the rightmost carpet, which four animals walked onto the carpets?



4. On the fourth floor of the castle, the owner would like to arrange magic carpets as shown below. However, teal magic carpets are incredibly expensive! Suggest a new arrangement, using only green and pink magic carpets, that will have the same behaviour as the owner's desired arrangement. That is, for every possible combination of three animals that could walk across the leftmost carpets, the animal that would walk off the rightmost carpet will be the same in both arrangements.



More Info:

Check out the CEMC at Home webpage on Wednesday, May 6 for a solution to Magic Carpets.

The green, teal, and pink magic carpets *look* and *act* like AND, OR, and NOT gates, respectively. Check out Escape Room on the 2019 Beaver Computing Challenge for a similar problem and more information about gates.

CEMC at Home

Grade 9/10 - Wednesday, April 29, 2020
Magic Carpets - Solution

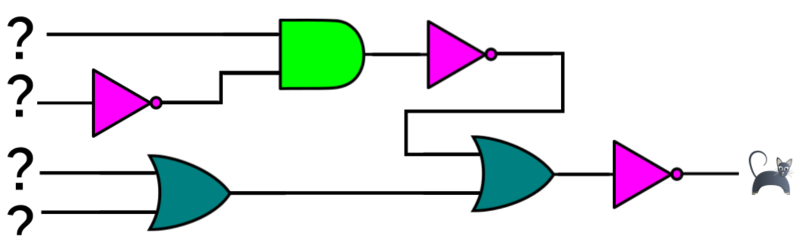
1. A dog will walk off the green carpet and a cat will walk off the pink carpet. Therefore, a dog and a cat will walk on the teal carpet, and so a cat will walk off the teal carpet (rightmost carpet).

2. It must be the case that a cat walks *off* the missing carpet since only a cat walking on the rightmost (pink) carpet will cause a dog to walk off. What animals walk *on* the missing carpet? At the top-left, a cat walks off the pink carpet and so two cats walk on the teal carpet. This means one cat walks off the teal carpet and approaches the missing carpet. At the bottom-left, a dog walks off the green carpet and approaches the missing carpet. So a cat and a dog walk on the missing carpet and a cat walks off. This can only happen if the missing carpet is teal.

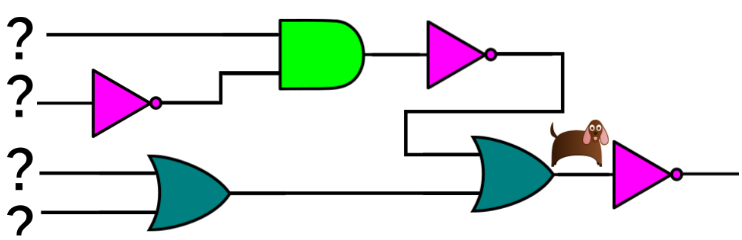
3. From top to bottom the animals are cat, dog, dog, and dog.

   One way to obtain this answer is to try all possible combinations of animals until you find one that results in a cat walking off the rightmost carpet. How many combinations would you have to try? Since there are 4 spots for animals, and each spot could be a cat or a dog, there are $2^4 = 16$ combinations. Testing each combination in turn may not be the best way to proceed.

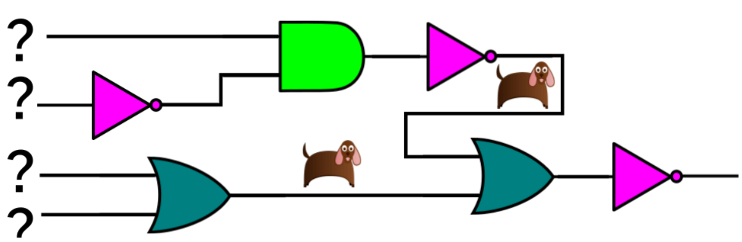
   An alternative way is to work backwards. Imagine that you have a video of the animals walking across the carpets, and you play the video in reverse. Here is a series of images that show the cat on the right moving backwards over the carpets.



The only way a cat can walk off a pink carpet is if a dog walks on.



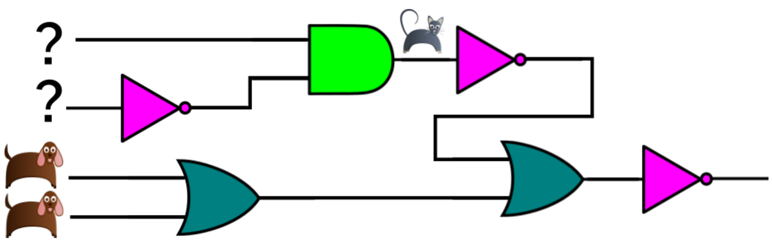
The only way a dog can walk off a teal carpet is if two dogs walk on.



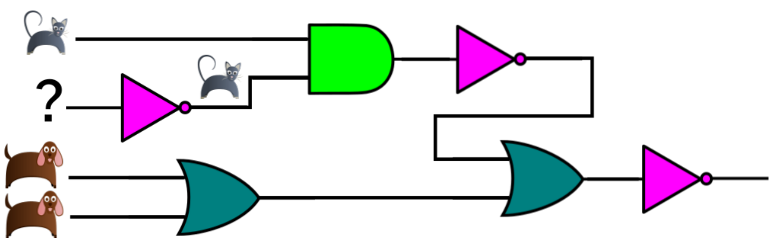
The only way a dog can walk off a pink carpet is if a cat walks on.

AND

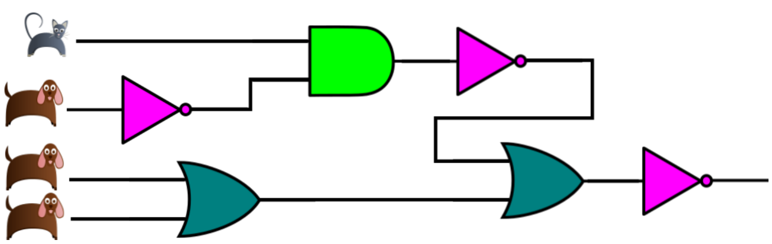
The only way a dog can walk off a teal carpet is if two dogs walk on.



The only way a cat can walk off a green carpet is if two cats walk on.

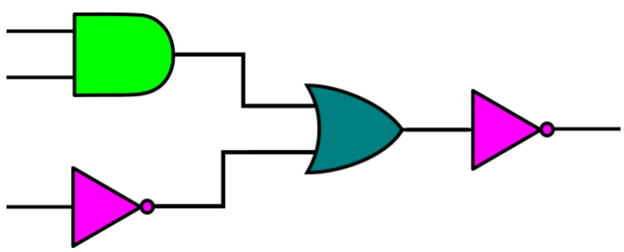


The only way a cat can walk off a pink carpet is if a dog walks on.

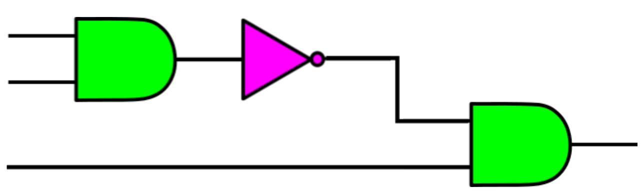


Therefore, this shows the only option for the starting animals.

4. The given arrangement with the teal carpet



is equivalent to the following arrangement that uses no teal carpets:



You may have found a different arrangement than this that also works.

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 30, 2020
A De-Light-Ful Machine

A machine has 2020 lights and 1 button. Each button press changes the state of exactly 3 of the lights. That means if the light is currently on, it turns off, and if the light is currently off, it turns on. Before each button press, the user selects which 3 lights will change their state.

To begin with, all the lights on the machine are off. What is the fewest number of button presses required in order for all the lights to be on?

Hint: Start by thinking about a machine with fewer lights.

More Info:

Check the CEMC at Home webpage on Thursday, May 7 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, May 7.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week
Problem D and Solution
A De-Light-Ful Machine

Problem

A machine has 2020 lights and 1 button. Each button press changes the state of exactly 3 of the lights. That means if the light is currently on, it turns off, and if the light is currently off, it turns on. Before each button press, the user selects which 3 lights will change their state. To begin with, all the lights on the machine are off. What is the fewest number of button presses required in order for all the lights to be on?

Hint: Start by thinking about a machine with fewer lights.

Solution

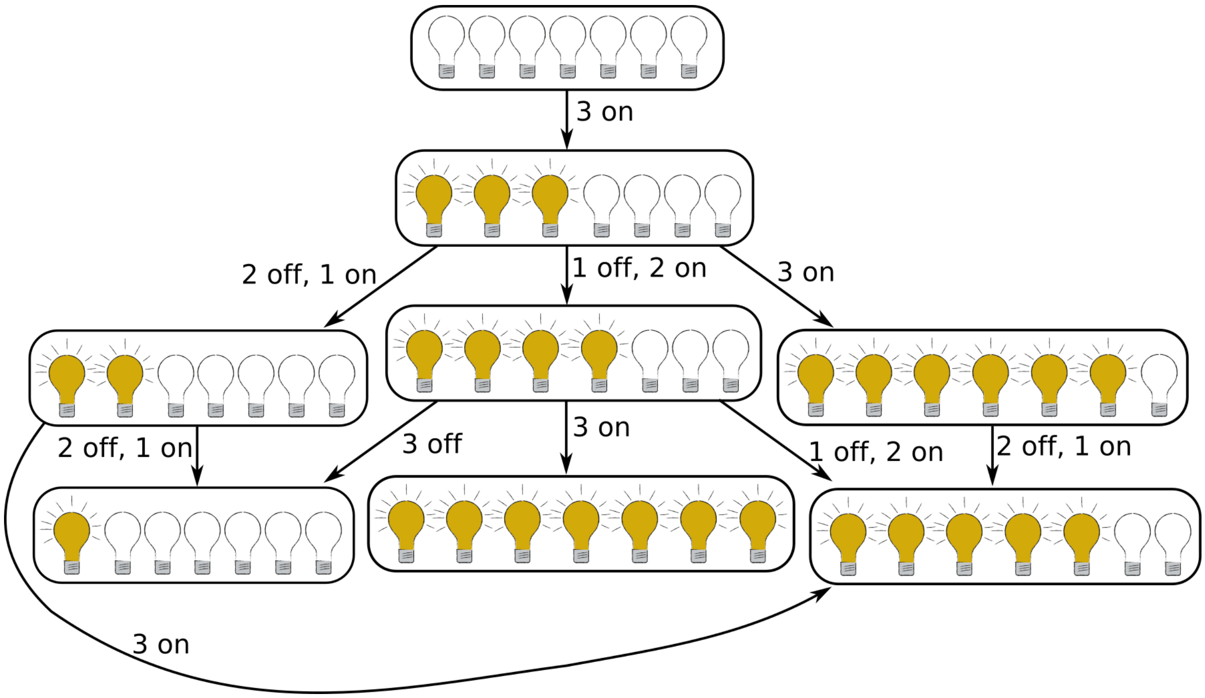
To turn on all the lights with the fewest number of button presses, we should turn on 3 lights with each button press, and not turn any lights off.

- The first button press would turn on 3 lights.
- The second button press would turn on 3 more lights, bringing the total to 6 lights on.
- The third button press would turn on 3 more lights, so now there would be 9 lights on.
- And so on . . .

Continuing in this way we can see that the total number of lights on would always be a multiple of 3. However, since 2020 is not a multiple of 3, this tells us that at least 1 button press must turn some lights off. Since we want to press the button the fewest number of times, that means we want the fewest number of button presses to turn lights off.

Now suppose the button was pressed 671 times, and each time 3 lights turned on. Then there would be $671 \times 3 = 2013$ lights on in total. Let's look at the remaining 7 lights that are still off. We can draw a diagram to show all the possible outcomes for the next button presses until all 7 lights are on.

Note that the order of the lights does not matter. We are interested in how many lights are on, not which particular lights are on. To simplify our diagram, at each step we have moved all of the lights that are on to the left.



Note that if a button press reverses the press that was just made, we did not include this in our diagram, as this will not give us the fewest number of button presses.

We can see in the diagram that the shortest sequence of steps to get all the lights on would be:

1. Turn 3 lights on.
2. Turn 1 light off and 2 lights on.
3. Turn 3 lights on.

This takes 3 steps, which means 3 button presses. If we add this to the 671 button presses to get to this point, that tells us there are $671 + 3 = 674$ button presses in total. We note that only 1 of these 674 button presses turns lights off. Since we know that at least 1 button press must turn some lights off, that tells us we cannot turn all the lights on using fewer button presses.

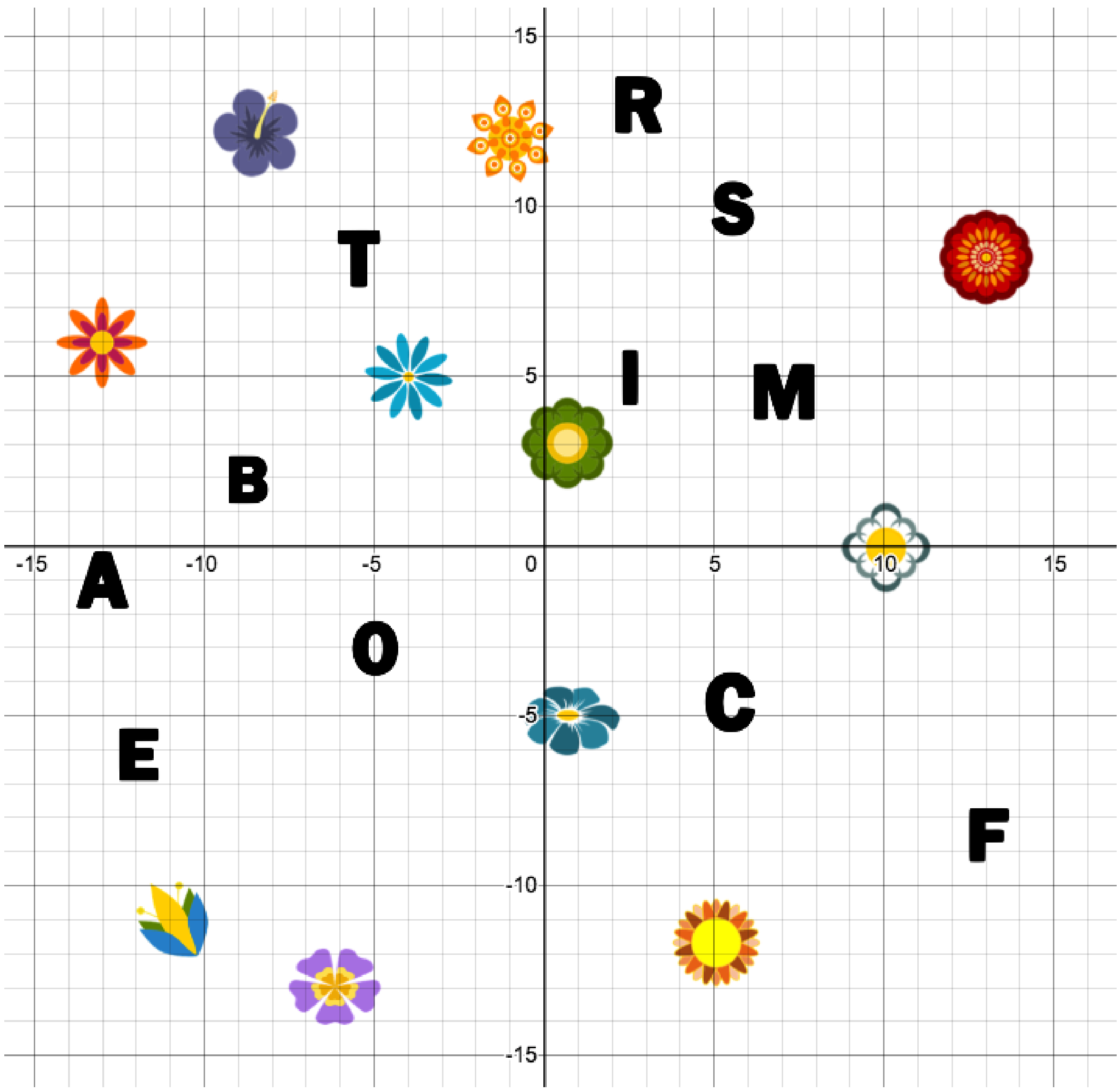
Therefore, 674 button presses are required to turn on all the lights.

CEMC at Home

Grade 9/10 - Friday, May 1, 2020
Flowers, Letters and Lines

Instructions: Eleven lines are described on the next page either by an equation or with other information. Carefully graph these lines on the grid below using a ruler. Each line should pass through *exactly one flower and one letter*. Match each flower to the letter that lies on its line to answer the riddle below. *The table on the next page might be useful to help organize your work.*

Riddle: What does the letter A have in common with a flower?



Answer:

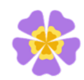
___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___ !

Lines:

$y = 5x$	$y = \frac{1}{9}x - 10$	$x + 2y - 16 = 0$
$y + 15 = -\frac{1}{3}(x - 15)$	$y = \frac{17}{2}$	$x - 5y - 10 = 0$
A line that has a slope of -2 and a y-intercept of 10	A line that passes through the points $(-15, 40)$ and $(-5, -20)$	A line that has a slope of -1 and passes through the point $(9, -8)$
$y + 4 = 3(x - 1)$	A vertical line that passes through the point $(-13, 0)$	

You may need to re-arrange a given equation or do some additional calculations to make the information about the given line more useful for graphing it.

You may find the table below useful in organizing your work.

Flower											
Letter											

More Info:

Check the CEMC at Home webpage on Friday, May 8 for a solution to Flowers, Letters and Lines.

For more practice with graphing linear equations, check out this lesson in the CEMC Courseware. There are also other lessons you may wish to review in the Linear Relations unit.

CEMC at Home

Grade 9/10 - Friday, May 1, 2020
Flowers, Letters and Lines - Solution

The eleven linear equations in $y = mx + b$ form are:

1. $y = 5x$	2. $y = \frac{1}{9}x - 10$	3. $x + 2y - 16 = 0$ $y = -\frac{1}{2}x + 8$
4. $y + 15 = -\frac{1}{3}(x - 15)$ $y = -\frac{1}{3}x - 10$	5. $y = \frac{17}{2}$	6. $x - 5y - 10 = 0$ $y = \frac{1}{5}x - 2$
7. A line that has a slope of -2 and a y-intercept of 10 $y = -2x + 10$	8. A line that passes through the points $(-15, 40)$ and $(-5, -20)$ $y = -6x - 50$	9. A line that has a slope of -1 and passes through the point $(9, -8)$ $y = -x + 1$
10. $y + 4 = 3(x - 1)$ $y = 3x - 7$	11. A vertical line that passes through the point $(-13, 0)$ $x = -13$	

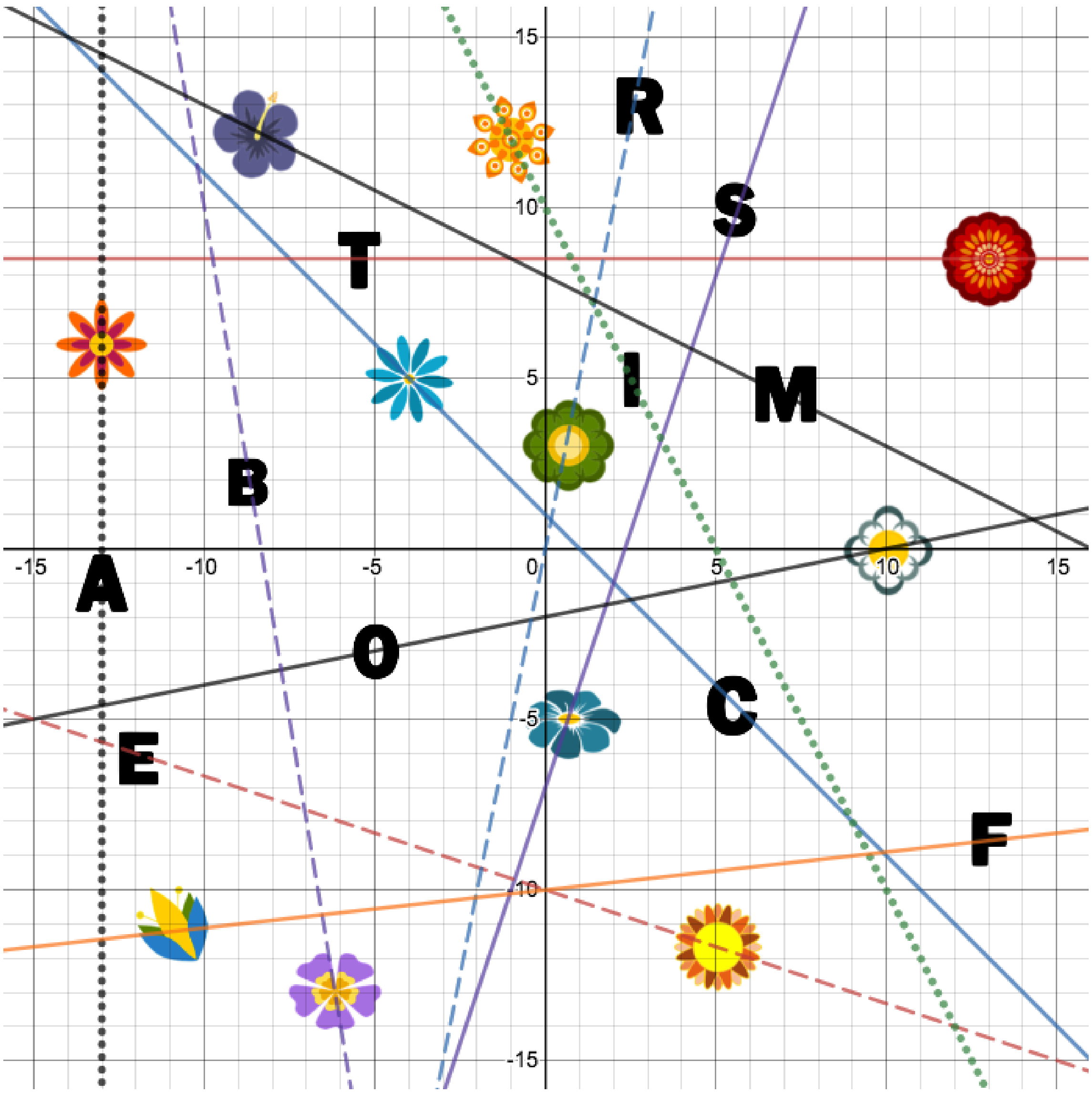
After graphing each linear equation (see next page for the graph), we see that each line goes through exactly one flower and one letter. Using the graph, we can fill in the table below:

Flower											
Letter	T	O	M	E	S	I	A	C	R	B	F
Equation	5.	6.	3.	4.	10.	7.	11.	9.	1.	8.	2.

Using the information from above, we can answer the riddle *"What does the letter A have in common with a flower?"*

A B E E C O M E S A F T E R I T !

Click on the graph to explore it further!



CEMC at Home

Grade 9/10 - Monday, May 4, 2020
Contest Day 1

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Euclid Contest, #2(a)

The three-digit positive integer m is odd and has three different digits. If the hundreds digit of m equals the product of the tens digit and ones (units) digit of m, what is m?

2020 Canadian Team Mathematics Contest, Team Problem #6

On Fridays, the price of a ticket to a museum is \$9. On one particular Saturday, there were 200 visitors to the museum, which was twice as many visitors as there were the day before. The total money collected from ticket sales on that particular Saturday was $\frac{4}{3}$ as much as the day before. The price of tickets on Saturdays is \$$k$. Determine the value of k.

More Info:

Check out the CEMC at Home webpage on Monday, May 11 for solutions to the Contest Day 1 problems.

CEMC at Home

Grade 9/10 - Monday, May 4, 2020
Contest Day 1 - Solution

Solutions to the two contest problems are provided below, including a video for the first problem.

2020 Euclid Contest, #2(a)

The three-digit positive integer m is odd and has three different digits. If the hundreds digit of m equals the product of the tens digit and ones (units) digit of m, what is m?

Solution:

Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 is must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

Video

Visit the following link to view a discussion of a solution to the first contest problem:
https://youtu.be/dJ6d0ILAGwE

2020 Canadian Team Mathematics Contest, Team Problem #6

On Fridays, the price of a ticket to a museum is \$9. On one particular Saturday, there were 200 visitors to the museum, which was twice as many visitors as there were the day before. The total money collected from ticket sales on that particular Saturday was $\frac{4}{3}$ as much as the day before. The price of tickets on Saturdays is $\$k$. Determine the value of k.

Solution:

There were 200 visitors on Saturday, so there were 100 visitors the day before. Since tickets cost \$9 on Fridays, the total money collected on Friday was \$900.

Therefore, the amount of money collected from ticket sales on the Saturday was $\frac{4}{3}(\$900) = \$1200$.

Since there were 200 visitors on Saturday, the price of tickets on that particular Saturday was $\frac{\$1200}{200} = \$6$. Therefore, the value of k is 6.

It turns out that you do not need to know the number of visitors to the museum on the Saturday to solve the problem. If this number (200) changes, but all other conditions in the problem are kept the same, then the answer will still be $k = 6$. Can you see why?

CEMC at Home

Grade 9/10 - Tuesday, May 5, 2020
Patterns in Arithmetic

The questions included in this activity can be solved by looking for a pattern and using it to get at the solution.

Problem 1: A Wizard's assistant is paid in an unusual way. The assistant's paycheque for the first week is one dollar. At the end of each week after the first week, the assistant is paid the amount of money earned the previous week plus two dollars for every week worked so far. What is the assistant's paycheque, in dollars, for the fifty-second week?

To get started, calculate the paycheque for a particular earlier week, say the 9^{th} *or* 10^{th} *week, looking for a pattern while you do so.*

Problem 2: Suppose that the integer N is the value of the following sum (with 52 terms):

$$1 + 11 + 101 + 1001 + 10001 + \cdots + 1\overbrace{000\ldots000}^{\text{50 zeroes}}1$$

When N is calculated and written as a single integer, what is the sum of its digits?

To get started, consider how each term in the sum is formed.

Problem 3: Using only the digits $1, 2, 3, 4,$ and 5, a sequence is created. The beginning of the sequence in shown below.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, \ldots$$

The sequence starts with one 1, followed by two 2s, then three 3s, four 4s, five 5s, six 1s, seven 2s, and so on. What is the 1000th term in the sequence?

Some patterns can be tough to explain precisely using few words. The description above is likely sufficient to relay to you how the sequence is defined, but does not precisely define the remaining terms in the sequence. Can you come up with a more formal way to describe how this sequence is defined?

Discovering the correct patterns can lead you to the correct answers for these problems. Think about how you can justify that the pattern you discover in each of the problems is in fact correct.

More Info:

Check out the CEMC at Home webpage on Tuesday, May 12 for a solution to Patterns in Arithmetic.

CEMC at Home

Grade 9/10 - Tuesday, May 5, 2020
Patterns in Arithmetic - Solution

Problem 1: A Wizard's assistant is paid in an unusual way. The assistant's paycheque for the first week is one dollar. At the end of each week after the first week, the assistant is paid the amount of money earned the previous week plus two dollars for every week worked so far. What is the assistant's paycheque, in dollars, for the fifty-second week?

Solution:

Looking at the first few weeks' salaries we have:

Week	Pay ($)
1	1
2	$1 + 2(2) = 5$
3	$5 + 2(3) = 11$
4	$11 + 2(4) = 19$
$\vdots$	$\vdots$

The above table may not provide enough information, so let's expand things a bit in another table:

Week	Pay ($)
1	1
2	$1 + 2(2) = 5$
3	$1 + 2(2) + 2(3) = 11$
4	$1 + 2(2) + 2(3) + 2(4) = 19$
$\vdots$	$\vdots$

At this point, it may be easier to see a pattern beginning to emerge. We might guess that the next row in the table will have $1+2(2)+2(3)+2(4)+2(5) = 19+10 = 29$ in the right column. Indeed, the pay in the 5^{th} week is 2×5 dollars more than it was on the 4^{th} week so will be $1+2(2)+2(3)+2(4)+2(5)$. This pattern will continue and so we can deduce a formula for calculating any week's pay directly.

In particular, the pay for the 52^{nd} week can be calculated as follows:

$$1 + 2(2) + 2(3) + \cdots + 2(51) + 2(52) = 1 + 2(2 + 3 + 4 + \cdots + 51 + 52)$$

We are left with the problem of computing the sum

$$S = 2 + 3 + 4 + \cdots + 51 + 52$$

where S represents the sum of the integers from 2 to 52. This can be done in a number of ways.

If you know the formula for the sum of the integers from 1 to n (it is said that Gauss found this formula on his own as a child) then you could use it here. Alternatively, you could notice that the numbers in the sum form an arithmetic sequence and use a formula for summing such a list of numbers.

We will calculate S as follows: First write out the sum S twice, with the second sum written with the terms in the opposite order.

$$\begin{array}{rcl} S &=& 2+\ 3+\ 4+\cdots+50+51+52 \\ +S &=& 52+51+50+\cdots+\ 4+\ 3+\ 2 \\ \hline 2S &=& 54+54+54+\cdots+54+54+54 \end{array}$$

Adding the terms on the left side gives $2S$. Adding the terms on the right side gives

$$54+54+54+\cdots+54+54+54,$$

the sum of 51 copies of 54. Therefore,

$$2S = 54+54+54+\cdots+54+54+54 = 51(54) = 2754$$

This means the assistant's pay for the 52$^{\text{nd}}$ week is

$$1+2(2+3+4+\cdots+51+52) = 1+2S = 1+2754 = 2755$$

dollars.

Problem 2: Suppose that the integer N is the value of the following sum (with 52 terms):

$$1+11+101+1001+10001+\cdots+1\overbrace{000\ldots000}^{50\text{ zeroes}}1$$

When N is calculated and written as a single integer, what is the sum of its digits?

Solution:

There are 52 numbers to be added together, and all of them have a 1 in the units position. One way to approach this problem is to first compute $N-52$ by subtracting 1 from each term. That is,

$$\begin{aligned} N-52 &= (1-1)+(11-1)+(101-1)+(1001-1)+\cdots+(1\overbrace{000\ldots000}^{50\text{ zeroes}}1-1) \\ &= 0+10+100+1000+\cdots+1\overbrace{000\ldots000}^{51\text{ zeroes}} \\ &= \overbrace{111\ldots111}^{51\text{ ones}}0 \end{aligned}$$

We can now simply add 52 to both sides to get

$$N = \overbrace{111\ldots111}^{50\text{ ones}}62$$

Therefore, the sum of the digits of N is $50\times 1+6+2 = 58$.

Problem 3: Using only the digits $1, 2, 3, 4,$ and 5, a sequence is created. The beginning of the sequence in shown below.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, \ldots$$

The sequence starts with one 1, followed by two 2s, then three 3s, four 4s, five 5s, six 1s, seven 2s, and so on. What is the 1000th term in the sequence?

Solution:

Let's think of the sequence as being in "blocks". That is, the first block consists of 1s, the second block consists of 2s, the third block consists of 3s, and so on.
To answer the question, we first need to determine in which block of digits the 1000^{th} term lies.
Note that each block consists of one more term than the previous block and that the first block consists of one term. Convince yourself that after you have written down the first n blocks in the sequence, you will have written down exactly

$$1 + 2 + 3 + \cdots + (n-1) + n$$

terms in total. How many blocks must be written down before you reach the block containing the 1000^{th} term? To answer this, we need to find when the sum above reaches 1000.

In the solution to Problem 1, we mentioned, but did not use, the formula for the sum of the positive integers from 1 to n. We shall make use of it here. The formula is:

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

What is the smallest positive integer n satisfying $\dfrac{n(n+1)}{2} \geq 1000$? Equivalently, what is the smallest positive integer n satisfying $n(n+1) \geq 2000$? We could test values of n starting at $n = 1$ until we find such an integer n, but this might take quite a while. We also "know" that n cannot be too small. For example, your intuition probably tells you that n is at least 10, or maybe even at least 20, so you might not start at $n = 1$ and instead start at some larger integer.

Here is a way to choose a "good" starting integer. Note that for any positive integer n, $(n+1)^2$ is larger than $n(n+1)$, so the n we seek must satisfy $(n+1)^2 \geq 2000$. Using a calculator, you can check that $\sqrt{2000} \approx 44.7$. Since $n+1$ is an integer, this means $n+1 \geq 45$, so $n \geq 44$. Thus, we need not check any values of n that are less than 44. With $n = 44$, we have $n(n+1) = 44(45) = 1980$ which is not ≥ 2000. With $n = 45$, we have $n(n+1) = 45(46) = 2070$, which is ≥ 2000. This means the smallest positive integer n with $n(n+1) \geq 2000$ is $n = 45$, and we only had to check two values of n to find it!

We can now see that the first 44 blocks (in total) contain $\dfrac{44(45)}{2} = 990$ terms, and the first 45 blocks (in total) contain $\dfrac{45(46)}{2} = 1035$ terms. This means the 1000^{th} term occurs in the 45^{th} block.

To determine what the 1000^{th} term will be, we need to determine what digit makes up the 45^{th} block.
Note that the 1^{st} block, 6^{th} block, 11^{th} block, and so on, consist of 1s.
The 2^{nd} block, 7^{th} block, 12^{th} block, and so on, consist of 2s.
The 3^{rd} block, 8^{th} block, 13^{th} block, and so on, consist of 3s.
The 4^{th} block, 9^{th} block, 14^{th} block, and so on, consist of 4s.
The 5^{th} block, 10^{th} block, 15^{th} block, and so on, consist of 5s.

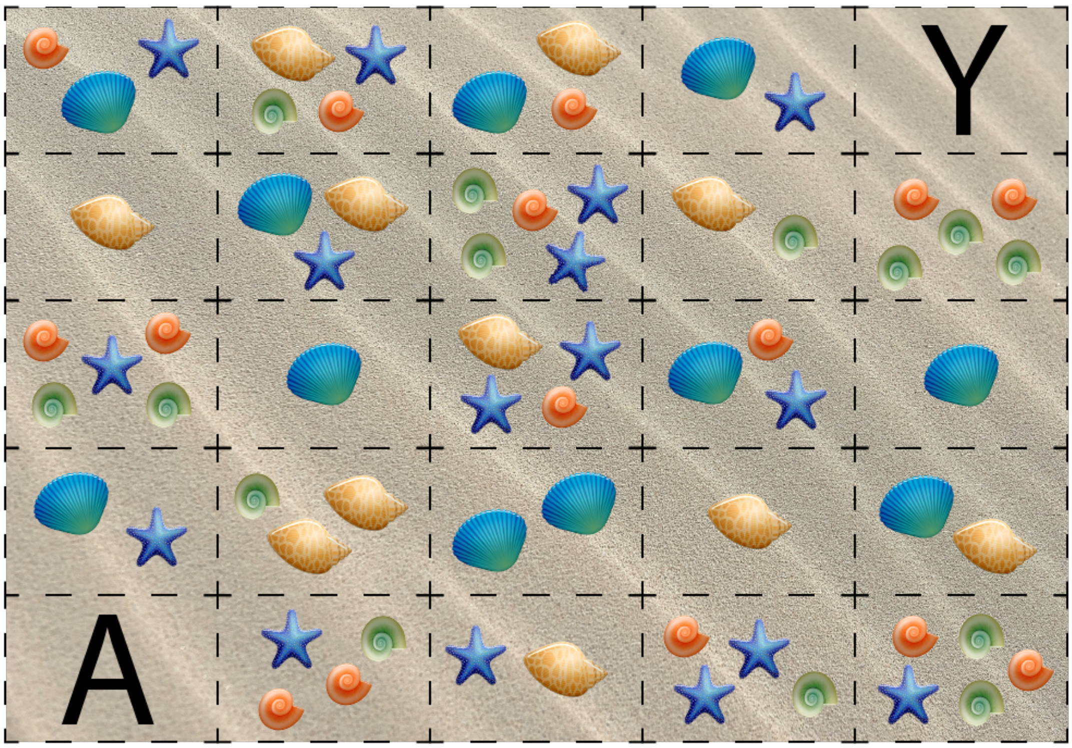
In particular, when n is a multiple of 5, the n^{th} block consists of 5s. Since the 1000^{th} term is in the 45^{th} block, the 1000^{th} term will be the digit 5.

CEMC at Home

Grade 9/10 - Wednesday, May 6, 2020

Sheldon's Shells

Sheldon is walking along the beach collecting shells. The shells are scattered across the beach in different areas as shown below.



Sheldon starts in the area marked "A" and ends in the area marked "Y". After collecting all of the shells in an area he either moves up or moves right to a new area. He does not move left or down.

Questions:

1. There are three different paths that Sheldon could take from A to the area located two to the right and one up from A (containing two blue shells). What is the largest number of shells that Sheldon could collect on his way from A to this area (not including these two blue shells)?
2. Sheldon stops part way during a trip from A to Y and notices that he has collected 8 shells so far, including the shells in the area in which he has stopped. What are all of the possible areas in which Sheldon may have stopped?
3. Consider all possible paths that Sheldon could take from A to Y. What is the maximum number of shells that Sheldon could collect on his way from A to Y?

More Info:

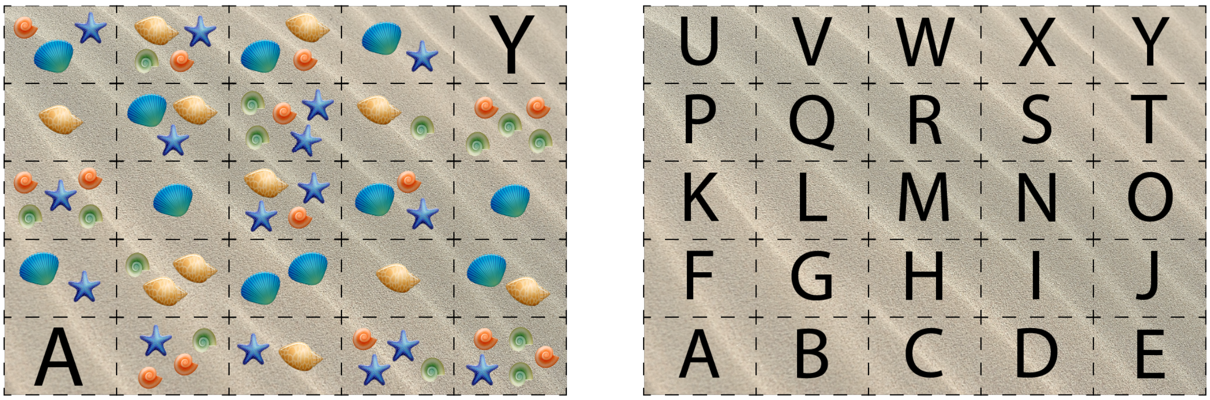
Check out the CEMC at Home webpage on Wednesday, May 13 for a solution to Sheldon's Shells.

If you enjoyed this problem, check out the problem Coins and Monsters from the 2019 Beaver Computing Challenge, which is a similar problem but with an extra twist!

CEMC at Home

Grade 9/10 - Wednesday, May 6, 2020
Sheldon's Shells - Solution

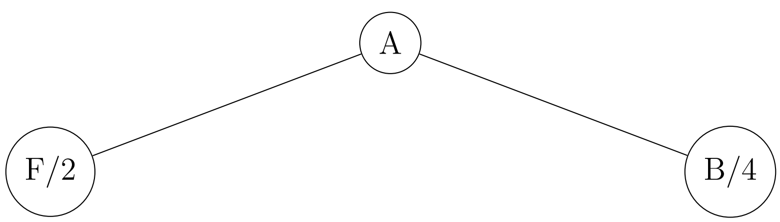
Label the areas of the beach as shown:



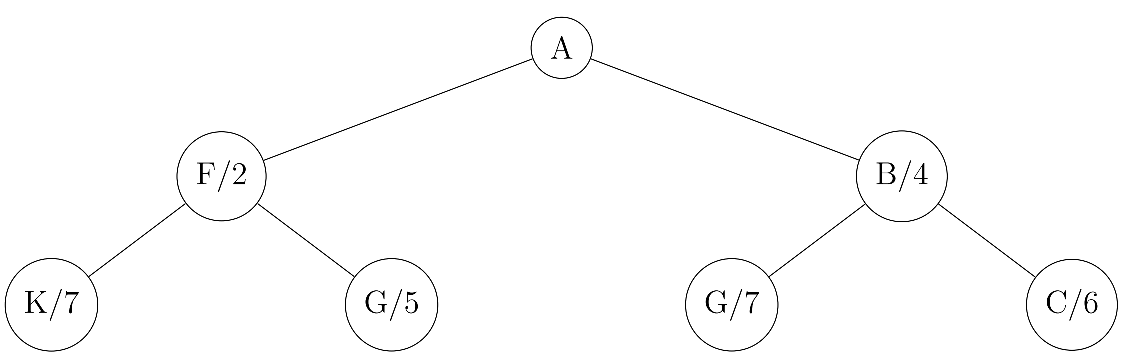
Recall that Sheldon only moves either up or right on his way from area A to area Y.

1. On his way from area A to the area with the two identical blue shells, Sheldon can collect $2 + 3 = 5$ shells (up, right, right), $4 + 3 = 7$ shells (right, up, right) or $4 + 2 = 6$ shells (right, right, up). This means the largest number of shells he can collect on this trip is 7.

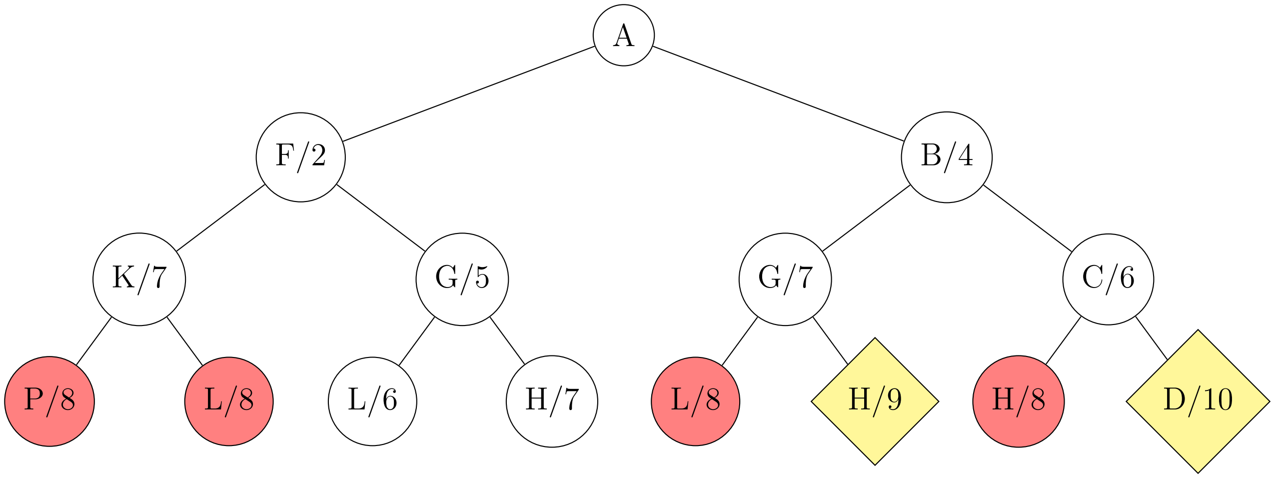
2. Sheldon stops after he has collected 8 shells. One way to figure out where Sheldon might have stopped is to build a tree starting with A and illustrating all the possible paths Sheldon could take. Since from A Sheldon can either move up to F (where he collects 2 shells) or right to B (where he collects 4 shells), the tree begins like this:



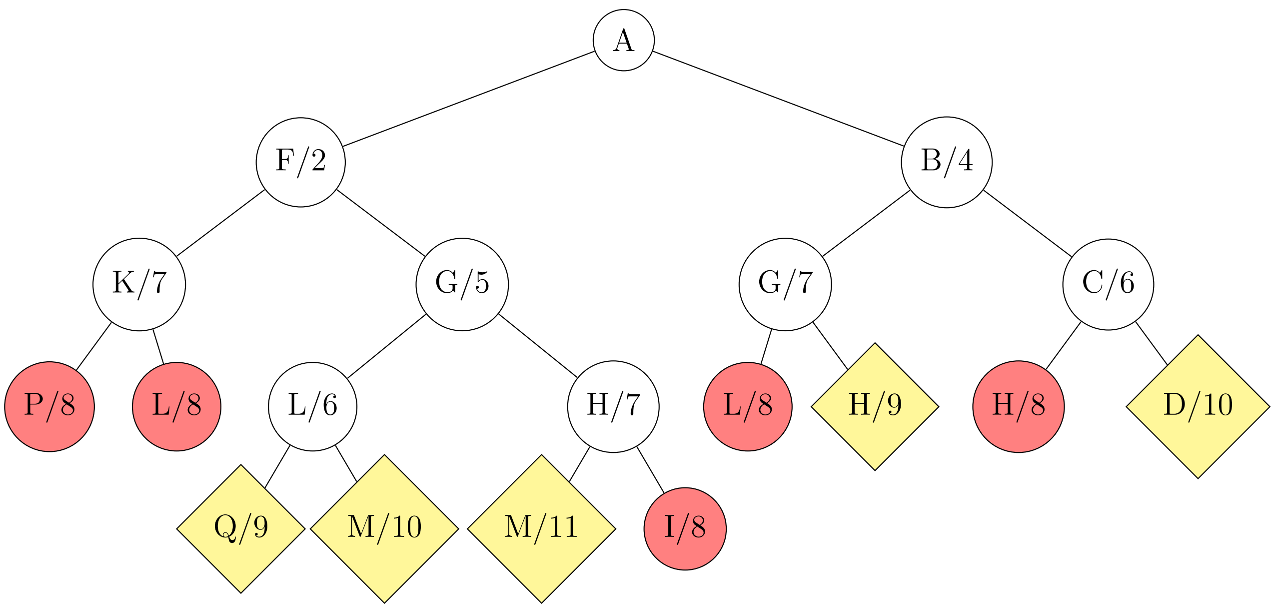
From F Sheldon can either move up to K (where he collects 5 more shells for a total of 7) or right to G (where he collects 3 more shells for a total of 5). From B Sheldon can either move up to G (where he collects 3 more shells for a total of 7) or right to C (where he collects 2 more shells for a total of 6). Those branches are added to the tree like this:



From left to right we can continue to add branches representing Sheldon's possible paths and cumulative amount of shells collected:



The areas shaded in red are areas on the path where Sheldon will have collected exactly 8 shells. We do not need to add any new branches from these areas since any further movement will result in collecting more than 8 shells. We also do not need to add any new branches from the yellow diamond areas since Sheldon has already exceeded 8 shells along these paths.



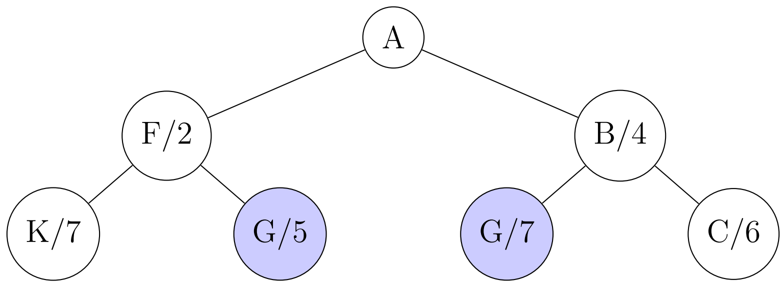
From this tree, we can see that the only areas Sheldon could have stopped in (after collecting exactly 8 shells) are P, L, I, or H.

3. Sheldon can collect 25 shells by following the path "A B G H M R S T Y". It turns out that this is the maximum number of shells that he can collect. How can we be sure of this?

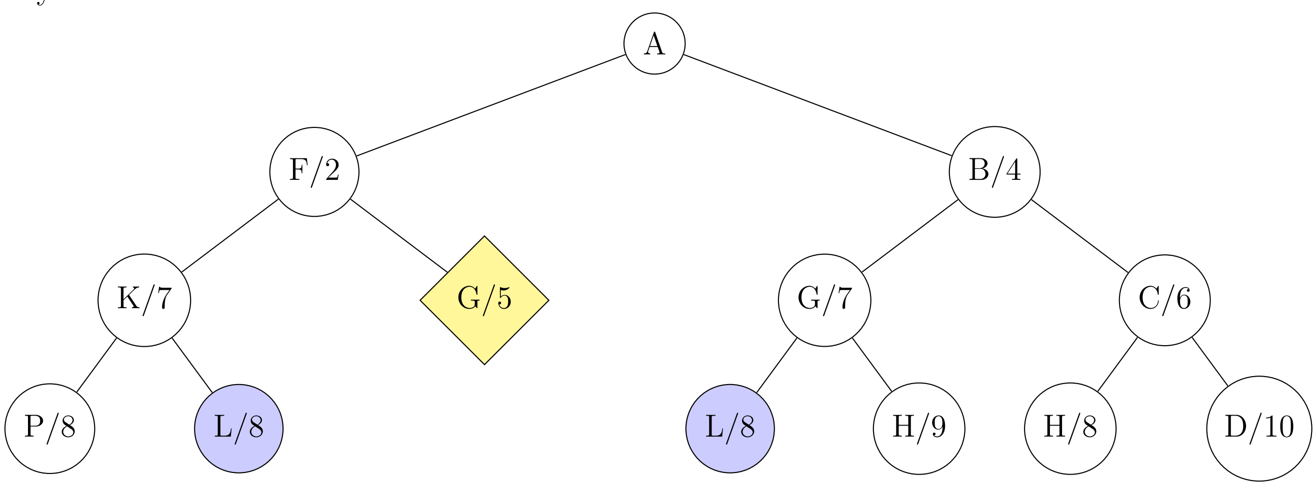
   One way to figure out the maximum number of shells that Sheldon could collect on his way from A to Y is to continue building the tree from the previous question until all paths reach Y. Then the bottom row of the tree can be scanned for the largest number.

   How many possible paths from A to Y are there? To get from A to Y Sheldon must move up four times and right four times, in some sequence. There are 70 different ways to arrange four "up"s and four "right"s, so there are 70 possible paths. That's going to make a big tree!

There are ways to simplify the tree if you would like to carry through with this approach. One way to simplify the tree is to "abandon" a path when a better path is found. For example, consider the beginning of the tree:



There are two ways Sheldon can get to area G. Taking the path "A B G" is better than taking the path "A F G" since it results in more shells. We can abandon path "A F G" by not adding any more branches to it.



In addition, we can simplify the tree by "merging" paths. Since two different paths to L result in the same number of shells (8), we do not lose any information by merging the paths together.

If we continue to expand, abandon, and merge paths we will eventually get a diagram which shows the maximum number of shells Sheldon can collect.

See the next page for the completed diagram for the tree approach.

Another way to approach a solution to this problem is to do calculations in a grid using the following observation:

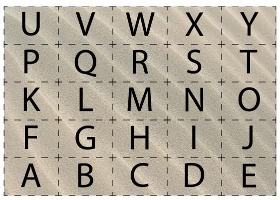
> The maximum number of shells that can be collected by Sheldon upon reaching a particular area of the beach is the number of shells in this area *plus* the maximum of
>
> - the total number of shells that can be collected upon reaching the section to the left, and
> - the total number of shells that can be collected upon reaching the section below.

We can compute each of these maximum numbers of shells starting at the bottom left area and working our way through the grid.

See the next page for the completed grid with these calculations.

Grid approach

In the grid below, the number of shells in each area is displayed in a small box in the bottom left corner of the corresponding square. There is only one way to get to each of areas F and B and the maximum numbers of shells you can collect upon reaching these areas are 2 and 4, respectively. These numbers are entered in the corresponding squares in the grid below. There are two ways to get to area G: through F (from the left) or through B (from below). We see that to collect the maximum number of shells upon reaching G, we want to travel through B, and in doing so we can collect $4+3=7$ shells. See if you can work your way through the rest of the calculations in the table.

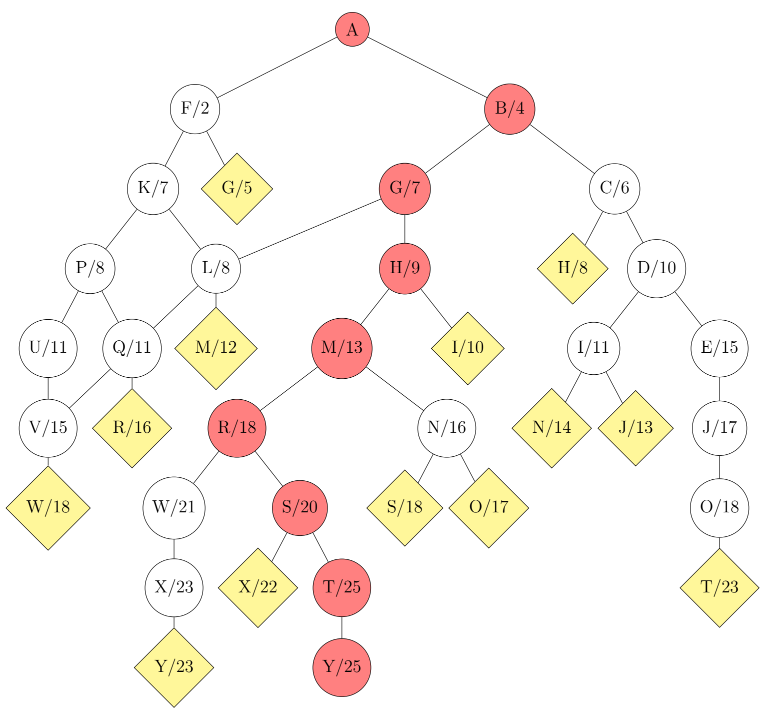


$8+3=11$ [3]	$11+4=15$ [4]	$18+3=21$ [3]	$21+2=23$ [2]	Y [0]
$7+1=8$ [1]	$8+3=11$ [3]	$13+5=18$ [5]	$18+2=20$ [2]	$20+5=25$ [5]
$2+5=7$ [5]	$7+1=8$ [1]	$9+4=13$ [4]	$13+3=16$ [3]	$17+1=18$ [1]
2 [2]	$4+3=7$ [3]	$7+2=9$ [2]	$10+1=11$ [1]	$15+2=17$ [2]
A [0]	4 [4]	$4+2=6$ [2]	$6+4=10$ [4]	$10+5=15$ [5]

Each arrow pointing to an area indicates whether the maximum comes from the section to the left or comes from the section below (or both). When we reach the top right area, we compute the value 25 which tells us that 25 is the maximum number of shells among all possibilities.

To collect this maximum number of shells, Sheldon should follow the path shown by the red arrows.

Tree approach



Looking at the bottom row, we see that the maximum number of shells is 25.

We can also see that the path Sheldon should take to collect 25 shells is "A B G H M R S T Y".

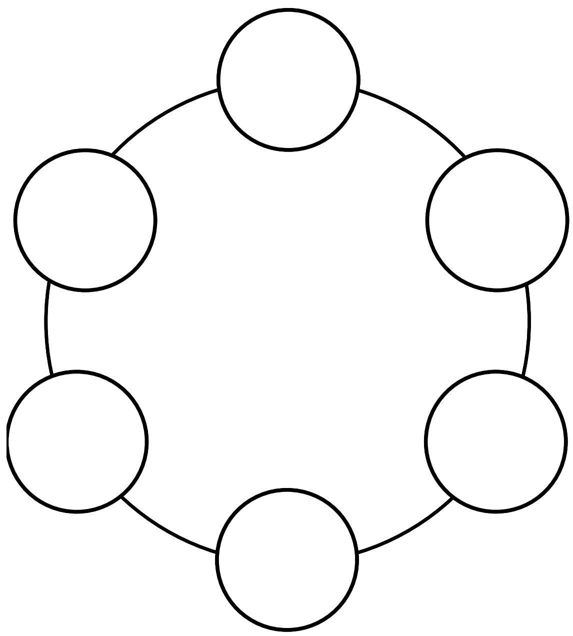
Can you see how the grid and the tree are related?

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, May 7, 2020
A Circle of Numbers

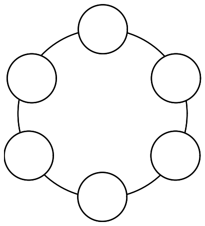
The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle below, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven.

In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.



More Info:

Check the CEMC at Home webpage on Thursday, May 14 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution emailed to you on Thursday, May 14.



Problem of the Week

Problem D and Solution

A Circle of Numbers

Problem

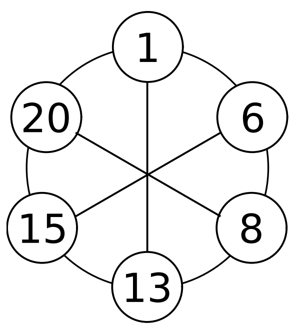
The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle above, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven. In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.

Solution

We will start by writing down all the pairs of numbers that add to a multiple of 7.

Sum of 7	Sum of 14	Sum of 21	Sum of 28	Sum of 35
1,6	1,13	6,15	13,15	15,20
	6,8	8,13	8,20	
		1,20		

To show these connections visually, we can write the numbers in a circle and draw a line connecting numbers that add to a multiple of 7.

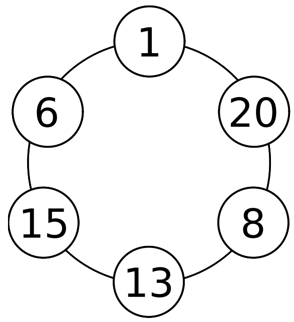


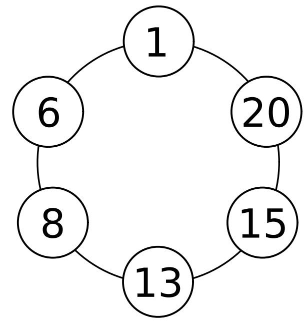
We will now determine all the different arrangements by looking at various cases. Note that in order for two arrangements to be different, at least some of the numbers need to be adjacent to different numbers.

Now, consider the possibilities for the numbers adjacent to 1. Since 6, 13, and 20 are the only numbers in our list that add with 1 to make a multiple of 7, there are three possible cases: 1 adjacent to 6 and 20, 1 adjacent to 6 and 13, and 1 adjacent to 13 and 20. We consider each case separately.

Case 1: 1 is adjacent to 6 and 20

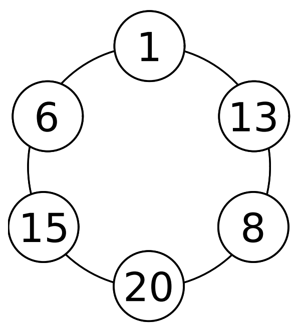
In this case, we can see from our table that 13 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

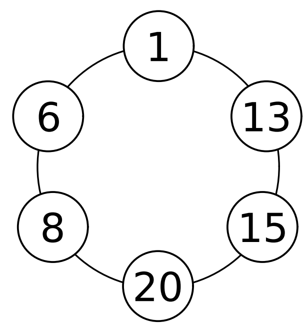




Case 2: 1 is adjacent to 6 and 13

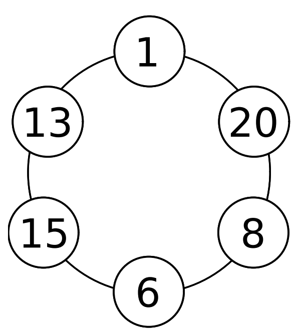
In this case, we can see from our table that 20 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

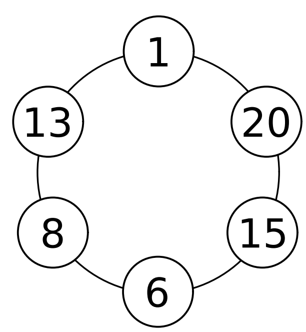




Case 3: 1 is adjacent to 13 and 20

In this case, we can see from our table that 6 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.





Therefore, we have found that there are 6 different arrangements. These are the arrangements shown in Cases 1, 2 and 3 above.

CEMC at Home

Grade 9/10 - Friday, May 8, 2020
Unsolved Problems

Odd Perfect Numbers

A *perfect number* is a positive integer that is equal to the sum of its *proper* positive divisors. A divisor is proper if it is smaller than the number itself. For example, the proper positive divisors of 6 are 1, 2, and 3. Since $1 + 2 + 3 = 6$, 6 is a perfect number. The proper divisors of 28 are 1, 2, 4, 7, and 14. Since $1 + 2 + 4 + 7 + 14 = 28$, 28 is also a perfect number.

(a) Is 120 a perfect number?

(b) Is 496 a perfect number?

(c) Are there any odd perfect numbers?

The answer to part (c) is currently, *we don't know!* The existence of an odd perfect number is an unsolved problem. To date, mathematicians have checked all positive integers up to at least 10^{1500} and have not yet found a single positive integer that is both odd and a perfect number. Does this mean that no such number exists? To conclusively solve this problem of whether or not an odd perfect number exists, we need to either find an example of an odd perfect number, or come up with a justification (a proof) that all perfect numbers are even. To date, we do not have either.

The Goldbach Conjecture

Consider the integer 16. It can be expressed as the sum of two prime numbers: $16 = 5 + 11$.

There is more than one way to express 16 *as the sum of two prime numbers. Can you find them all?*

Remember that a prime number is an integer p that is greater than 1 and whose only positive divisors are 1 and p (itself). The first few prime numbers are $2, 3, 5, 7, 11, \ldots$

What numbers can be expressed as the sum of two prime numbers in at least one way?

(a) Express 34 as the sum of two prime numbers.

(b) Express 52 as the sum of two prime numbers.

(c) Explain why 41 cannot be written as the sum of two prime numbers.

(d) Choose any even integer greater than 2 and express it as the sum of two prime numbers.

The Goldbach conjecture states that every even integer greater than 2 can be expressed as the sum of two prime numbers. It is called a conjecture and not a theorem because all evidence uncovered so far suggests it is true, but it has not been proven. To date, mathematicians have checked all even integers greater than 2 up to around 10^{18}, and have found a way to express each of these integers as the sum of two prime numbers. Does this mean that all even integers greater than 2 must have this property? To conclusively solve this problem, we need to either find an even number greater than 2 that *cannot* be written as the sum of two prime numbers, or come up with a justification (a proof) that all even numbers greater than 2 *can* be written in this way. To date, we do not have either.

Hailstone Sequence

Consider a sequence where the first term is a positive integer, n, and each subsequent term is obtained as follows:

- If the previous term is *even*, then the next term is one half of the previous term.
- If the previous term is *odd*, then the next term is three times the previous term, plus one.

This sequence is known as a *hailstone sequence* and the Collatz conjecture states that no matter what positive integer, n, is chosen for the first term, the sequence will always eventually reach 1. It is called a conjecture and not a theorem because it believed by many but has not been proven.

If you have experience programming, try writing a computer program that takes as input a positive integer, n, and outputs the hailstone sequence, stopping when the sequence reaches 1.

If the program doesn't stop, can you conclude the sequence never reaches 1? Why or why not?

Alternatively, you can experiment with the sequence using a computer program that we have written in Python, by following the instructions below.

Instructions:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open this free online Python interpreter in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. This allows you to interact with the program using the black panel on the right. Enter a positive integer and observe the hailstone sequence.
5. If you encounter an error, or you want to explore a different seqeuence, you can hit *run* to begin again.

More Info:

Check out the CEMC at Home webpage on Friday, May 15 for a discussion of these questions.

CEMC at Home

Grade 9/10 - Friday, May 8, 2020
Unsolved Problems - Solution

Odd Perfect Numbers

(a) Is 120 a perfect number?

The proper divisors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, and 60.
Since $1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 20 + 24 + 30 + 40 + 60 = 240$ and not 120, 120 is not a perfect number.

(b) Is 496 a perfect number?

The proper divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 248.
Since $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$, 496 is a perfect number.

(c) Are there any odd perfect numbers?

The answer to this question is currently unknown. It is known that there are no odd perfect numbers between 1 and 10^{1500}.

The Goldbach Conjecture

(a) Express 34 as the sum of two prime numbers.

There are four different ways to express 34 as the sum of two prime numbers:

$$34 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17$$

(b) Express 52 as the sum of two prime numbers.

There are three different ways to express 52 as the sum of two prime numbers:

$$52 = 5 + 47 = 11 + 41 = 23 + 29$$

(c) Explain why 41 cannot be written as the sum of two prime numbers.

You could write out a list of all primes less than 41 and check that no pair adds to 41. A faster way to approach this problem is as follows: Since 41 is odd, if 41 can be written as a sum of two primes then it must be an even prime plus an odd prime. The only even prime is 2 and so the only possibility is $41 = 2 + p$ where p is an odd prime. But we can see that p must be $41 - 2 = 39$ which is not prime.

(d) Choose any even integer greater than 2 and express it as the sum of two prime numbers.

It is not known whether there exists a general strategy that can be followed to write any even integer greater than 2 as the sum of two prime numbers. It is known that there is a way to write any even integer from 4 to around 10^{18} as the sum of two primes numbers.

Hailstone Sequence

Consider a computer program that takes as input a positive integer, n, and outputs the hailstone sequence, stopping when the sequence reaches 1. *If the program doesn't stop, can you conclude the sequence never reaches 1? Why or why not?*

If the program doesn't stop you cannot conclude that the sequence never reaches 1. That *might* be the reason why it never stopped, but there are two other possibilities as well.

- There might be an error in the program. For example, the sequence might not be generated correctly.
- You might not have waited long enough. It is possible that the number 1 is a term in the sequence but it does not appear for a very long time. Perhaps the program would stop eventually, given enough time.

You may have noticed that mathematicians have checked many more numbers in an effort to find an odd perfect number (up to around 10^{1500}) than they have checked to see if the Goldbach conjecture is true (up to around 10^{18}). You may want to do some research to get an idea of why this might be the case. Is there some reason to believe that it is much easier to check if a number is perfect than to check if a number is the sum of two prime numbers? How difficult is it to check these conditions for very large numbers?

CEMC at Home
Grade 9/10 - Monday, May 11, 2020
Contest Day 2

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Team Problem #9

How many times does the digit 0 appear in the integer equal to 20^{10}?

2020 Canadian Team Mathematics Contest, Individual Problem #7

Twenty-seven unit cubes are each coloured completely black or completely red. The unit cubes are assembled into a larger cube. If $\frac{1}{3}$ of the surface area of the larger cube is red, what is the smallest number of unit cubes that could have been coloured red?

More Info:

Check out the CEMC at Home webpage on Thursday, May 21 for solutions to the Contest Day 2 problems.

CEMC at Home

Grade 9/10 - Monday, May 11, 2020
Contest Day 2 - Solution

Solutions to the two contest problems are provided below, including a video for the second problem.

2020 Canadian Team Mathematics Contest, Team Problem #9

How many times does the digit 0 appear in the integer equal to 20^{10}?

Solution:

By factoring and using exponent rules, we have $20^{10} = (2 \times 10)^{10} = 2^{10} \times 10^{10}$.
Therefore, $20^{10} = 1024 \times 10^{10}$, which is the integer 1024 followed by ten zeros.
Thus, 20^{10} has eleven digits that are 0. That is, 10 zeros at the end and one coming from the 1024 at the beginning.

2020 Canadian Team Mathematics Contest, Individual Problem #7

Twenty-seven unit cubes are each coloured completely black or completely red. The unit cubes are assembled into a larger cube. If $\frac{1}{3}$ of the surface area of the larger cube is red, what is the smallest number of unit cubes that could have been coloured red?

Solution:

Since $\sqrt[3]{27} = 3$, the dimensions of the larger cube must be $3 \times 3 \times 3$.
Therefore, each side of the larger cube has area $3 \times 3 = 9$.
A cube has 6 faces, so the total surface of the cube is made up of $9 \times 6 = 54$ of the 1 by 1 squares from the faces of the unit cubes.
Since $\frac{1}{3}$ of the surface area is red, this means $\frac{54}{3} = 18$ of these unit squares must be red.

The unit cube at the centre of the larger cube has none of its faces showing, the 6 unit cubes in the centres of the outer faces have exactly 1 face showing, the 12 unit cubes on the edge but not at a corner have 2 faces showing, one of each of two adjacent sides, and the 8 unit cubes at the corners each have 3 faces showing.
For any unit cube, there are either 0, 1, 2, or 3 of its faces showing on the surface of the larger cube. This means at most three faces of any unit cube are on the surface of the larger cube. Thus, there must be at least 6 cubes painted red in order to have 18 red unit squares on the surface of the larger cube.

There are 8 unit cubes on the corners, so if we colour exactly 6 unit cubes red and the other 21 black, then arrange the cubes into a $3 \times 3 \times 3$ cube so that the 6 red unit cubes are at the corners, there will be exactly 18 of the unit squares on the surface coloured red.
Therefore, the answer is 6.

Video

Visit the following link for an explanation of the solution to the second contest problem:
https://youtu.be/K9ax9uQESME

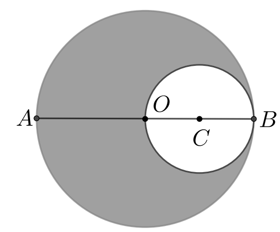
CEMC at Home

Grade 9/10 - Tuesday, May 12, 2020
Shady Circles

For each problem, use the information and diagram given to **find the area of the shaded region**. Express your answers as simplified exact numbers. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

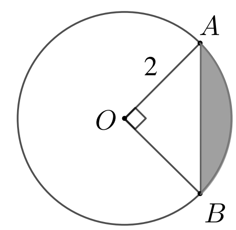
Problem 1

Two circles are centred at C and O as shown. AB is a diameter of the larger circle. OB is a diameter of the smaller circle. The larger circle has a diameter of 20.



Problem 2

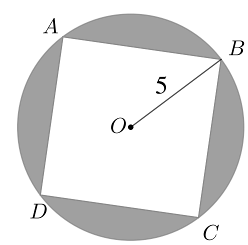
The circle with centre O has a radius of 2. Points A and B are on the circle and $\angle AOB = 90°$ as shown.



Problem 3

Square $ABCD$ is inscribed in the circle with centre O and radius 5 as shown.

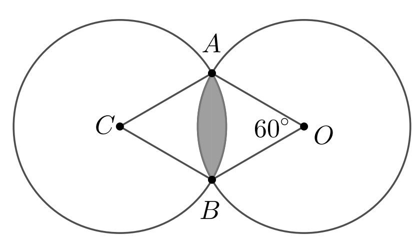
A square is inscribed in a circle if all four vertices of the square lie on the circle.



Problem 4

Two circles, each with a radius of 10, are centred at C and O as shown. The circles intersect at points A and B with $\angle AOB = 60°$.

Can you visualize what this diagram would look like if you pulled the circles apart?



More Info:

Check out the CEMC at Home webpage on Tuesday, May 19 for a solution to Shady Circles.

To review area calculations involving circles and triangles, visit the following videos in the CEMC courseware: area of triangles and area of circles.

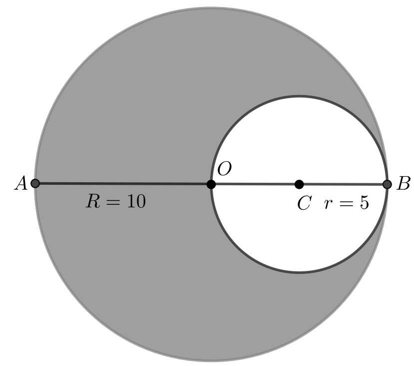
CEMC at Home

Grade 9/10 - Tuesday, May 12, 2020
Shady Circles - Solution

Problem 1 Solution

We will find the area of the shaded region by subtracting the area of the smaller circle from the area of the larger circle.
Let R be the radius of the larger circle and r be the radius of the smaller circle.

Since the diameter of the smaller circle is the radius of the larger circle, we have $R = \frac{20}{2} = 10$ and $r = \frac{10}{2} = 5$.



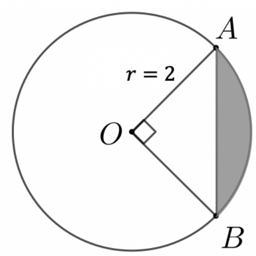
$A_{\text{larger}} = \pi R^2 = \pi(10)^2 = 100\pi$
$A_{\text{smaller}} = \pi r^2 = \pi(5)^2 = 25\pi$

Therefore, the area of the shaded region is $100\pi - 25\pi = 75\pi$.

Problem 2 Solution

To find the area of the shaded region, we will find the area of the sector of the circle with arc AB and subtract the area of $\triangle AOB$. Note that the triangle is a right isosceles triangle and therefore, the base and height are both equal to the radius which is 2.

$A_{wholeCircle} = \pi r^2 = \pi(2)^2 = 4\pi$
$A_{sector} = \left(\dfrac{90}{360}\right) 4\pi = \pi$
$A_{triangle} = \dfrac{bh}{2} = \dfrac{(2)(2)}{2} = 2$



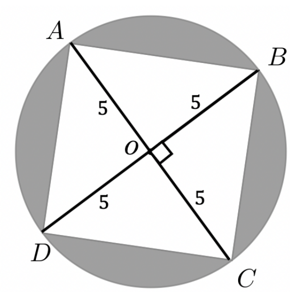
Therefore, the area of the shaded region is $\pi - 2$.

Problem 3 Solution

There are a few different ways to approach this problem. We will outline two approaches. Each of these approaches relies upon the following facts that we will not prove:

1) The two diagonals of the inscribed square intersect at the centre, O, of the circle.

2) The two diagonals of the inscribed square bisect each other and meet at right angles.



Approach 1: Recognize that the shaded region in this problem consists of four identical shaded regions, each having an area that can be calculated by subtracting the area of a triangle from the area of a sector of a the circle (as in Problem 2).

The final calculation is as follows: Area $= 4\left(\frac{\pi(5)^2}{4} - \frac{5^2}{2}\right) = 25\pi - 50$.

Approach 2: Recognize that the area of the shaded region is the area of the circle minus the area of the inscribed square.

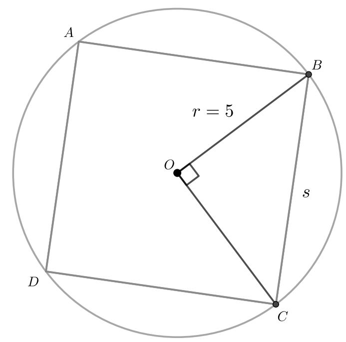
The area of the circle $\pi r^2 = \pi(5)^2 = 25\pi$.

Let s be the side length of the square as shown in the figures below.

Note that $\triangle BOC$ is a right isosceles triangle. Therefore, its base and height are both equal to the radius which is $r = 5$. We can calculate the value of s^2 as follows:

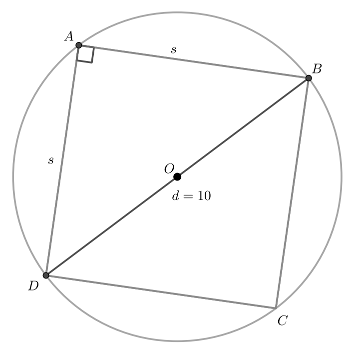
Using the Pythagorean Theorem on $\triangle BOC$, we get

$$\begin{aligned} s^2 &= r^2 + r^2 \\ s^2 &= (5)^2 + (5)^2 \\ s^2 &= 25 + 25 \\ s^2 &= 50 \end{aligned}$$



Alternatively, using the Pythagorean Theorem on $\triangle BAD$, with diameter $d = 10$, we get

$$\begin{aligned} d^2 &= s^2 + s^2 \\ 10^2 &= 2s^2 \\ 100 &= 2s^2 \\ 50 &= s^2 \end{aligned}$$

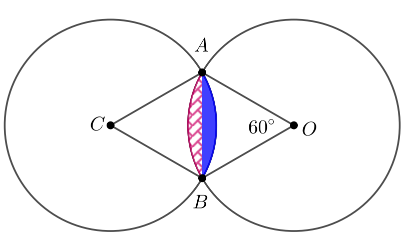


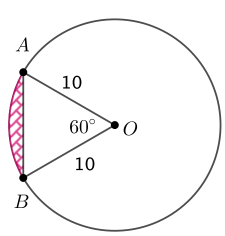
Each of these calculations tells us that the area of the square is $s^2 = 50$.

Therefore, the area of the shaded region is $25\pi - 50$.

Problem 4 Discussion

One way to find the area of the shaded region is to observe that it is made up of two identical regions as shown below. You can find the area of each of the regions using a similar method to that in the solution to Problem 2, although the area of the triangle will not be as easy to calculate in this case. We leave the details to you, but give the key values in the calculations here.





Area of triangle AOB is $\frac{1}{2}(10)(\sqrt{75})$

Area of sector AOB is $\frac{60}{360}(\pi(10)^2)$

CEMC at Home

Grade 9/10 - Thursday, May 21, 2020
Contest Day 3

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Team Problem #5

What is the smallest eight-digit positive integer that has exactly four digits which are 4?

2020 Canadian Team Mathematics Contest, Team Problem #14

Jeff caught 21 fish, each having a mass of at least 0.2 kg. He noticed that the average mass of the first three fish that he caught was the same as the average mass of all 21 fish. The total mass of the first three fish was 1.5 kg. What is the largest possible mass of any one fish that Jeff could have caught?

More Info:

Check out the CEMC at Home webpage on Monday, May 25 for solutions to the Contest Day 3 problems.

CEMC at Home

Grade 9/10 - Thursday, May 21, 2020
Contest Day 3 - Solution

Solutions to the two contest problems are provided below.

2020 Canadian Team Mathematics Contest, Team Problem #5

What is the smallest eight-digit positive integer that has exactly four digits which are 4?

Solution:

The smallest eight-digit number is 10 000 000.
The smallest 10 000 eight-digit numbers are those of the form $10\,00a\,bcd$ where a, b, c, and d are digits. Among the 10 000 smallest eight-digit numbers, 10 004 444 is the only one that has four digits that are equal to 4.
This means the smallest eight-digit positive integer that has exactly four digits which are 4 is 10 004 444.

2020 Canadian Team Mathematics Contest, Team Problem #14

Jeff caught 21 fish, each having a mass of at least 0.2 kg. He noticed that the average mass of the first three fish that he caught was the same as the average mass of all 21 fish. The total mass of the first three fish was 1.5 kg. What is the largest possible mass of any one fish that Jeff could have caught?

Solution:

Since the total mass of the first three fish is 1.5 kg, the average mass of the first three fish is 0.5 kg.
Let M be the total mass of all of the fish. Since the average mass of the first three fish is the same as the average mass of all of the fish, this means $\frac{M}{21} = 0.5$ kg or $M = 10.5$ kg.
Since the first three fish have a total mass of 1.5 kg, this means the last 18 fish that Jeff caught have a total mass of 10.5 kg $-$ 1.5 kg $=$ 9 kg.
If 17 of these 18 fish have as small a mass that is as possible, the 18^{th} of these fish will have a mass that is as large as possible.
The smallest possible mass is 0.2 kg, so the total mass of 17 fish, each having as small a mass as possible, is 17×0.2 kg $= 3.4$ kg.
The largest possible mass of any fish that Jeff could have caught is 9 kg $-$ 3.4 kg $=$ 5.6 kg.

CEMC at Home

Grade 9/10 - Friday, May 22, 2020
Binary Puzzles

A binary puzzle is a type of logic puzzle that is done on an $n \times n$ grid where n is even. Your task is the complete the grid by filling in all empty cells according to the following rules:

1. Each cell in the grid must contain either the digit 0 or the digit 1.
2. The grid cannot have three or more consecutive cells, either horizontally or vertically, that all contain the same digit.
3. Half of the entries in each row of the grid must be 0s and the other half of the entries must be 1s. The same is true of each column.
4. No two rows in the grid can be identical, and no two columns can be identical.

Example: Let's complete the following row in a binary puzzle grid following the first three rules. *To ensure rule 4 is satisfied, we would have to consider all of the rows and columns in the grid at once.*

1	1			0		1	

Notice that the third cell must contain a 0, otherwise the row would have three consecutive cells containing 1s (which would violate the second rule).

1	1	0		0		1	

Now, notice that the fourth cell must contain a 1, otherwise the row would have three consecutive cells containing 0s (which would violate the second rule).

1	1	0	1	0		1	

The row now has four 1s and two 0s. According to the third rule, the remaining two cells must contain 0s.

1	1	0	1	0	0	1	0

Therefore, there is only way to complete this particular row according to the rules.

Note that there are other ways to reason how this row must be completed. For example, we could start by observing that the original partially completed row already has three 1s and so we must use exactly one 1 and three 0s while filling in the row. We could then argue that we have no choice but to place the 1 in the fourth cell. Can you see why? What must happen if we place the 1 in a different cell?

Try the four binary puzzle grids on the next page.

These puzzles increase in difficulty but can all be completed using solid reasoning. Enjoy!

More Info:

Check out the CEMC at Home webpage on Friday, May 29 for solutions to these Binary Puzzles.

The title *Binary Puzzles* is a reference to the binary number system. This system only uses the digits 0 and 1. Certain branches of mathematics as well as many electronics, including computers, make use of the binary number system.

PUZZLE 1

	0			1			
1			0			1	0
1		1					0
		1	1		1		
1					1		
		0		0			1
	1			0			1
0			1				

PUZZLE 2

		1	1			1	
							0
0		0		1		1	1
0					0		
		0				1	
			0				
1				1		1	

PUZZLE 3

			1	1			0
	1						
0		0					
						0	
1		1				1	
	0			1		1	
			1		0		

PUZZLE 4

0			0		1		
		1					1
				0		0	
				0	0		
0				1		1	
	1		1				1

CEMC at Home

Grade 9/10 - Friday, May 22, 2020
Binary Puzzles - Solution

PUZZLE 1

0	0	1	0	1	1	0	1
1	1	0	0	1	0	1	0
1	0	1	1	0	0	1	0
0	0	1	1	0	1	0	1
1	1	0	0	1	1	0	0
1	0	0	1	0	0	1	1
0	1	1	0	0	1	0	1
0	1	0	1	1	0	1	0

PUZZLE 2

0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0
1	0	1	1	0	1	0	0
0	1	0	0	1	0	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
0	1	1	0	0	1	0	1
1	0	0	1	1	0	1	0

PUZZLE 3

1	0	0	1	1	0	1	0
0	1	1	0	0	1	0	1
0	1	0	1	0	1	1	0
1	0	1	0	1	0	0	1
0	1	0	1	0	1	0	1
1	0	1	0	0	1	1	0
1	0	1	0	1	0	1	0
0	1	0	1	1	0	0	1

PUZZLE 4

0	1	1	0	0	1	1	0
1	0	1	0	1	0	0	1
1	0	0	1	0	1	0	1
0	1	0	1	1	0	1	0
1	0	1	0	1	1	0	0
1	0	0	1	0	0	1	1
0	1	1	0	1	0	1	0
0	1	0	1	0	1	0	1

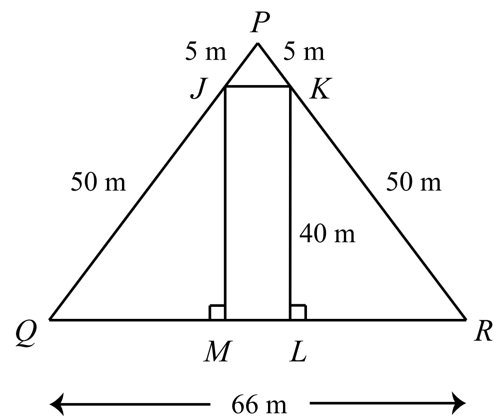
CEMC at Home

Grade 9/10 - Monday, May 25, 2020
Contest Day 4

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Fryer Contest, #2

In the diagram, rectangle $JKLM$ is drawn with its vertices on the sides of $\triangle PQR$ so that $PJ = PK = 5$ m, $JQ = KR = 50$ m, $KL = 40$ m, and $QR = 66$ m, as shown.



(a) What is the length of LR?

(b) What is the length of ML?

(c) Determine the height of $\triangle PJK$ drawn from P to JK.

(d) Determine the fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$.

More Info:

Check out the CEMC at Home webpage on Monday, June 1 for solutions to the Contest Day 4 problems.

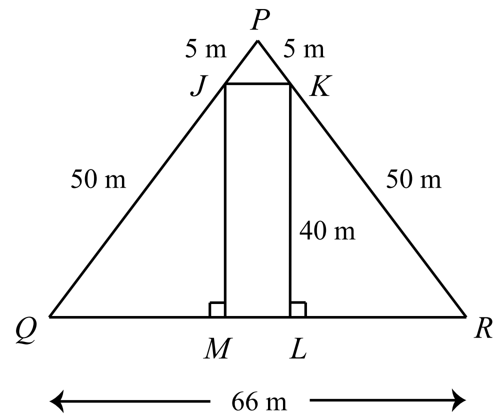
CEMC at Home

Grade 9/10 - Monday, May 25, 2020
Contest Day 4 - Solution

A solution to the contest problem is provided below, along with an accompanying video.

2020 Fryer Contest, #2

In the diagram, rectangle $JKLM$ is drawn with its vertices on the sides of $\triangle PQR$ so that $PJ = PK = 5$ m, $JQ = KR = 50$ m, $KL = 40$ m, and $QR = 66$ m, as shown.



(a) What is the length of LR?

(b) What is the length of ML?

(c) Determine the height of $\triangle PJK$ drawn from P to JK.

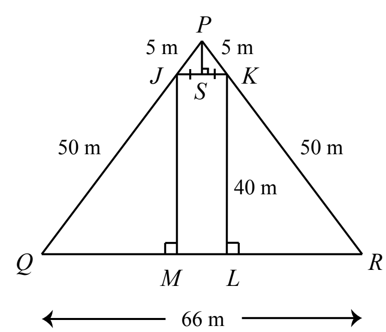
(d) Determine the fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$.

Solution:

(a) In $\triangle KLR$, we have $\angle KLR = 90^\circ$ and by using the Pythagorean Theorem, we get $LR^2 = 50^2 - 40^2 = 900$ and so $LR = \sqrt{900} = 30$ m (since $LR > 0$).

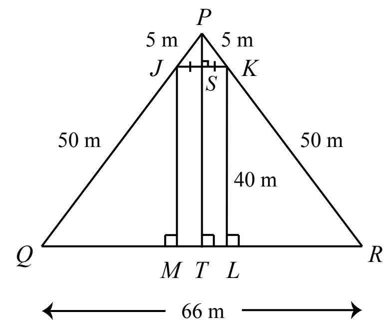
(b) We begin by showing that $\triangle JMQ$ is congruent to $\triangle KLR$.
Since $JKLM$ is a rectangle, then $JM = KL = 40$ m.
In addition, hypotenuse JQ has the same length as hypotenuse KR, and so $\triangle JMQ$ is congruent to $\triangle KLR$ by HS congruence.
Thus, $MQ = LR = 30$ m and so $ML = 66 - 30 - 30 = 6$ m.

(c) Since $PJ = PK = 5$ m, $\triangle PJK$ is isosceles and so the height, PS, drawn from P to JK bisects JK, as shown.
Since $JKLM$ is a rectangle, then $JK = ML = 6$ m and so $SK = \frac{JK}{2} = 3$ m.
Using the Pythagorean Theorem in $\triangle PSK$, we get $PS^2 = 5^2 - 3^2 = 16$ and so $PS = 4$ m (since $PS > 0$).
Thus the height of $\triangle PJK$ drawn from P to JK is 4 m.



See the next page for a solution to part (d) and a link to the video.

(d) We begin by determining the area of $\triangle PQR$.
Construct the height of $\triangle PQR$ drawn from P to T on QR, as shown.
Since PT is perpendicular to QR, then PT is parallel to KL (since KL is also perpendicular to QR).
By symmetry, PT passes through S, and so the height PT is equal to $PS + ST = PS + KL$ or $4 + 40 = 44$ m.
The area of $\triangle PQR$ is $\frac{1}{2} \times QR \times PT = \frac{1}{2} \times 66 \times 44 = 1452$ m^2.
The area of $JKLM$ is $ML \times KL = 6 \times 40 = 240$ m^2.
The fraction of the area of $\triangle PQR$ that is covered by rectangle $JKLM$ is $\frac{240}{1452} = \frac{20}{121}$.



Video
Visit the following link to view a discussion of a solution to this contest problem:
https://youtu.be/gMpResbow9E

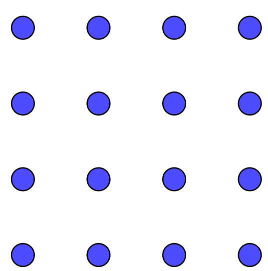
CEMC at Home

Grade 9/10 - Tuesday, May 26, 2020
Perfect Squares

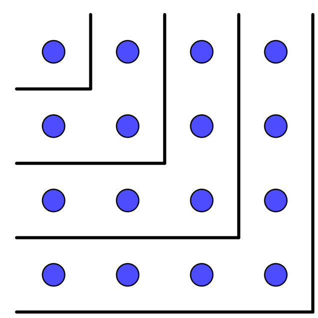
A *perfect square* (or *square number*) is an integer that is the square of an integer. In other words, an integer s is a perfect square if $s = n^2$ for some integer n. There are many different ways to illustrate a perfect square and they often involve the geometric notion of a square.

Consider the perfect square $16 = 4^2$

Since $16 = 4^2$ we can illustrate the perfect square 16 by drawing 16 dots arranged in a 4 by 4 (square) grid as shown.



What do you notice if we group the dots in the grid as shown below? Can you see how this illustrates another way to build the number 16?



You can check directly that $16 = 4^2$ is the sum of the first 4 positive odd integers: $16 = 1+3+5+7$. The illustrations above give an idea of why this is true. If you drew a similar illustration of the perfect square $25 = 5^2$, and grouped the 25 dots as shown above, what would you observe?

Fun Fact: The perfect square $s = n^2$ (where n is a positive integer) is equal to the sum of the first n consecutive positive odd integers. *Take some time to think about why this fact is true.*

Problems: Use the fact above to find an efficient way to answer each of the following questions.

1. What is the sum of the first 99 consecutive positive odd integers?
2. If 1225 is the sum of the first m consecutive positive odd integers, what is the value of m?
3. What is the value of the sum $1 + 3 + 5 + \cdots + 141 + 143 + 145$?
4. What is the value of the sum $17 + 19 + 21 + \cdots + 207 + 209 + 211$?
5. What is the value of the sum $2 + 4 + 6 + \cdots + 296 + 298 + 300$?

More Info:

Check out the CEMC at Home webpage on Tuesday, June 2 for a solution to Perfect Squares.

The sum of the first n consecutive odd numbers, $1 + 3 + 5 + \ldots + (2n - 1)$, is an example of an *arithmetic series* where the first term is 1 and the common difference is 2. Check out this lesson in the CEMC Courseware for more information about arithmetic series.

CEMC at Home

Grade 9/10 - Tuesday, May 26, 2020
Perfect Squares - Solution

1. What is the sum of the first 99 consecutive positive odd integers?

   Solution: The sum of the first 99 consecutive positive odd integers is equal to $99^2 = 9801$.

2. If 1225 is the sum of the first m consecutive positive odd integers, what is the value of m?

   Solution: Since the sum of the first m consecutive positive odd integers is equal to m^2 we must have $m^2 = 1225$. Since m is positive, $m = \sqrt{1225} = 35$.

3. What is the value of the sum $1 + 3 + 5 + \cdots + 141 + 143 + 145$?

   Solution: This is the sum of the first n consecutive positive odd integers for some n. What is the value of n (the number of terms in the sum)? We can write odd numbers in the form $2k-1$ where k is an integer, and so we rewrite this sum as

   $$1 + 3 + \cdots + 143 + 145 = \big(2(1) - 1\big) + \big(2(2) - 1\big) + \cdots + \big(2(72) - 1\big) + \big(2(73) - 1\big)$$

   Rewriting the sum in this way allows us to count that there are 73 terms in the sum, and so $n = 73$. Since this sum is the sum of the first 73 consecutive positive odd integers, the sum must be equal to $73^2 = 5329$. (Note that n can be calculated as follows: $n = \frac{145+1}{2} = 73$.)

4. What is the value of the sum $17 + 19 + 21 + \cdots + 207 + 209 + 211$?

   Solution: First, we note that the given sum can be calculated as the following difference of sums:

   $$17+19+21+\cdots+207+209+211 = (1+3+5+\ldots+207+209+211) - (1+3+5+\ldots+11+13+15)$$

   You can verify that the sum $1+3+5+\ldots+207+209+211$ has $\frac{211+1}{2} = 106$ terms. Since this is the sum of the first 106 consecutive positive odd integers, the value of the sum is $106^2 = 11\,236$.

   You can verify that the sum $1+3+5+\ldots+11+13+15$ has $\frac{15+1}{2} = 8$ terms. Since this is the sum of the first 8 consecutive positive odd integers, the value of the sum is $8^2 = 64$.

   Therefore, $17 + 19 + 21 + \cdots + 207 + 209 + 211 = 11\,236 - 64 = 11\,172$.

5. What is the value of the sum $2 + 4 + 6 + \cdots + 296 + 298 + 300$?

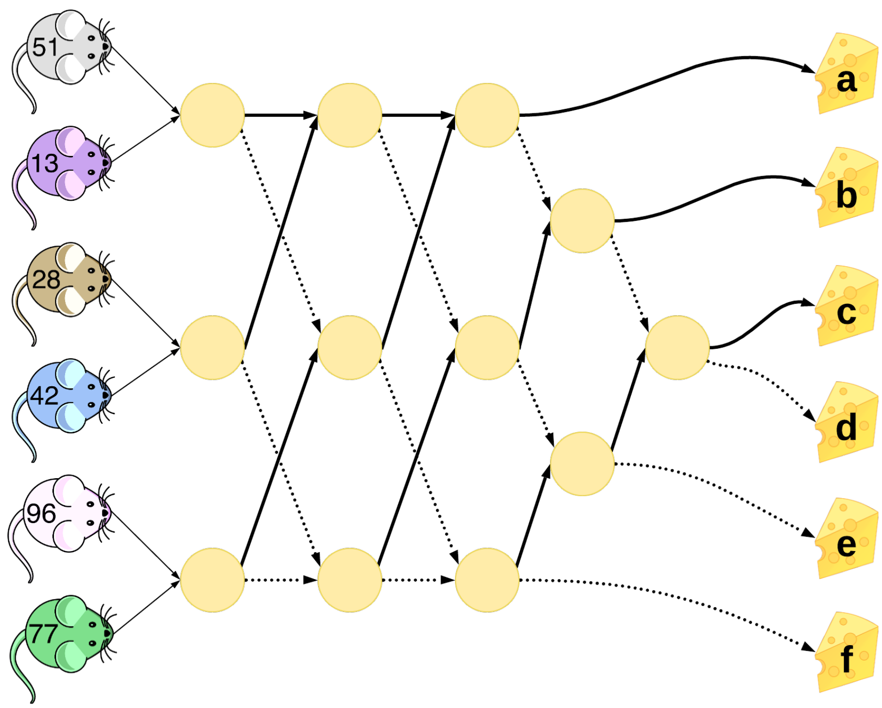
   Solution: The terms in this sum are consecutive positive **even** integers. There are $\frac{300}{2} = 150$ terms in the sum. To create a sum of consecutive positive **odd** integers, we can rewrite each term as an odd number plus 1, and then collect all the extra 1s as follows:

   $$\begin{aligned}
   &2 + 4 + 6 + \ldots + 296 + 298 + 300\\
   &= (1 + 1) + (3 + 1) + (5 + 1) + \ldots + (295 + 1) + (297 + 1) + (299 + 1)\\
   &= 1 + 3 + 5 + \ldots + 295 + 297 + 299 + (1 + 1 + 1 + \ldots + 1)\\
   &= 150^2 + 150\\
   &= 22\,650
   \end{aligned}$$

CEMC at Home

Grade 9/10 - Wednesday, May 27, 2020
Mixed Up Mice

Six numbered mice are moving through the network of paths shown below in order to reach the cheese. To start, the mice line up randomly on the left side of the network. Then, each mouse moves along a path by following the arrows. When a mouse reaches a yellow circle it waits for another mouse to arrive. When another mouse arrives at the circle, the two mice compare their numbers. The mouse with the smaller number follows the solid arrow out of the circle, while the mouse with the larger number follows the dotted arrow out of the circle.



Questions

1. After all of the mice move through the network of paths shown above, which mouse ends up with which piece of cheese?

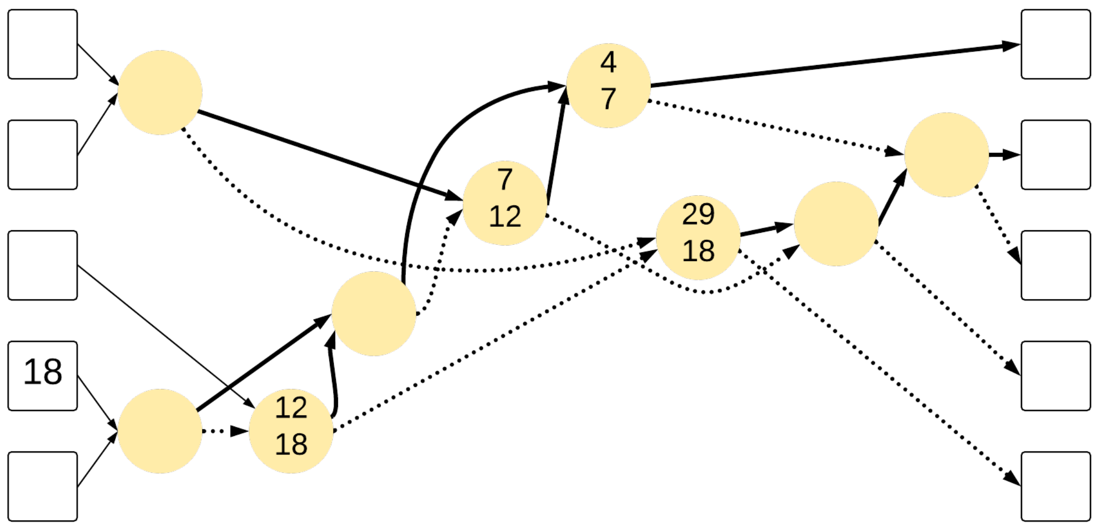
2. Now, line up the mice again at the start of the network, but change their starting order. After all of the mice move through the network of paths again, which mouse ends up with which cheese? Try repeating this a few times, each time with a different starting order for the mice. What do you notice?

3. What would happen if, upon leaving the circles, the mice with the smaller numbers followed the dotted arrows and the mice with the larger numbers followed the solid arrows? Explain.

Think about questions 1, 2, and 3 before moving on to the next page.

The network shown on the first page is an example of a *sorting network* for six numbers. There are six *inputs* on the left side of the network and six *outputs* on the right side of the network. Each yellow circle represents a *comparison* between two inputs (i.e. which number is larger?) and produces two outputs as demonstrated by the arrows. As the inputs move through the network, they are reordered (or *sorted*) according to their relative sizes.

4. Suppose you were given the following sorting network for five numbers. This network follows the same rules as the network on the first page. The inputs on the left side of the network are the numbers 4, 7, 12, 18, and 29, in some order. Using the information given below about how the numbers moved through the network, is it possible to determine the starting order of the five input numbers?



5. Draw a possible sorting network that sorts exactly four numbers.

   Can you draw two different sorting networks that sort exactly four numbers?

   Activity: Try drawing out your sorting network on a driveway with sidewalk chalk, or find a way to lay it out on a floor (what can you use for the circles and the arrows?). Ask your family members to pick a card from one suit in a deck of cards and randomly line up at the start. Play through a few rounds to convince yourself that you have a proper sorting network.

More Info:

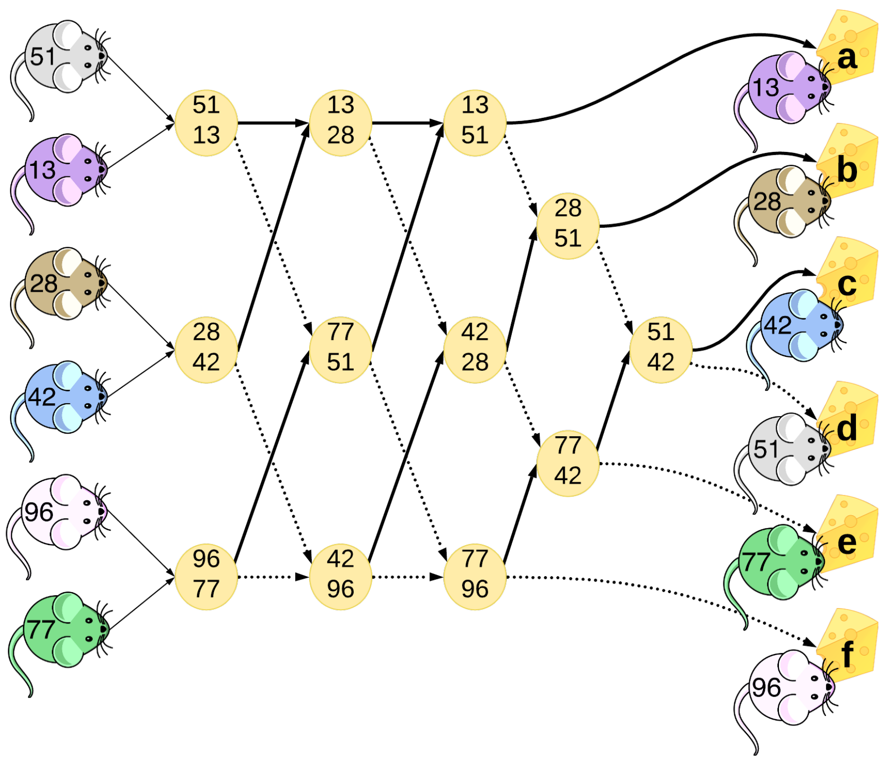
Check out the CEMC at Home webpage on Wednesday, June 3 for a solution to Mixed Up Mice.

CEMC at Home

Grade 9/10 - Wednesday, May 27, 2020
Mixed Up Mice - Solution

In computer science, a sorting network sorts a fixed number of items into a specific order which is predetermined. In this activity, you explored examples of sorting networks that sorted integers.

1. The solution below shows the path that each mouse travelled. Each yellow circle shows numbers that are being compared, and the mouse with the smaller number follows the *solid* line out of the circle, while the mouse with the larger number follows the *dotted* line.



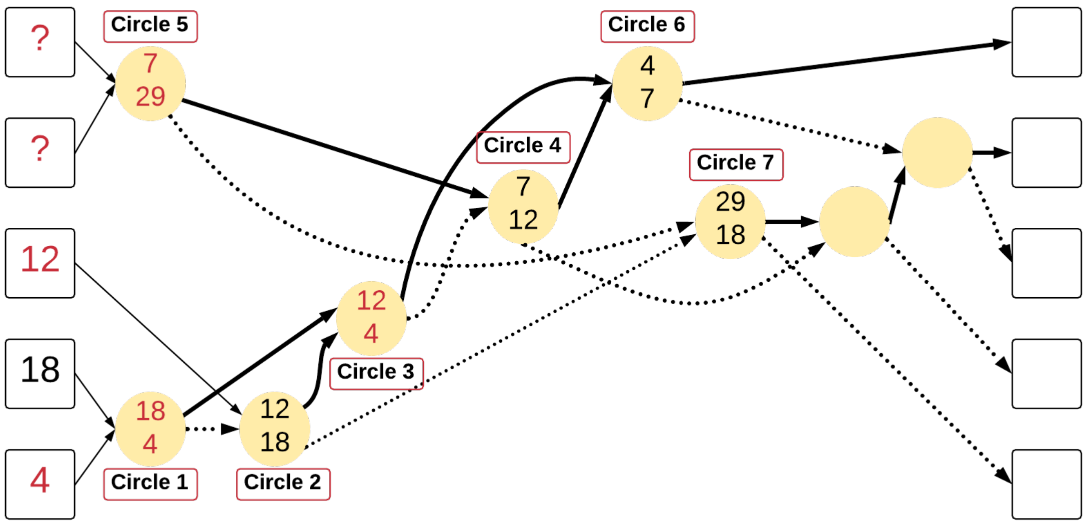
2. Regardless of how the numbers are arranged at the start, each time the mice reach the cheese, their numbers are sorted from smallest to largest, reading from top to bottom. This is not a coincidence as the above network is a *sorting network* for six numbers.

3. If instead the mouse with the smaller number follows the *dotted* line and the mouse with the larger number follows the *solid* line out of the circle, then the mice would end up being sorted from largest to smallest (reading top to bottom).

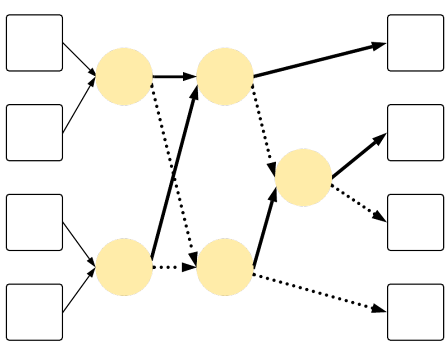
   Note: The numbers we have chosen for the mice are all distinct, and the network rules currently do not specify what to do if two mice have the same number. A computer cannot typically handle this lack of clarity. Instead, it needs to be given clear instructions to cover all possible scenarios. For example, for this sorting network, we could say that if two equal numbers are compared, then the mouse that comes from above follows the solid line out of the circle, while the mouse that comes from below follows the dotted arrow of the circle. If our only goal is to produce a sorted list of numbers, then it does not matter how we choose to handle the comparison of two equal numbers. However, if mice with equal numbers have other information associated with them, then how we make this choice could become more important.

4. It is not possible to completely determine the original order of the five inputs. How much *can* we determine about the order? There are a number of ways to approach this question. One possible approach is described below.

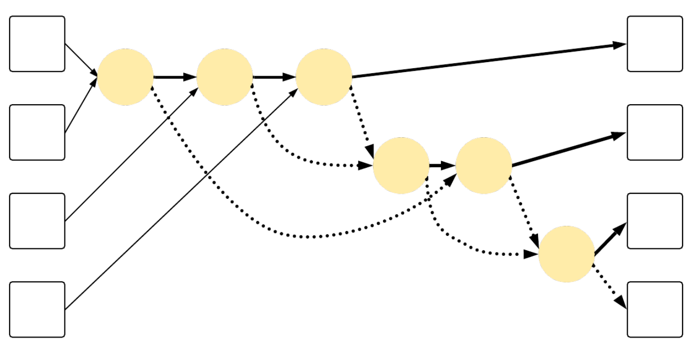
Since we are given the starting position for 18, we know that *Circle 1* has to contain 18 and another number. Next, we see that *Circle 2* contains 12 and 18. From here, we can trace back to the start of the network and place the 12 in the third input from the top. Looking at *Circle 2* again, we know that 12 is the smaller number, so it would proceed to *Circle 3*. From *Circle 4* we can determine that the 7 traces back to *Circle 5* as the 12 traces back to *Circle 3*. Similarly, looking at *Circle 6* we conclude that the 4 traces back to *Circle 3*. Following the arrows even further back to the start of the network, we determine that the 4 was the bottom input. Last, looking at *Circle 7* we determine that 29 comes from *Circle 5* as 18 traces back to *Circle 2*. We cannot determine the original order of inputs 7 and 29.



5. Below are two different sorting networks for four numbers. The first sorting network takes advantage of **parallel processing* while the second does not.



Sorting Network - Parallel Processing
Some comparisons occur simultaneously



Sorting Network - Sequential Processing
Only one comparison occurs at a time

*When a sorting network is used, it can be possible to perform some comparisons simultaneously, also known as *parallel processing*. This can speed up a sorting process overall. For example, in the sorting network from problem 1., the comparison between 51 and 13 occurs at the same time as the comparison between 28 and 42 as well as the comparison between 96 and 77. A common use of parallel processing with sorting networks is in the design of hardware.

CEMC at Home

Grade 9/10 - Thursday, May 28, 2020
Repetition By Product

A positive integer is to be placed in each box below.

Integers may be repeated, but the product of any four adjacent integers is always 120.

Determine all possible values for x.

		2			4			x			3		

More Info:

Check out the CEMC at Home webpage on Friday, May 29 for two different solutions to Repetition By Product.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 9/10 - Thursday, May 28, 2020
Repetition By Product - Solution

Problem:

A positive integer is to be placed in each box below.

Integers may be repeated, but the product of any four adjacent integers is always 120.

Determine all possible values for x.

		2			4			x			3		

Solution:

In both Solution 1 and Solution 2, let a_1 be the integer placed in the first box, a_2 the integer placed in the second box, a_4 the integer placed in the fourth box, and so on, as shown below.

a_1	a_2	2	a_4	a_5	4	a_7	a_8	x	a_{10}	a_{11}	3	a_{13}	a_{14}

Solution 1

Consider boxes 3 to 6. Since the product of any four adjacent integers is 120, we have $2 \times a_4 \times a_5 \times 4 = 120$. Therefore, $a_4 \times a_5 = \frac{120}{2\times 4} = 15$. Since a_4 and a_5 are positive integers, there are four possibilities: $a_4 = 1$ and $a_5 = 15$, or $a_4 = 15$ and $a_5 = 1$, or $a_4 = 3$ and $a_5 = 5$, or $a_4 = 5$ and $a_5 = 3$.

In each of the four cases, we will have $a_7 = 2$. We can see why by considering boxes 4 to 7. We have $a_4 \times a_5 \times 4 \times a_7 = 120$, or $15 \times 4 \times a_7 = 120$, since $a_4 \times a_5 = 15$. Therefore, $a_7 = \frac{120}{15\times 4} = 2$.

<u>Case 1:</u> $a_4 = 1$ and $a_5 = 15$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $15 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{15\times 4\times 2} = 1$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $4 \times 2 \times 1 \times x = 120$, or $x = \frac{120}{4\times 2} = 15$.
Let's check that $x = 15$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 9 to 12. If $x = 15$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{15\times 3} = \frac{8}{3}$. But a_{10} and a_{11} must both be integers, so is not possible for $a_{10} \times a_{11} = \frac{8}{3}$. Therefore, it must not be possible for $a_4 = 1$ and $a_5 = 15$, and so we find that there is no solution for x in this case.

<u>Case 2:</u> $a_4 = 15$ and $a_5 = 1$

Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $1 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{4\times 2} = 15$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times 2\times 15} = 1$.
Let's check that $x = 1$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 15$ and $x = 1$, then $a_{10} = \frac{120}{2\times 15\times 1} = 4$. Similarly, $a_{11} = \frac{120}{15\times 1\times 4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 1 \times 4 \times 2 \times 3 = 24 \neq 120$. Therefore, it must not be possible for $a_4 = 15$ and $a_5 = 1$. There is no solution for x in this case.

<u>Case 3:</u> $a_4 = 3$ and $a_5 = 5$
Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $5 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{5\times4\times2} = 3$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times3} = 5$.
Let's check that $x = 5$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 7 to 10. Since $a_7 = 2$, $a_8 = 3$ and $x = 5$, then $a_{10} = \frac{120}{2\times3\times5} = 4$. Similarly, $a_{11} = \frac{120}{3\times5\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 5 \times 4 \times 2 \times 3 = 120$. Therefore, the condition that $a_{12} = 3$ is satisfied in the case where $a_4 = 3$ and $a_5 = 5$. If we continue to fill out the entries in the boxes, we obtain the entries shown in the diagram below.

5	4	2	3	5	4	2	3	5	4	2	3	5	4

We see that $x = 5$ is a possible solution. However, is it the only solution? We have one final case to check.

<u>Case 4:</u> $a_4 = 5$ and $a_5 = 3$
Consider boxes 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $3 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{3\times4\times2} = 5$.
Next, consider boxes 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times5} = 3$.
Let's check that $x = 3$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider boxes 9 to 12. If $x = 3$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{3\times3} = \frac{40}{3}$. But a_{10} and a_{11} must both be integers, so it is not possible for $a_{10} \times a_{11} = \frac{40}{3}$. Therefore, it must not be possible for $a_4 = 5$ and $a_5 = 3$, and so we find that there is no solution for x in this case.

Therefore, the only possible value for x is $x = 5$.

Solution 2

You may have noticed a pattern for the a_i's in Solution 1. We will explore this pattern.

a_1	a_2	2	a_4	a_5	4	a_7	a_8	x	a_{10}	a_{11}	3	a_{13}	a_{14}

Since the product of any four integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$. Since both sides are divisible by $a_2a_3a_4$, and each is a positive integer, then $a_1 = a_5$.
Similarly, $a_2a_3a_4a_5 = a_3a_4a_5a_6 = 120$, and so $a_2 = a_6$.
In general, $a_na_{n+1}a_{n+2}a_{n+3} = a_{n+1}a_{n+2}a_{n+3}a_{n+4}$, and so $a_n = a_{n+4}$.
We can use this along with the given information to fill out the boxes as follows:

x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $4 \times 2 \times 3 \times x = 120$ and so $x = \frac{120}{4\times2\times3} = 5$.

CEMC at Home

Grade 9/10 - Friday, May 29, 2020
Circle Splash

A group of friends have devised a fun way to stay cool on a hot summer day – bursting water balloons!

Here is how they play:

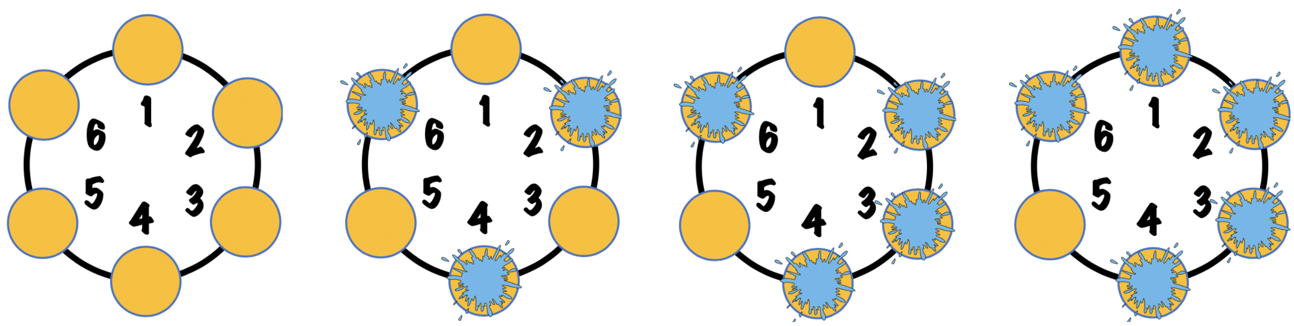
- Everyone stands in a circle. One friend is assigned the number 1 (position 1), and then the other friends in the circle are numbered 2, 3, 4, etc. moving clockwise around the circle until you get back to position 1.
- Starting with the person in position 1 and moving clockwise around the circle, every second person who is still dry bursts a water balloon over their head. (This is explained further below.)
- The last person remaining dry wins the game!

Example: Here is how the game plays out if the circle contains six friends.

The first time around the circle, friends 2, 4, and 6 get water balloons.

Since friend 2 is already out, friend 3 gets the next water balloon.

Since friends 4 and 6 are already out, friend 1 gets the next water balloon, and so friend 5 wins!



Questions:

For each instance of the game below, determine the position of the friend that will win the game.

1. The game is played with 7 friends.
2. The game is played with 16 friends.
3. The game is played with 41 friends.

Challenge: Can you determine the position of the friend that wins the game if the game is played with n friends, where n is a positive integer that is at least 2?

More Info:

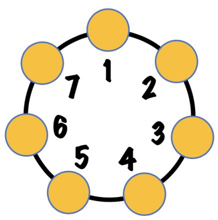
Check out the CEMC at Home webpage on Friday, June 5 for a solution to Circle Splash.

CEMC at Home

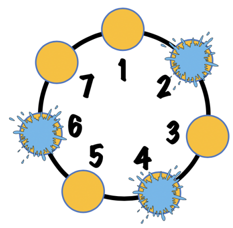
Grade 9/10 - Wednesday, May 13, 2020
Circle Splash - Solution

1. When the game is played with 7 friends, the friend in position 7 will win (remain dry). We illustrate this using the following images:

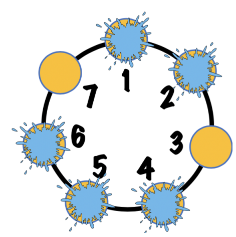
Start



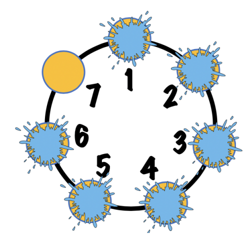
The first time around the circle, friends 2, 4, and 6 get water balloons.



The second time around the circle, friends 1 and 5 get water balloons.



The third time around the circle, friend 3 gets a water balloon and so friend 7 wins!



2. When the game is played with 16 friends, the friend in position 1 will win (remain dry). Instead of drawing pictures to illustrate the solution, we start with the sequence of numbers from 1 to 16 and cross off the numbers when this position gets a water balloon.

   Each time we travel through the numbers, we cross out every second number that is not already crossed out. Each time we reach 16, we loop around to 1.

Start:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
First pass:	1, ~~2~~, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~
Second pass:	1, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, ~~7~~, ~~8~~, 9, ~~10~~, ~~11~~, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~
Third pass:	1, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~8~~, 9, ~~10~~, ~~11~~, ~~12~~, ~~13~~, ~~14~~, ~~15~~, ~~16~~
Fourth pass:	1, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~8~~, ~~9~~, ~~10~~, ~~11~~, ~~12~~, ~~13~~, ~~14~~, ~~15~~, ~~16~~

   The numbers were crossed out in the following order: 2, 4, 6, 8, 10, 12, 14, 16, 3, 7, 11, 15, 5, 13, 9.

   This leaves 1 at the end.

   Can you see a pattern forming here?

3. When the game is played with 41 friends, the friend in position 19 will remain dry. This can be determined by simulating the game as was done in the solution to either question 1 or question 2, although it will take longer in this case. After discussing the challenge problem, we will return to this answer.

Challenge: Can you determine the position of the friend that wins the game if the game is played with n friends, where n is a positive integer that is at least 2?

Solution:

When the game is played with n friends, a simulation will not work. We need to step back and analyze the game more generally.

Here are some observations:

- The position of the friend who remains dry will never be even since all even numbers are eliminated during the first pass around the circle.
- If a circle has an even number of friends in it, then after one pass, we will be on position 1 again and the number of friends left dry in the circle will be half of what it was at the start.

This second observation above leads to some more observations:

- If, after one pass, the circle still has an even number of friends that are dry, then after pass two, we will be on position 1 again and the number of friends left dry in the circle will be half of what it was.
- If after every pass (except for the final one), the circle continues to have an even number of friends that are dry, eventually we will land on position 1 with only two friends left dry in the circle. On the final pass, the friend in position 1 remains dry and the other friend gets wet.
- In order for the circle to have an even number of friends left dry after every pass (except for the final one), the number of friends at the start, n, needs to be a power of 2.
- We can conclude then that if n is a power of 2, then the friend in position 1 will always remain dry. (Notice that our answer to question 2 agrees with this.)

What if n is not a power of 2? At some point during the game, eliminating people one at a time, the number of friends left dry in the circle *will become* a power of 2. For example, if we started with 12 friends, then after four water balloons, the circle will be left with $8 = 2^3$ people that are still dry. When this happens, whichever position is "next" in the circle acts like position 1 (in the case where n is a power of 2) and will be the position that remains dry for the rest of the game.

Given n, how many people need to be eliminated so that the number of people remaining dry in the circle is a power of 2? Rewrite n as a power of 2 plus m as follows:

$$n = 2^k + m$$

where 2^k is the largest power of 2 less than or equal to n and m is a non-negative integer. Thus, after eliminating m people, there will be 2^k people remaining dry in the circle.

The position that remains dry for the rest of the game is the position in the circle that comes immediately after the mth person that is eliminated. What position is this? Since on the first trip around the circle every second person is eliminated, and it must be the case that $2m < n$, the mth person to be eliminated is in position $2m$. This means the "next" position is position $2m + 1$.

Can you see why it must be the case that $2m < n$ *and why this is important for our argument above?*

We can conclude that if n is not a power of 2 and we rewrite n as $n = 2^k + m$ (as outlined earlier), then the person that remains dry is the person in position $2m + 1$.

Note 1: If n is a power of 2 then we have $m = 0$ and hence $2m + 1 = 2(0) + 1 = 1$. So this formula tells us the position that remains dry in the case where n is a power of 2 as well.

Note 2: This formula agrees with our earlier answers. In question 1, we have $n = 7$ which can be rewritten as $7 = 4 + 3 = 2^2 + 3$. This means $m = 3$ and so the position that remains dry is $2m + 1 = 2(3) + 1 = 7$. In question 2, we have $n = 16$ which can be rewritten as $16 = 2^4 + 0$. This means $m = 0$ and so $2m + 1 = 2(0) + 1 = 1$. In question 3, we have $n = 41$ which can be rewritten as $n = 32 + 9 = 2^5 + 9$. This means $m = 9$ and so $2m + 1 = 2(9) + 1 = 19$.

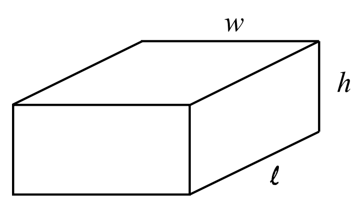
CEMC at Home

Grade 9/10 - Monday, June 1, 2020
Contest Day 5

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Galois Contest, #2

For a rectangular prism with length ℓ, width w, and height h as shown, the surface area is given by the formula $A = 2\ell w + 2\ell h + 2wh$ and the volume is given by the formula $V = \ell wh$.



(a) What is the surface area of a rectangular prism with length 2 cm, width 5 cm, and height 9 cm?

(b) A rectangular prism with height 10 cm has a square base. The volume of the prism is 160 cm^3. What is the side length of the square base?

(c) A rectangular prism has a square base with area 36 cm^2. The surface area of the prism is 240 cm^2. Determine the volume of the prism.

(d) A rectangular prism has length k cm, width $2k$ cm, and height $3k$ cm, where $k > 0$. The volume of the prism is x cm^3. The surface area of the prism is x cm^2. Determine the value of k.

More Info:

Check out the CEMC at Home webpage on Monday, June 8 for a solution to the Contest Day 5 problem.

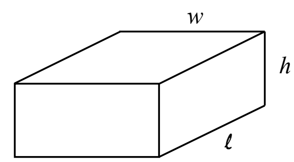
CEMC at Home

Grade 9/10 - Monday, June 1, 2020
Contest Day 5 - Solution

A solution to the contest problem is provided below.

2020 Galois Contest, #2

For a rectangular prism with length ℓ, width w, and height h as shown, the surface area is given by the formula $A = 2\ell w + 2\ell h + 2wh$ and the volume is given by the formula $V = \ell wh$.



(a) What is the surface area of a rectangular prism with length 2 cm, width 5 cm, and height 9 cm?

(b) A rectangular prism with height 10 cm has a square base. The volume of the prism is 160 cm^3. What is the side length of the square base?

(c) A rectangular prism has a square base with area 36 cm^2. The surface area of the prism is 240 cm^2. Determine the volume of the prism.

(d) A rectangular prism has length k cm, width $2k$ cm, and height $3k$ cm, where $k > 0$. The volume of the prism is x cm^3. The surface area of the prism is x cm^2. Determine the value of k.

Solution:

(a) The surface area of a rectangular prism is given by the formula $A = 2\ell w + 2\ell h + 2wh$.
Thus, the rectangular prism with length 2 cm, width 5 cm, and height 9 cm has surface area $2(2)(5) + 2(2)(9) + 2(5)(9) = 20 + 36 + 90 = 146$ cm^2.

(b) The volume of a rectangular prism is given by the formula $V = \ell wh$.
If the rectangular prism has a square base, then $\ell = w$ and so $V = \ell^2 h$.
Substituting $V = 160$ cm^3 and $h = 10$ cm, we get $160 = \ell^2(10)$ or $\ell^2 = 16$, and so $\ell = 4$ cm (since $\ell > 0$).
Therefore, the side length of the square base of a rectangular prism with height 10 cm and volume 160 cm^3 is 4 cm.

(c) If a rectangular prism has a square base, then $\ell = w$.
Since the area of the base is 36 cm^2, then $36 = \ell \cdot w = \ell^2$, and so $\ell = w = \sqrt{36} = 6$ cm (since $\ell > 0$).
If the surface area of this prism is 240 cm^2, then substituting the values of ℓ and w, we get $240 = 2(6)(6) + 2(6)h + 2(6)h$ or $240 = 72 + 24h$, and so $h = \dfrac{240 - 72}{24} = 7$ cm.
Thus, the volume of the prism is $\ell wh = (6)(6)(7) = 252$ cm^3.

See the next page for a solution to part (d).

(d) Substituting into the formula for volume, we get $x = k(2k)(3k)$ or $x = 6k^3$.
Substituting into the formula for surface area, we get $x = 2(k)(2k) + 2(k)(3k) + 2(2k)(3k)$ or $x = 4k^2 + 6k^2 + 12k^2 = 22k^2$.
Equating the two expressions that are each equal to x and solving, we get

$$\begin{aligned} 6k^3 &= 22k^2 \\ 6k^3 - 22k^2 &= 0 \\ 2k^2(3k - 11) &= 0 \end{aligned}$$

Since $k > 0$, then $3k - 11 = 0$ and so $k = \frac{11}{3}$.

CEMC at Home

Grade 9/10 - Tuesday, June 2, 2020
Famous Mathematicians

Throughout human history, many mathematicians have made significant contributions to the subject. These important historical figures often lead fascinating lives filled with interesting stories. Five of these mathematicians are listed below.

Al-Khwārizmī	He was a 9th century mathematician from Baghdad where he was the head of the library *House of Wisdom*. The title of one of his books gave us the word "algebra".
Blaise Pascal	He was a 17th French century mathematician whose work laid the foundation for the modern theory of probability. He also contributed greatly to the areas of physics and religion.
William Tutte	He was born in England and was an important codebreaker during World War II. In 1962, he started working at the University of Waterloo where his work greatly shaped the area of graph theory.
Grigori Perelman	A current Russian mathematician who was awarded the Fields Medal in 2006, but declined it. His work in the area of geometry is important for the Poincaré conjecture, a famous result about topology.
Maryam Mirzakhani	She was the first woman to be awarded the prestigious Fields Medal. She was born in Iran and then studied and worked in the United States before breast cancer took her life in 2017.

Choose two of these five mathematicians and for each one you choose:

1. Do some online research to determine an additional interesting fact about the mathematician.
2. Try to find a connection between something you have studied in a recent mathematics class and the mathematical work of this historical figure.
3. If you had the chance to go back in time and meet this mathematician, what question would you ask them?

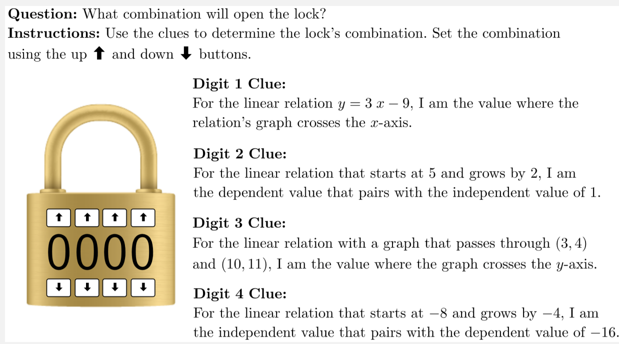
More Info: The CEMC Pascal Math Contest is named in honour of Blaise Pascal.

CEMC at Home

Grade 9/10 - Wednesday, June 3, 2020
Interact with Mathematics

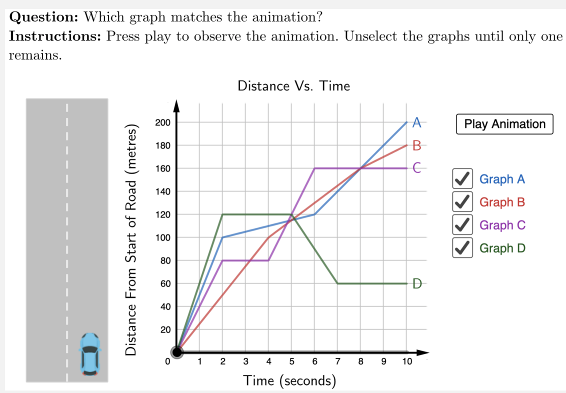
Technology can help us make mathematical discoveries and learn about mathematical objects. Three online examples of this from different areas of mathematics are featured below.

Unknown Linear Values: Use the properties of linear relations to find a lock's combination.



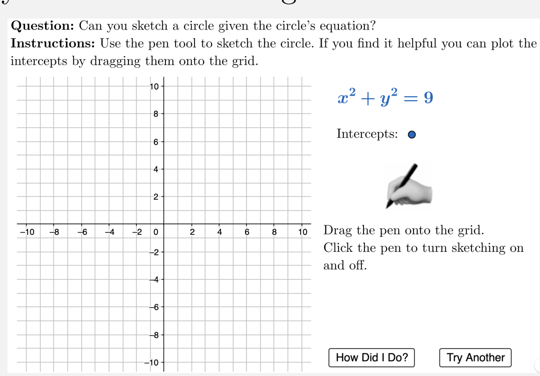
Link to App: https://www.geogebra.org/m/nzvvj3dh

Story Graphs: A car is driving down a road. Identify the graph that describes the car's trip.



Link to App: https://www.geogebra.org/m/n4mtvugh

Sketching a Circle: Try your hand at drawing a freehand circle.



Link to App: https://www.geogebra.org/m/ukedzvnb

More Info: CEMC courseware lessons feature hundreds of interactive mathematics applications. For the Grade 9/10/11 CEMC courseware, an interactive library has been built which allows you to perform a keyword search and/or display only the applications from a given strand, unit or lesson.

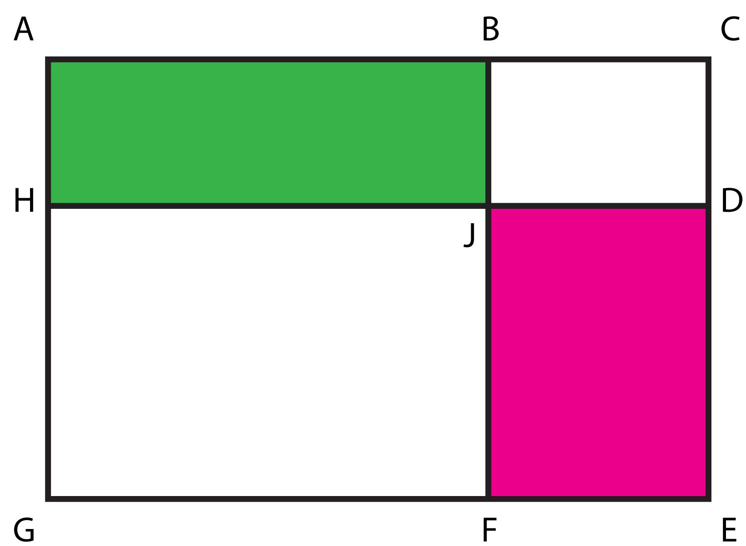
CEMC at Home

Grade 9/10 - Thursday, June 4, 2020
Maximize the Area

Two rectangles, $ABJH$ and $JDEF$, with integer side lengths, share a common corner at J such that HJD and BJF are perpendicular line segments. The two rectangles are enclosed by a larger rectangle $ACEG$, as shown.

The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

Determine the largest possible area of the rectangle $ACEG$. Note that the diagram is not intended to be to scale.



More Info:

Check out the CEMC at Home webpage on Friday, June 5 for a solution to Maximize the Area.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

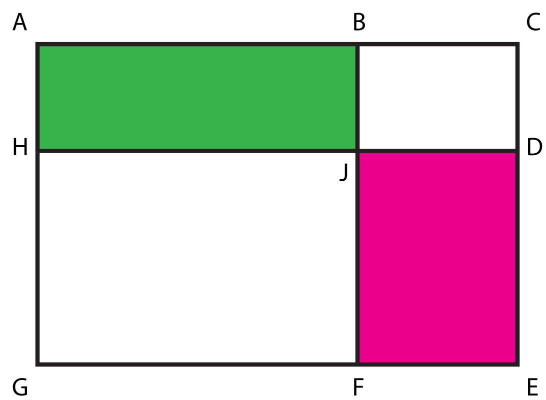
Grade 9/10 - Thursday, June 4, 2020
Maximize the Area - Solution

Problem:

Two rectangles, $ABJH$ and $JDEF$, with integer side lengths, share a common corner at J such that HJD and BJF are perpendicular line segments. The two rectangles are enclosed by a larger rectangle $ACEG$, as shown.

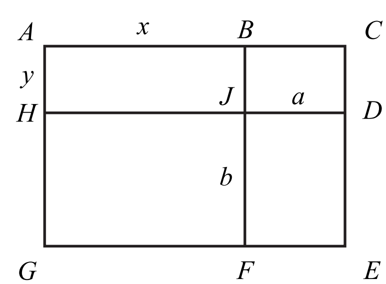
The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

Determine the largest possible area of the rectangle $ACEG$. Note that the diagram is not intended to be to scale.



Solution:

Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.



Then,

$$\begin{aligned} AB &= HJ = GF = x, \\ AH &= BJ = CD = y, \\ BC &= JD = FE = a, \text{ and} \\ HG &= JF = DE = b. \end{aligned}$$

Also,

$$\begin{aligned} \text{area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG) \\ &= 6 + ya + 15 + xb \\ &= 21 + ya + xb \end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

To maximize the area, we need to pick the values of x, y, a, b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

Case 1: $x = 1$ cm, $y = 6$ cm and $a = 1$ cm, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42 \text{ cm}^2$$

Case 2: $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112 \text{ cm}^2$$

Case 3: $x = 1$ cm, $y = 6$ cm and $a = 3$ cm, $b = 5$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44 \text{ cm}^2$$

Case 4: $x = 1$ cm, $y = 6$ cm and $a = 5$ cm, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54 \text{ cm}^2$$

Case 5: $x = 2$ cm, $y = 3$ cm and $a = 1$, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54 \text{ cm}^2$$

Case 6: $x = 2$ cm, $y = 3$ cm and $a = 15$, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68 \text{ cm}^2$$

Case 7: $x = 2$ cm, $y = 3$ cm and $a = 3$, $b = 5$ cm

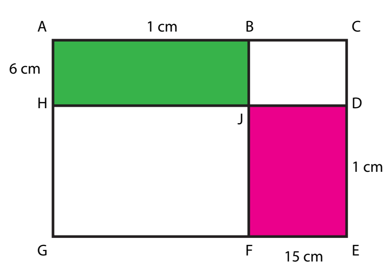
$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40 \text{ cm}^2$$

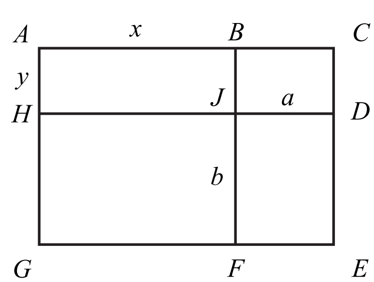
Case 8: $x = 2$ cm, $y = 3$ cm and $a = 5$, $b = 3$ cm

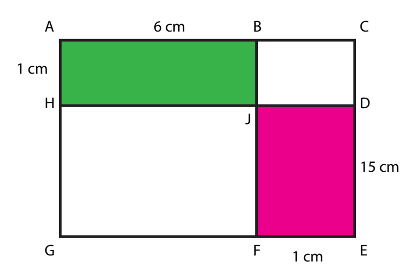
$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42 \text{ cm}^2$$

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.







CEMC at Home

Grade 9/10 - Friday, June 5, 2020
Math and CS in the News

Most weeks, our CEMC Homepage provides a link to a story in the media about mathematics and/or computer science. These stories show us how important mathematics and computer science are in today's world. They are a great source for discussions.

Using this article from CBC News, think about the following questions. (URL also provided below.)

1. What do you think someone means when they say that humans are more *intelligent* than other members of the animal kingdom?
2. What is artificial intelligence? Can you name two ways in which you use artificial intelligence yourself?
3. What advantages and disadvantages do you see to artificial intelligence?
4. Predict the future: How will artificial intelligence change in 20 years?

URL of the article:
https://www.cbc.ca/news/technology/artificial-intelligence-human-brain-to-merge-in-2030s-says-futurist-kurzweil-1.3100124

More Info:

A full archive of past posts can be found in our Math and CS in the News Archive. Similar resources for other grades may also be of interest.

CEMC at Home

Grade 9/10 - Monday, June 8, 2020
Contest Day 6

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Fryer Contest, #3

In a *Dlin sequence*, the first term is a positive integer and each term after the first is calculated by adding 1 to the previous term in the sequence, then doubling the result. For example, the first seven terms of the Dlin sequence with first term 4 are:

$$4, 10, 22, 46, 94, 190, 382$$

(a) The 5^{th} term in a Dlin sequence is 142. What are the 4^{th} and 6^{th} terms in the sequence?

(b) Determine all possible first terms which give a Dlin sequence that includes 1406.

(c) Which possible first terms from 10 to 19 inclusive produce a Dlin sequence in which all terms after the first have the same ones (units) digit?

(d) Determine the number of positive integers between 1 and 2020, inclusive, that can be the third term in a Dlin sequence.

More Info:

Check out the CEMC at Home webpage on Monday, June 15 for a solution to the Contest Day 6 problem.

CEMC at Home

Grade 9/10 - Monday, June 8, 2020
Contest Day 6 - Solution

A solution to the contest problem is provided below.

2020 Fryer Contest, #3

In a *Dlin sequence*, the first term is a positive integer and each term after the first is calculated by adding 1 to the previous term in the sequence, then doubling the result. For example, the first seven terms of the Dlin sequence with first term 4 are:

$$4, 10, 22, 46, 94, 190, 382$$

(a) The 5^{th} term in a Dlin sequence is 142. What are the 4^{th} and 6^{th} terms in the sequence?

(b) Determine all possible first terms which give a Dlin sequence that includes 1406.

(c) Which possible first terms from 10 to 19 inclusive produce a Dlin sequence in which all terms after the first have the same ones (units) digit?

(d) Determine the number of positive integers between 1 and 2020, inclusive, that can be the third term in a Dlin sequence.

Solution:

(a) If the 5^{th} term in a Dlin sequence is 142, then the 6^{th} term is $(142 + 1) \times 2 = 143 \times 2 = 286$.
To determine the 4^{th} term in the sequence given the 5^{th}, we "undo" adding 1 followed by doubling the result by first dividing the 5^{th} term by 2 and then subtracting 1 from the result.
To see this, consider that if two consecutive terms in a Dlin sequence are a followed by b, then $b = (a + 1) \times 2$.
To determine the operations needed to find a given b (that is, to move backward in the sequence), we rearrange this equation to solve for a.

$$\begin{aligned} b &= (a+1) \times 2 \\ \tfrac{b}{2} &= a+1 \\ \tfrac{b}{2} - 1 &= a \end{aligned}$$

Thus if the 5^{th} term in the sequence is 142, then the 4^{th} term is $\frac{142}{2} - 1 = 71 - 1 = 70$.
(We may check that the term following 70 is indeed $(70 + 1) \times 2 = 142$.)

(b) If the 1^{st} term is 1406, then clearly this is a Dlin sequence that includes 1406.
If the 2^{nd} term in a Dlin sequence is 1406, then the 1^{st} term in the sequence is $\frac{1406}{2} - 1 = 703 - 1 = 702$.
If the 3^{rd} term in a Dlin sequence is 1406, then the 2^{nd} term is 702 (as calculated in the line above) and the 1^{st} term in the sequence is $\frac{702}{2} - 1 = 351 - 1 = 350$.
If the 4^{th} term in a Dlin sequence is 1406, then the 3^{rd} term is 702, the 2^{nd} term is 350, and the 1^{st} term in the sequence is $\frac{350}{2} - 1 = 175 - 1 = 174$.

At this point, we see that 174, 350, 702, and 1406 are possible 1^{st} terms which give a Dlin sequence that includes 1406.
We may continue this process of working backward (dividing by 2 and subtracting 1) to determine all possible 1^{st} terms which give a Dlin sequence that includes 1406.

$$1406 \to 702 \to 350 \to 174 \to \frac{174}{2} - 1 = 86 \to \frac{86}{2} - 1 = 42 \to \frac{42}{2} - 1 = 20 \to \frac{20}{2} - 1 = 9$$

Attempting to continue the process beyond 9 gives $\frac{9}{2} - 1 = \frac{7}{2}$ which is not possible since the 1^{st} term in a Dlin sequence must be a positive integer (and so all terms are positive integers).
Thus, the possible 1^{st} terms which give a Dlin sequence that includes 1406 are 9, 20, 42, 86, 174, 350, 702, and 1406.

(c) Each of the integers from 10 to 19 inclusive is a possible first term, and so we must determine the ones digit of each term which follows each of these ten possible first terms.
If the 1^{st} term is 10, then the 2^{nd} term $(10 + 1) \times 2 = 22$ has ones digit 2, and the 3^{rd} term $(22 + 1) \times 2 = 46$ has ones digit 6.
If the 1^{st} term is 11, then the ones digit of the 2^{nd} term $(11+1) \times 2 = 24$ is 4, and the 3^{rd} term $(24 + 1) \times 2 = 50$ has ones digit 0.
Given each of the possible first terms, we list the ones digits of the 2^{nd} and 3^{rd} terms in the table below.

1^{st} term	10	11	12	13	14	15	16	17	18	19
Units digit of the 2^{nd} term	2	4	6	8	0	2	4	6	8	0
Units digit of the 3^{rd} term	6	0	4	8	2	6	0	4	8	2

From the table above, we see that the only ones digit which repeats itself is 8.
Thus, if the 1^{st} term in the sequence is 18 (has ones digit 8), then the 2^{nd} and 3^{rd} terms in the sequence have ones digit 8 and so all terms will have the same ones digit, 8.
Similarly, if the 1^{st} term in the sequence is 13 (has ones digit 3), then the 2^{nd} and 3^{rd} terms in the sequence have ones digit 8.
It then follows that all further terms after the first will have ones digit 8.
The 1^{st} terms (from 10 to 19 inclusive) which produce a Dlin sequence in which all terms after the 1^{st} term have the same ones digit are 13 and 18.

(d) If the 1^{st} term in a Dlin sequence is x, then the 2^{nd} term is $(x + 1) \times 2 = 2x + 2$, and the 3^{rd} term is $(2x + 2 + 1) \times 2 = (2x + 3) \times 2 = 4x + 6$.
For example, if $x = 1$ (note that this is the smallest possible 1^{st} term of a Dlin sequence), then the 3^{rd} term is $4 \times 1 + 6 = 10$, and if $x = 2$, the 3^{rd} term is $4 \times 2 + 6 = 14$.
What is the largest possible value of x (the 1^{st} term of the sequence) which makes $4x + 6$ (the 3^{rd} term of the sequence) less than or equal to 2020?
Setting $4x + 6$ equal to 2020 and solving, we get $4x = 2014$ and so $x = 503.5$.
Since the 1^{st} term of the sequence must be a positive integer, the 3^{rd} term cannot be 2020.
Similarly, solving $4x + 6 = 2019$, we get that x is not an integer and so the 3^{rd} term of a Dlin sequence cannot equal 2019.
When $4x + 6 = 2018$, we get $4x = 2012$ and so $x = 503$.
Thus, if a Dlin sequence has 1^{st} term equal to 503, then the 3^{rd} term of the sequence is a positive integer between 1 and 2020, namely 2018.
Further, 503 is the largest possible 1^{st} term for which the 3^{rd} term has this property.
Each 1^{st} term x will give a different 3^{rd} term, $4x + 6$.

Thus, to count the number of positive integers between 1 and 2020, inclusive, that can be the 3^{rd} term in a Dlin sequence, we may count the number of 1^{st} terms which give a 3^{rd} term having this property.
The smallest possible 1^{st} term is 1 (giving a 3^{rd} term of 10) and the largest possible 1^{st} term is 503 (which gives a 3^{rd} term of 2018).
Further, every value of x between 1 and 503 gives a different 3^{rd} term between 10 and 2018.
Thus, there are 503 positive integers between 1 and 2020, inclusive, that can be the 3^{rd} term in a Dlin sequence.

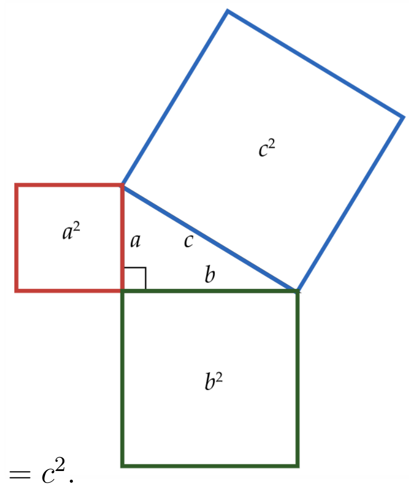
CEMC at Home

Grade 9/10 - Tuesday, June 9, 2020
Pythagorean Triples

The Pythagorean Theorem: In a right-angled triangle, where c represents the length of the hypotenuse and a and b represent the lengths of the two legs (the two shorter sides), the following equation is true:

$$a^2 + b^2 = c^2$$

Note: It is also true that any triangle with side lengths a, b, and c that satisfy the equation $a^2 + b^2 = c^2$ must be a right-angled triangle.



A *Pythagorean triple* is a triple of integers (a, b, c) that satisfies $a^2 + b^2 = c^2$.

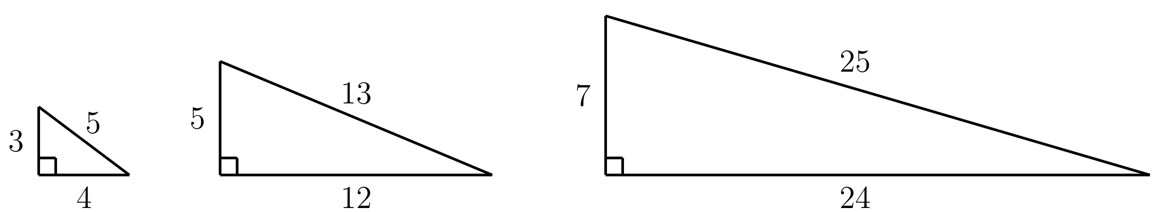
If a triangle is formed with integer side lengths a, b, and c, then this triangle is a right-angled triangle exactly when (a, b, c) is a Pythagorean triple.

Problem 1: The triple $(3, 4, 5)$ is a Pythagorean triple. If $a = 3$, $b = 4$, and $c = 5$, then

$$a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 \quad \text{and} \quad c^2 = 5^2 = 25$$

and therefore, $a^2 + b^2 = c^2$.

Show that $(5, 12, 13)$ and $(7, 24, 25)$ are also Pythagorean triples.



Is every positive integer a part of at least one Pythagorean triple? It turns out that the answer to this question is no. However, every positive integer that is *at least 3* is a part of a Pythagorean triple. Let's explore this idea.

Problem 2: Consider the Pythagorean triples from Problem 1: $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$. Notice that the integers in the leftmost coordinates of the triples are the odd integers 3, 5, and 7. Do you notice a pattern in the other two integers in each triple? The remaining two integers in each triple are consecutive integers: 4 and 5, 12 and 13, 24 and 25. Let's explore this pattern.

(a) Build a Pythagorean triple that includes the odd integer 9 by following these steps:

(i) Determine n such that $9^2 = 2n + 1$. (*Answer:* $n = \frac{81-1}{2} = 40$.)

(ii) Verify that $(n + 1)^2 - n^2 = 9^2$. (*Answer:* $41^2 - 40^2 = 1681 - 1600 = 81 = 9^2$.)

(iii) Write down a Pythagorean triple for which the smallest integer is 9. (*Answer:* Since $41^2 - 40^2 = 9^2$, we have $9^2 + 40^2 = 41^2$ and so $(9, 40, 41)$ is a Pythagorean triple.)

(b) Build a Pythagorean triple that includes the odd integer 11 by following these steps:

  (i) Determine n such that $11^2 = 2n + 1$.
  (ii) Verify that $(n + 1)^2 - n^2 = 11^2$.
  (iii) Write down a Pythagorean triple for which the smallest integer is 11.

(c) Use the ideas from (a) and (b) to build Pythagorean triples that include the next four odd integers: 13, 15, 17, 19.

Can you see how to do this for any odd integer? We explore this in the challenge problem.

Problem 3:

(a) Consider the Pythagorean triple $(3, 4, 5)$. Show that if you multiply each integer in the triple by 2, then you obtain another Pythagorean triple.

(b) Use the idea from (a) to build another Pythagorean triple that includes the odd integer 9.

(c) Show that for every positive integer n, the triple $(3n, 4n, 5n)$ is a Pythagorean triple.

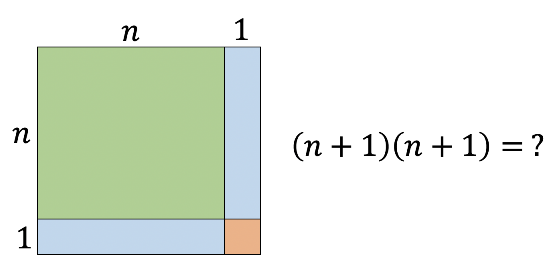
It is also true that $(5n, 12n, 13n)$ *and* $(7n, 24n, 25n)$ *are Pythagorean triples.*

(d) Use the ideas from Problem 2 and Problem 3 to show that every integer from 4 to 20 is part of at least one Pythagorean triple.

Challenge Problem: Think about how you might use some of these ideas to show that every integer that is at least 3 is part of a Pythagorean triple. One possible approach is outlined below, but there are others:

(a) Odd numbers:

  (i) Show that for every positive integer n, we have $(n + 1)^2 - n^2 = 2n + 1$.



  (ii) Use the identity from part (i) to explain why every odd integer that is at least 3 is part of a Pythagorean triple.

(b) Even numbers:

  (i) Show that for every positive integer n, we have $(n + 2)^2 - n^2 = 4n + 4$.
  (ii) Use the identity from part (i) to explain why every even integer that is at least 4 is part of a Pythagorean triple.

More Info:

Check out the CEMC at Home webpage on Tuesday, June 16 for a solution to Pythagorean Triples.

CEMC at Home

Grade 9/10 - Tuesday, June 9, 2020
Pythagorean Triples - Solution

Problem 1: The triple $(3, 4, 5)$ is a Pythagorean triple. Show that $(5, 12, 13)$ and $(7, 24, 25)$ are also Pythagorean triples.

Solution:

If $a = 5$, $b = 12$, and $c = 13$, then $a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$ and $c^2 = 13^2 = 169$. Therefore, $a^2 + b^2 = c^2$.

If $a = 7$, $b = 24$, and $c = 25$, then $a^2 + b^2 = 7^2 + 24^2 = 49 + 576 = 625$ and $c^2 = 25^2 = 625$. Therefore, $a^2 + b^2 = c^2$.

Problem 2:

(a) Build a Pythagorean triple that includes the odd integer 9 by following these steps:

- (i) Determine n such that $9^2 = 2n + 1$. (*Answer:* $n = \frac{81-1}{2} = 40$.)
- (ii) Verify that $(n + 1)^2 - n^2 = 9^2$. (*Answer:* $41^2 - 40^2 = 1681 - 1600 = 81 = 9^2$.)
- (iii) Write down a Pythagorean triple for which the smallest integer is 9. (*Answer:* Since $41^2 - 40^2 = 9^2$, we have $9^2 + 40^2 = 41^2$ and so $(9, 40, 41)$ is a Pythagorean triple.)

(b) Build a Pythagorean triple that includes the odd integer 11 by following these steps:

- (i) Determine n such that $11^2 = 2n + 1$.
- (ii) Verify that $(n + 1)^2 - n^2 = 11^2$.
- (iii) Write down a Pythagorean triple for which the smallest integer is 11.

(c) Use the ideas from (a) and (b) to build Pythagorean triples that include the next four odd integers: 13, 15, 17, 19.

Solution:

(b)
- (i) We have $11^2 = 121 = 2n + 1$ exactly when $n = \frac{121-1}{2} = 60$.
- (ii) When $n = 60$ we have $(n + 1)^2 - n^2 = 61^2 - 60^2 = 3721 - 3600 = 121 = 11^2$.
- (iii) Since $61^2 - 60^2 = 11^2$, we have $11^2 + 60^2 = 61^2$ and so $(11, 60, 61)$ is a Pythagorean triple involving the integer 11.

(c) For the integer 13: We have $13^2 = 169 = 2n + 1$ exactly when $n = \frac{169-1}{2} = 84$. We can verify that $85^2 - 84^2 = 13^2$ which means $13^2 + 84^2 = 85^2$ and so $(13, 84, 85)$ is a Pythagorean triple involving the integer 13.

Using this same method, we can also obtain the following Pythagorean triples:

$$(15, 112, 113), (17, 144, 145), (19, 180, 181)$$

Problem 3:

(a) Consider the Pythagorean triple $(3, 4, 5)$. Show that if you multiply each integer in the triple by 2, then you obtain another Pythagorean triple.

Solution:

If we multiply each integer in the triple $(3, 4, 5)$ by 2, then we obtain the triple $(6, 8, 10)$. We can check that this triple is a Pythagorean triple as follows: $6^2 + 8^2 = 36 + 64 = 100 = 10^2$.

(b) Use the idea from (a) to build another Pythagorean triple that includes the odd integer 9.

Solution:

If we multiply each integer in the triple $(3, 4, 5)$ by 3, then we obtain the triple $(9, 12, 15)$. We can check that this triple is a Pythagorean triple as follows: $9^2 + 12^2 = 81 + 144 = 225 = 15^2$.

Note that we have now found two different Pythagorean triples that involve the odd number 9: $(9, 40, 41)$ *and* $(9, 12, 15)$.

(c) Show that for every positive integer n, the triple $(3n, 4n, 5n)$ is a Pythagorean triple.

It is also true that $(5n, 12n, 13n)$ *and* $(7n, 24n, 25n)$ *are Pythagorean triples.*

Solution:

First we note that for every positive n, the numbers $3n$, $4n$, and $5n$ are positive integers. Also, we have $(3n)^2 + (4n)^2 = 9n^2 + 16n^2 = 25n^2 = (5n)^2$. This means that the triple $(3n, 4n, 5n)$ is a Pythagorean triple.

Note: In a similar way, we can show that $(5n)^2 + (12n)^2 = 25n^2 + 144n^2 = 169n^2 = (13n)^2$ *and* $(7n)^2 + (24n)^2 = 49n^2 + 576n^2 = 625n^2 = (25n)^2$.

(d) Use the ideas from Problem 2 and Problem 3 to show that every integer from 4 to 20 is part of at least one Pythagorean triple.

Solution:

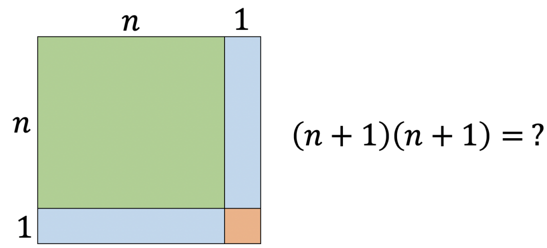
We provide at least one triple for each integer. Some of the triples given below have already been justified earlier. See if you can determine how the other triples were built using the ideas from Problem 2 and Problem 3. For example, the second triple given for 10 was obtained by multiplying each integer in the Pythagorean triple $(5, 12, 13)$ by 2 and using the idea from Problem 3(c).

4: $(3, 4, 5)$
5: $(3, 4, 5)$, $(5, 12, 13)$
6: $(6, 8, 10)$
7: $(7, 24, 25)$
8: $(6, 8, 10)$
9: $(9, 40, 41)$, $(9, 12, 15)$
10: $(6, 8, 10)$, $(10, 24, 26)$
11: $(11, 60, 61)$
12: $(5, 12, 13)$, $(9, 12, 15)$, $(12, 16, 20)$
13: $(13, 84, 85)$
14: $(14, 48, 50)$
15: $(9, 12, 15)$, $(15, 112, 113)$
16: $(12, 16, 20)$
17: $(17, 144, 145)$
18: $(18, 24, 30)$, $(18, 80, 82)$
19: $(19, 180, 181)$
20: $(12, 16, 20)$, $(20, 48, 52)$

Challenge Problem: Think about how you might use some of these ideas to show that every integer that is at least 3 is part of a Pythagorean triple. One possible approach is outlined below, but there are others:

(a) Odd numbers:

(i) Show that for every positive integer n, we have $(n+1)^2 - n^2 = 2n + 1$.



(ii) Use the identity from part (i) to explain why every odd integer that is at least 3 is part of a Pythagorean triple.

Solution:

(i) *Method 1:* Using the image provided, we see that the area of the largest square is represented by the quantity $(n+1)(n+1)$, the area of the medium square is represented by the quantity $(n)(n)$, the area of each of the two rectangles is represented by the quantity $(1)(n)$, and the area of the smallest square is represented by the quantity $(1)(1)$. Since the medium square, small square and two rectangular regions are used to form the larger square we must have

$$(n+1)(n+1) = (n)(n) + (1)(n) + (1)(n) + (1)(1)$$

This simplifies to $(n+1)^2 = n^2+2n+1$ which can be rearranged to give $(n+1)^2-n^2 = 2n+1$.
Method 2: Using the distributive property, we have $(n+1)(n+1) = (n+1)(n)+(n+1)(1)$ which means

$$(n+1)^2 = (n+1)(n+1) = (n+1)(n) + (n+1)(1) = n^2 + n + n + 1 = n^2 + 2n + 1$$

It follows that $(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$.
Method 3: If you have seen the formula for the difference of squares before, then you may see that

$$(n+1)^2 - n^2 = ((n+1) + n)((n+1) - n) = (2n+1)(1) = 2n + 1$$

(ii) We follow the method from Problem 2 for a general odd integer k that is *greater than* 1: Let $n = \frac{k^2-1}{2}$. Since k^2 must also be an odd integer that is greater than 1, $k^2 - 1$ must be an even integer that is greater than 0. This means n is a positive integer. Rearranging the equation gives $k^2 = 2n + 1$. Using the formula from (i), we get that

$$(n+1)^2 - n^2 = 2n + 1 = k^2$$

for this value of n. Rearranging the equation above gives

$$k^2 + n^2 = (n+1)^2$$

which shows that $(k, n, n+1)$ is a Pythagorean triple that includes the given odd integer k.

(b) Even numbers:

(i) Show that for every positive integer n, we have $(n+2)^2 - n^2 = 4n + 4$.

(ii) Use the identity from part (i) to explain why every even integer that is at least 4 is part of a Pythagorean triple.

Solution:

(i) Since $(n+2)^2 = (n+2)(n+2) = (n+2)(n) + (n+2)(2) = n^2 + 2n + 2n + 4 = n^2 + 4n + 4$ we have $(n+2)^2 - n^2 = (n^2 + 4n + 4) - n^2 = 4n + 4$.

(ii) To show this you can follow the method from part (a) of the challenge problem. We do not give a full solution here, but instead outline the steps using an example.

We can build a Pythagorean triple that includes the even integer 6 by following these steps:

- Determine n such that $6^2 = 4n + 4$:
  Solving we get $n = \frac{36-4}{4} = 8$.
- From part (i) above, we know that for this n we will have $(n+2)^2 - n^2 = 6^2$.
  We can verify this directly: $(8+2)^2 - 8^2 = 10^2 - 8^2 = 100 - 64 = 36 = 6^2$.
- This works shows that $(6, 8, 10)$ is a Pythagorean triple.

Suppose that k is an even integer that is *greater than* 2. If you can find find a positive integer n such that $k^2 = 4n + 4$, then the steps above show that $(k, n, n+2)$ is a Pythagorean triple. Can you see why there will always be such a value of n?

If $k > 2$ and is even then $k^2 > 4$ and is a multiple of 4, and so $k^2 - 4 > 0$ and is a multiple of 4. It follows that $n = \frac{k^2-4}{4}$ is a positive integer and satisfies $k^2 = 4k + 4$ as needed!

Parts (a) and (b) of the challenge problem show that every integer that is at least 3 is part of a Pythagorean triple. Can you explain why the integers 1 and 2 cannot be part of Pythagorean triples?

CEMC at Home

Grade 9/10 - Wednesday, June 10, 2020
Interplanetary Bases

You are part of an interplanetary mission to catalogue the number of elements present on two of Jupiter's moons. Six different astronauts have reported the number of elements they discovered, but to play a prank on you, they reported their findings using different *number systems* than you are used to. Making matters worse, they refuse to tell you exactly what number systems they have used.

Astronaut	**Total Number of Elements Discovered**	**Base used**	**Number of Elements Discovered on Moon 1**	**Number of Elements Discovered on Moon 2**
Afon	131			
Breanna	105			
Cheng	221			
Denisa	56			
Eka	105			
Fergus	221			

Each of the astronauts reported their numbers using a number system with some *base* b.

You normally use a *base 10* number system. In the base 10 system, you can use any of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8,$ or 9 to form the digits in your numbers. In this base, the numeral 763 represents the integer with value

$$7 \times 100 + 6 \times 10 + 3 \times 1 = 7 \times 10^2 + 6 \times 10^1 + 3 \times 10^0$$

Each astronaut used a number system with a positive integer b as the base, with $4 \leq b \leq 7$. In the base b system, they can use any of the digits $0, 1, \ldots, (b-1)$ to form the digits in their numbers. In this base, the numeral ABC (where A, B, and C are digits) represents the integer with value

$$A \times b^2 + B \times b + C \times 1 = A \times b^2 + B \times b^1 + C \times b^0$$

For example, in the system with base $b = 4$, the numeral 203 represents the integer with value

$$2 \times 4^2 + 0 \times 4 + 3 \times 1$$

but the numerals 124 and 552 have no meaning in this system since they contain digits larger than 3.

Problem 1: The astronauts gave you two clues about the overall report:

- Exactly two of the astronauts reported their numbers in base $b = 4$.
- All six astronauts discovered (and reported discovering) the same number of elements in total.

Determine how many elements were discovered on the two moons, in total (giving your answer in the usual base 10 system), and determine which base each of the astronauts was using to report their findings.

Problem 2: You are now told that each astronaut discovered 16 distinct elements on Moon 1 and discovered 3 elements common to both moons, with these numbers given in base 10. Using this information, fill in the blank cells in the table above. *Make sure to write the numbers in the right base!*

More Info:

Check out the CEMC at Home webpage on Wednesday, June 17 for a solution to Interplanetary Bases.

CEMC at Home

Grade 9/10 - Wednesday, June 10, 2020
Interplanetary Bases - Solution

Set-up: Six different astronauts have reported the number of elements they discovered on two moons, but reported their findings using different *number systems* than you are used to.

Astronaut	Total Number of Elements Discovered	Base used	Number of Elements Discovered on Moon 1	Number of Elements Discovered on Moon 2
Afon	131			
Breanna	105			
Cheng	221			
Denisa	56			
Eka	105			
Fergus	221			

Each astronaut used a number system with a positive integer b as the base, with $4 \leq b \leq 7$. In the base b system, they can use any of the digits $0, 1, \ldots, (b-1)$ to form the digits in their numbers. In this base, the numeral ABC (where A, B, and C are digits) represents the integer with value

$$A \times b^2 + B \times b + C \times 1 = A \times b^2 + B \times b^1 + C \times b^0$$

Problem 1: The astronauts gave you two clues about the overall report:

- Exactly two of the astronauts reported their numbers in base $b = 4$.
- All six astronauts discovered (and reported discovering) the same number of elements in total.

Determine how many elements were discovered on the two moons, in total, and determine which base each of the astronauts was using to report their findings.

Solution:

If two astronauts reported their findings using the same number system, then they will have reported identical numerals. The only numerals that appear more than once in the table above are 105 and 221. This means one of these two numerals must represent the total number of elements discovered written in base $b = 4$. Since a base 4 representation of a number can only use the digits 0, 1, 2, and 3, the numeral 105 cannot be a base 4 representation of any number. This means the total number of elements discovered is represented in base 4 as 221. In base 4, the numeral 221 represents the integer $2 \times 4^2 + 2 \times 4 + 1 = 32 + 8 + 1 = 41$.

We now know that Cheng and Fergus reported their findings in base 4, and that there were 41 elements discovered in total. There are several ways to approach the problem from here.

Consider the numeral 131. We know this is not in base 4 because the total in base 4 is represented by 221. The largest digit in 131 is 3, so this numeral makes sense in all three other bases: 5, 6, and 7. In base 5, it represents the integer $1 \times 5^2 + 3 \times 5^1 + 1 = 25 + 15 + 1 = 41$. Notice that in base 6, the numeral 131 would represent the integer $1 \times 6^2 + 3 \times 6 + 1 = 36 + 18 + 1 = 55$ and in base 7 it would represent $7^2 + 3 \times 7 + 1 = 71$. We conclude that the numeral 131 is in base 5 and that Afon used base 5.

This means the numeral 105 represents the total in either base 6 or base 7. In base 6, it represents the integer $1 \times 6^2 + 0 \times 6 + 5 = 36 + 5 = 41$. In base 7, it represents the integer $1 \times 7^2 + 0 \times 7 + 5 = 54$. We conclude that the numeral 105 is in base 6 and so Breanna and Eka used base 6.

Finally, we suspect that 56 is in base 7, and indeed, $5 \times 7 + 6 = 41$, so we conclude that Denisa reported in base 7.

We can also solve this problem using algebra once we have determined that the total number of elements is 41. For example, we know that the numeral 56 is the base b representation of 41 for some b between 4 and 7 inclusive. This means $41 = 5b + 6$, or $35 = 5b$, which can be solved for b to get $b = 7$. Similarly, to determine in which base the numeral 131 represents the number 41, we can solve the equation $41 = 1 \times b^2 + 3 \times b + 1$ for b. Rearranging, we get $40 = b^2 + 3b$. If you have experience factoring quadratics, you can solve this equation for b. If not, since you only have four possibilities for b, you can check $b = 4, 5, 6, 7$ and find that $b = 5$ is the only solution among these four choices. Finally, the numeral reported by Breanna leads to the equation $41 = b^2 + 5$ or $36 = b^2$. Since b is positive, this means $b = 6$.

Problem 2: You are now told that each astronaut discovered 16 distinct elements on Moon 1 and discovered 3 elements common to both moons, with these numbers given in base 10. Using this information, fill in the blank cells in the table above.

Solution:

The number of elements that were discovered on only Moon 1 is $16 - 3 = 13$, which means there must have been 28 elements discovered on Moon 2. Filling in the table means finding the base 4, 5, 6, and 7 representations of 16 and 28. These representations are given in the table below. To help explain how these numerals were obtained, we show the work for the first row below the table.

Astronaut	**Total Number of Elements Discovered**	**Base used**	**Number of Elements Discovered on Moon 1**	**Number of Elements Discovered on Moon 2**
Afon	131	5	31	103
Breanna	105	6	24	44
Cheng	221	4	34	130
Denisa	56	7	22	40
Eka	105	6	24	44
Fergus	221	4	34	130

To represent the integer 16 in base 5, we first note that its base 5 representation must have at most two digits. This is because the numeral ABC represents the integer with value $A \times 5^2 + B \times 5 + C$ which is at least 25 (assuming A is a positive digit). Thus, we seek digits A and B between 0 and 4 inclusive so that $A \times 5 + B = 16$. It is easy to check that digits $A = 3$ and $B = 1$ satisfy this equation, and in fact, it is true that no other pair of integers between 0 and 4 inclusive satisfies the equation. Therefore, the numeral 31 is the base 5 representation of the integer 16.

To represent the integer 28 in base 5, notice that 5^3 and hence all larger powers of 5 are greater than 28, so the base 5 representation of 28 has at most three digits. As well, the two-digit numeral with the largest value in base 5 is 44 which represents the integer $4 \times 5 + 4 = 24$, so this means the base 5 representation of 28 has exactly three digits. Therefore, we seek integers A, B, and C all between 0 and 4 inclusive satisfying $A \times 5^2 + B \times 5^1 + C \times 5^0 = 28$ or $25A + 5B + C = 28$. We know that $A \geq 1$ since the representation has three digits, but if $A \geq 2$, then $25A \geq 50 > 28$, so this means we must have $A = 1$. The equation then simplifies to $5B + C = 3$, and the only solution to this equation where B and C are integers between 0 and 4 inclusive is $B = 0$ and $C = 3$. Therefore, the base 5 representation of the integer 28 is 103.

There are systematic ways of expressing numbers in various bases. You may wish to do an internet search to learn more about this.

CEMC at Home

Grade 9/10 - Thursday, June 11, 2020
PRODUCTivity

The *"digit product"* of a positive integer is the product of the individual digits of the integer.

For example, the digit product of 234 is $2 \times 3 \times 4 = 24$. Other numbers also have a digit product of 24. For example, 2223, 113 181 and 38 each have a digit product of 24. The number 38 is the smallest positive integer with a digit product of 24.

There are many positive integers whose digit product is 2000.

Determine the smallest positive integer whose digit product is 2000.

More Info:

Check out the CEMC at Home webpage on Friday, June 12 for a solution to PRODUCTivity.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 9/10 - Thursday, June 11, 2020
PRODUCTivity - Solution

Problem:

The *"digit product"* of a positive integer is the product of the individual digits of the integer.

For example, the digit product of 234 is $2 \times 3 \times 4 = 24$. Other numbers also have a digit product of 24. For example, 2223, 113 181 and 38 each have a digit product of 24. The number 38 is the smallest positive integer with a digit product of 24.

There are many positive integers whose digit product is 2000.

Determine the smallest positive integer whose digit product is 2000.

Solution:

Let N be the smallest positive integer whose digit product is 2000.

In order to find N, we must find the minimum possible number of digits whose product is 2000. This is because if the integer a has more digits than the integer b, then $a > b$.
Once we have determined the digits that form N, then the integer N is formed by writing those digits in increasing order.

Note that the digits of N cannot include 0, or else the digit product of N would be 0.
Also, the digits of N cannot include 1, otherwise we could remove the 1 and obtain an integer with fewer digits (and thus, a smaller integer) with the same digit product. Therefore, the digits of N will be between 2 and 9, inclusive.

Since the digit product of N is 2000, we will use the prime factorization of 2000 to help determine the digits of N:

$$2000 = 2^4 \times 5^3$$

In order for a digit to have a factor of 5, the digit must equal 5. Therefore, three of the digits of N are 5.

The remaining digits of N must have a product of $2^4 = 16$. We need to find a combination of the smallest number of digits whose product is 16. We cannot have one digit whose product is 16, but we can have two digits whose product is 16. In particular, $16 = 2 \times 8$ and $16 = 4 \times 4$.

Therefore, N has 5 digits. They are 5, 5, 5, 2, 8 or 5, 5, 5, 4, 4. In order for N to be as small as possible, its digits must be in increasing order. The smallest positive integer formed by the digits 5, 5, 5, 2, 8 is 25 558. The smallest positive integer formed by the digits 5, 5, 5, 4, 4 is 44 555.

Since $25\ 558 < 44\ 555$, the smallest N is 25 558. That is, the smallest positive integer with a digit product of 2000 is 25 558.

CEMC at Home

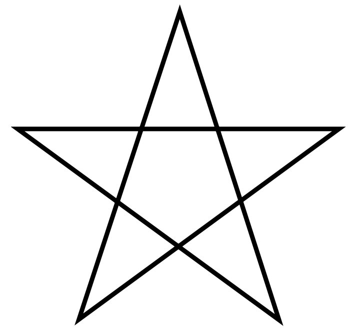
Grade 9/10 - Friday, June 12, 2020
Pieces of a Star

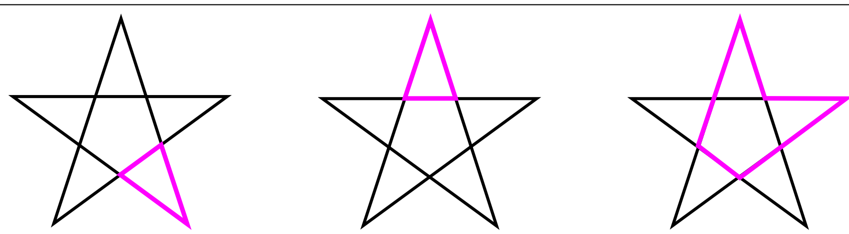
A *polygon* is a two-dimensional closed figure formed by line segments. A *simple polygon* is a polygon that does not "intersect itself". We can think about tracing out the boundary of a simple polygon: we only travel in straight lines and the only time we return to a point for a second time is when we have finished tracing out the entire boundary.

Problem:

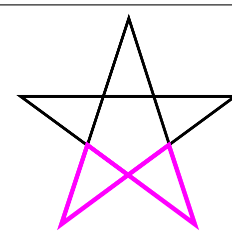
There are 37 simple polygons to be found in the 5-pointed star shown on the right. Describe the 37 simple polygons and explain why there are no more.

The sides of your polygons should all be line segments appearing in the star. To get you started, below are three examples of simple polygons that can be found in the star, along with a polygon that is not simple.





Three simple polygons
(allowed)



Non-simple polygon
(not allowed)

Hints for counting polygons:

1. What simple polygons can you find that are not congruent to any of the examples shown above?
2. How many simple polygons do not include the interior of the pentagon?
3. The number of simple polygons that include the interior of the pentagon is a power of 2.

Follow-up Discussion:

Were you able to find a systematic way of counting the simple polygons in the 5-pointed star? Could this same strategy be used to count the simple polygons in a 6-pointed star or a 7-pointed star?

In general, an n-pointed star, for an integer $n \geq 5$, consists of an n-gon in the centre, with n triangles attached to its sides pointing outwards. Think about the following:

Follow-up Question: How many simple polygons are there in an n-pointed star?

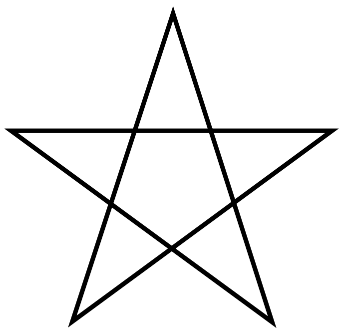
More Info:

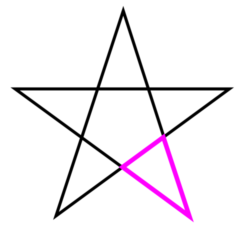
Check out the CEMC at Home webpage on Friday, June 19 for a solution to Pieces of a Star.

CEMC at Home

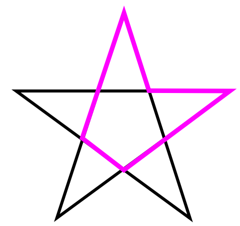
Grade 9/10 - Friday, June 12, 2020
Pieces of a Star - Solution

Problem: There are 37 simple polygons to be found in the 5-pointed star shown. Describe the 37 simple polygons and explain why there are no more.





Example 1



Example 2

Solution: First, we note that there are 5 simple polygons in the star that do not include the pentagon in the centre: the 5 triangles that form the 5 points of the star. One such triangle is shown in Example 1 above.

Every other simple polygon in the star will consist of the pentagon in the centre plus some of the 5 triangles which form the points of the star. One of these polygons is formed by choosing *none* of the triangles. In this case we get the pentagon in the centre. Another of these polygons is formed by choosing *all* of the triangles. In this case we get the entire 5-pointed star (or its boundary). Another of these polygons is formed by choosing the two triangles at the top right. In this case we get the pentagon shown in Example 2.

We could try to systematically draw all of these polygons, but we can count them without doing so. Each such polygon either includes a particular triangle or it does not. This means when drawing one of these polygons, we have two choices for each triangle: in or out. This leads to $2^5 = 32$ different possibilities for which triangles we include, and hence there must be 32 simple polygons that include the pentagon in the centre.

Putting this all together, there are 5 simple polygons that do not include the pentagon (the 5 outer triangles) and 32 simple polygons that include the pentagon. Therefore, there are $5 + 32 = 37$ simple polygons in the star, in total.

Follow-up Question: How many simple polygons are there in an n-pointed star?

Solution: We can think of an n-pointed star as an n-gon with n triangles around it. The n triangles form the n points of the star.

Some of the simple polygons in the star are the n triangles that form the n points of the star. Every other simple polygon will consist of the n-gon in the centre plus some of the n triangles that form the points of the star. Since there are n triangles, and for each triangle we have 2 choices (include or not include), there are 2^n such polygons.

Therefore, in total, we have $n + 2^n$ simple polygons in the n-pointed star.

In particular, when $n = 6$ we have $6 + 2^6 = 70$ and so there are 70 simple polygons in the 6-pointed star, and when $n = 7$ we have $7 + 2^7 = 135$ and so there are 135 simple polygons in the 7-pointed star.

CEMC at Home

Grade 9/10 - Monday, June 15, 2020
History of Computing

Computers can be found on our desks, in our pockets and even in our refrigerators! This is remarkable because modern computers have been around for less than 100 years. During this time, there has been a constant stream of new discoveries and advances in technology.

Use this online tool to arrange the following list of events in the history of computer science from earliest to most recent.

A. Deep Blue is the first computer program to beat a human world chess champion.

B. The Harvard Mark I mechanical computer is built and is used for military purposes during World War II.

C. Sun Microsystems develops the Java programming language.

D. The ASCII is developed to create standard binary codes for 128 different characters.

E. Computers are used to determine that a perfect winning strategy does not exist for the game of checkers.

F. The first email is sent. It is sent from Ray Tomlinson to Ray Tomlinson.

G. Konrad Zuse designs the Z3 electromechanical computer which is considered the first automatic programmable computer.

H. The Altair 8800 is the first personal computer to sell in large numbers.

I. A robot named Elektro is built which responds to voice commands.

J. Guido van Rossum creates and releases the Python programming language.

K. Doug Engelbart invents the computer mouse.

L. Animators create Cindy, the first human-like CGI (computer generated imagery) movie character.

More Info:

Our webpage Computer Science and Learning to Program is the best place to find the CEMC's computer science resources.

CEMC at Home

Grade 9/10 - Tuesday, June 16, 2020
Can You Find the Terms?

Can you find all of the given mathematics and computer science terms in the grid? Good Luck!

U	A	L	O	B	A	R	A	P	B	Y	F	N	X	R
S	L	O	P	E	A	J	S	Y	N	T	A	X	X	F
R	C	V	E	W	F	U	N	C	T	I	O	N	A	M
R	O	N	C	W	I	N	A	E	L	O	O	B	H	E
O	N	L	O	O	P	T	F	V	V	Q	V	T	K	V
L	D	O	T	P	C	A	N	H	E	C	I	D	E	U
U	I	A	Z	G	O	V	F	I	R	R	K	J	F	J
I	T	O	U	K	O	L	K	N	O	O	T	A	V	B
Q	I	S	V	H	U	T	Y	G	T	P	C	E	X	T
U	O	R	G	H	L	A	L	N	O	T	D	I	X	F
X	N	Y	D	L	X	A	R	T	O	K	H	I	P	L
D	A	K	T	P	Q	N	K	R	G	M	E	D	M	D
K	L	I	H	T	K	U	I	C	A	N	I	T	V	I
E	X	P	O	N	E	N	T	A	Q	Y	Y	A	Z	O
K	T	E	V	S	G	N	I	T	S	E	T	T	L	S

EXPONENT	MIDPOINT	SYNTAX
POLYNOMIAL	VERTEX	ALGORITHM
SLOPE	ARRAY	CONDITIONAL
PARABOLA	LOOP	FUNCTION
FACTORING	BOOLEAN	TESTING

More Info:

Check the CEMC at Home webpage on Wednesday, June 17 for the solution to Can You Find the Terms?

CEMC at Home

Grade 9/10 - Tuesday, June 16, 2020
Can You Find the Terms? - Solution

U	A	L	O	B	A	R	A	P	B	Y	F	N	X	R
S	L	O	P	E	A	J	S	Y	N	T	A	X	X	F
R	C	V	E	W	F	U	N	C	T	I	O	N	A	M
R	O	N	C	W	I	N	A	E	L	O	O	B	H	E
O	N	L	O	O	P	T	F	V	V	Q	V	T	K	V
L	D	O	T	P	C	A	N	H	E	C	I	D	E	U
U	I	A	Z	G	O	V	F	I	R	R	K	J	F	J
I	T	O	U	K	O	L	K	N	O	O	T	A	V	B
Q	I	S	V	H	U	T	Y	G	T	P	C	E	X	T
U	O	R	G	H	L	A	L	N	O	T	D	I	X	F
X	N	Y	D	L	X	A	R	T	O	K	H	I	P	L
D	A	K	T	P	Q	N	K	R	G	M	E	D	M	D
K	L	I	H	T	K	U	I	C	A	N	I	T	V	I
E	X	P	O	N	E	N	T	A	Q	Y	Y	A	Z	O
K	T	E	V	S	G	N	I	T	S	E	T	T	L	S

EXPONENT	MIDPOINT	SYNTAX
POLYNOMIAL	VERTEX	ALGORITHM
SLOPE	ARRAY	CONDITIONAL
PARABOLA	LOOP	FUNCTION
FACTORING	BOOLEAN	TESTING

CEMC at Home

Grade 9/10 - Monday, June 15, 2020
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CEMC at Home

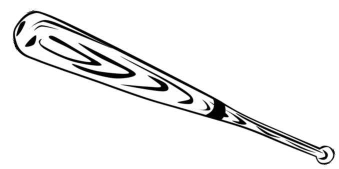
Grade 9/10 - Wednesday, June 17, 2020
The Standings

In a softball league with four teams, each team has played every other team 4 times.

Each team earned 3 points for a win, 1 point for a tie and no points for a loss.

The total accumulated points were:

Lions	22
Tigers	19
Mounties	14
Royals	12



How many games ended in a win and how many games ended in a tie?

More Info:

Check out the CEMC at Home webpage on Thursday, June 18 for a solution to The Standings.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 9/10 - Wednesday, June 17, 2020
The Standings - Solution

Problem:

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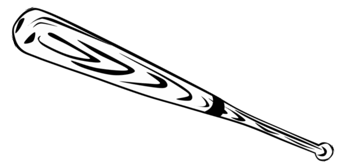


How many games ended in a win and how many games ended in a tie?

Solution:

We begin by calculating the total number of games played. Since each team played every other team 4 times, each team played $3 \times 4 = 12$ games. Since there are four teams, a total of $\frac{4\times 12}{2} = 24$ games were played. We divide by 2 since each game is counted twice. For example, the Lions playing the Tigers is the same as the Tigers playing the Lions.

In games where one team won and one team lost, one team earned 3 points and the other 0 points, so a total of 3 points were awarded. In games that resulted in a tie, both teams earned 1 point, so a total of 2 points were awarded.

If there were 0 ties, then 24 games would result in $24 \times 3 = 72$ points being awarded. However, $22 + 19 + 14 + 12 = 67$ points were actually awarded in all of the games. Since a total of 3 points were awarded when there was a win and a total of 2 points were awarded when there was a tie, every point below 72 must represent a tie. Since $72 - 67 = 5$, there must have been 5 ties. Since 24 games were played, $24 - 5 = 19$ games resulted in a win.

Therefore, there were 19 games that ended in a win and 5 games ended in a tie.

We should check that there is a combination of wins, ties and losses that satisfies the conditions in the problem. Indeed, one possibility is:

Team Name	Wins	Ties	Losses	Total Points
Lions	7	1	4	22
Tigers	6	1	5	19
Mounties	3	5	4	14
Royals	3	3	6	12
TOTALS	19	10	19	67

Notice that in the chart there are a total of 10 ties. That means that 5 games ended in a tie and a total of 10 points were awarded for ties.

CEMC at Home

Grade 9/10 - Thursday, June 18, 2020
Games and Puzzles

The CEMC has created lots of resources that we hope you have found interesting over the last few months. We also know that there are lots of online games and puzzles created by other organizations that make use of mathematics and logic. We've highlighted three examples below that you can explore for more mathematical fun!

Fraction Game from NCTM (https://www.nctm.org)

To make moves in this game, you need to use logic and number sense involving fractions.

The Remainders Game from NRICH (https://nrich.maths.org)

Use your knowledge of remainders to figure out a mystery number.

Slitherlink Puzzles by Krazydad (https://krazydad.com)

In a Slitherlink Puzzle, you connect horizontally or vertically adjacent dots to form a meandering path that forms a single loop, without crossing itself, or branching.

You can find other interesting mathematics related games and puzzles online. Share your favourites using any forum you are comfortable with.

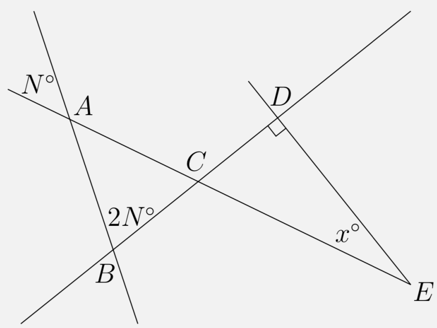
CEMC at Home
Grade 9/10 - Friday, June 19, 2020
Relay Day - Part 1

As part of the CEMC's Canadian Team Mathematics Contest, students participate in Math Relays. Just like a relay in track, you "pass the baton" from teammate to teammate in order to finish the race, but in the case of a Math Relay, the "baton" you pass is actually a number!

Read the following set of problems carefully.

Problem 1: Let A be the number of multiples of 5 between 1 to 2020 inclusive and B be the number of multiples of 20 between 1 and 2020 inclusive. What is the value of $10A \div B$?

Problem 2: *Replace N below with the number you receive.*
Four line segments intersect in points A, B, C, D, and E, as shown. The measure of $\angle CED$ is x°. What is the value of x?



Problem 3: *Replace N below with the number you receive.*
Armen paid \$190 to buy movie tickets for a group of N people, consisting of some adults and some children. Movie tickets cost \$5 for children and \$9 for adults. How many children's tickets did he buy?

Notice that you can answer Problem 1 without any additional information.

In order to answer Problem 2, you first need to know the mystery value of N. The value of N used in Problem 2 will be the *answer* to Problem 1. (For example, if the answer you got for Problem 1 was 5 then you would start Problem 2 by replacing N with 5 in the problem statement.)

Similarly, you need the answer to Problem 2 to answer Problem 3. The value of N in Problem 3 is the *answer* that you got in Problem 2.

Now try the relay! You can use this tool to check your answers.

Follow-up Activity: Can you come up with your own Math Relay?
What do you have to think about when making up the three problems in the relay?

In Part 1 of this resource, you were asked to complete a relay on your own. But, of course, relays are meant to be completed in teams! In a team relay, three different people are in charge of answering the problems. Player 1 answers Problem 1 and passes their answer to Player 2; Player 2 takes Player 1's answer and uses it to answer Problem 2; Player 2 passes their answer to Player 3; and so on.

In Part 2 of this resource, you will find instructions on how to run a relay game for your friends and family. We will provide a relay for you to use, but you can also come up with your own!

CEMC at Home

Grade 4 to 12 - Friday, June 19, 2020
Relay Day - Part 2

Relay for Family and Friends

In Part 1 of this resource, you learned how to do a Math Relay. Now, why not try one out with family and friends!

You can put together a relay team and

- play just for fun, not racing any other team, or
- compete against another team in your household (if you have at least 6 people in total), or
- compete with a team from another family or household by
  - timing your team and comparing times with other teams to declare a winner, or
  - competing live using a video chat.

Here are the instructions for how to play.

Relay Instructions:

1. Decide on a team of three players for the relay. The team will be competing together.
2. Find someone to help administer the relay; let's call them the "referee".
3. Each teammate will be assigned a number: 1, 2, or 3. Player 1 will be assigned Problem 1, Player 2 will be assigned Problem 2, and Player 3 will be assigned Problem 3.
4. The three teammates should not see any of the relay problems in advance and should not talk to each other during the relay.
5. Right before the relay starts, the referee should hand out the correct relay problem to each of the players, with the problem statement face down (not visible).
6. The referee will then start the relay. At this time *all three players* can start working on their problems.

   Think about what Player 2 and Player 3 can do before they receive the value of N (the answer from the previous question passed to them by their teammate).
7. When Player 1 thinks they have the correct answer to Problem 1, they record their answer on the answer sheet and pass the sheet to Player 2. When Player 2 thinks they have the correct answer to Problem 2, they record their answer to the answer sheet and pass the sheet to Player 3. When Player 3 thinks they have the correct answer to Problem 3, they record their answer on the answer sheet and pass the sheet to the referee.

8. If all three answers passed to the referee are correct, then the relay is complete! If at least one answer is incorrect, then the referee passes the sheet back to Player 3.

9. At any time during the relay, the players on the team can pass the answer sheet back and forth between them, as long as they write nothing but their current answers on it and do not discuss anything. (For example, if Player 2 is sure that Player 1's answer must be incorrect, then Player 2 can pass the answer sheet back to Player 1, silently. This is a cue for Player 1 to check their work and try again.)

See the next page for a relay for family and friends! This includes instructions for the referee. You can also come up with your own relays to play. You can find many more relays from past CTMC contests on the CEMC's Past Contests webpage.

Sample answer sheets are provided below for you to use for your relays if you wish.

Answer Sheets:

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Problem 1 Answer	
Problem 2 Answer	
Problem 3 Answer	

Relay For Three

Instructions for the Referee:

1. Multiple questions at different levels of difficulty are given for the different relay positions.
   - Assign one of the first three problems (marked "Problem 1") to Player 1.
   - Assign one of the next three problems (marked "Problem 2") to Player 2.
   - Assign one of the last three problems (marked "Problem 3") to Player 3.

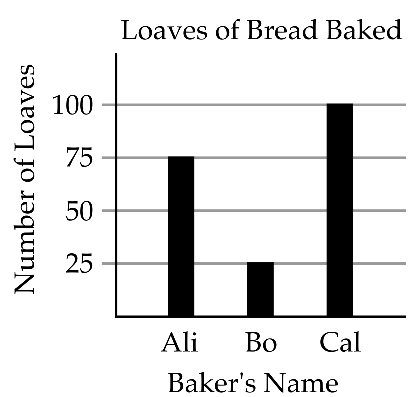
   Choose a problem so that each player is comfortable with the level of their question. The level of difficulty of each question is represented using the following symbols:

   - (circle) These questions should be accessible to most students in grade 4 or higher.
   - (square) These questions should be accessible to most students in grade 7 or higher.
   - (diamond) These questions should be accessible to most students in grade 9 or higher.
2. Use this tool to find the answers for the relay problems in advance.

Relay Problems (to cut out):

Problem 1 (circle)

The graph shows the number of loaves of bread that three friends baked. How many loaves did Bo bake?

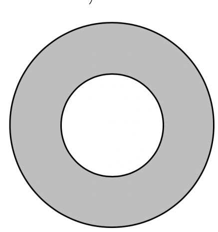


Problem 1 (square)

An equilateral triangle has sides of length $x+4$, $y+11$, and 20. What is the value of $x+y$?

Problem 1 (diamond)

In the figure shown, two circles are drawn. If the radius of the larger circle is 10 and the area of the shaded region (in between the two circles) is 75π, then what is the *square* of the radius of the smaller circle?



Problem 2 ●

Replace N below with the number you receive.
Kwame writes the whole numbers in order from 1 to N (including 1 and N). How many times does Kwame write the digit '2'?

Problem 2 ■

Replace N below with the number you receive.
The total mass of three dogs is 43 kilograms. The largest dog has a mass of N kilograms, and the other two dogs have the same mass. What is the mass of each of the smaller dogs?

Problem 2 ◆

Replace N below with the number you receive.
The points $(6, 16)$, $(8, 22)$, and (x, N) lie on a straight line. Find the value of x.

Problem 3 ●

Replace N below with the number you receive.
You have some boxes of the same size and shape. If N oranges can fit in one box, how many oranges can fit in two boxes, in total?

Problem 3 ■

Replace N below with the number you receive.
One morning, a small farm sold 10 baskets of tomatoes, 2 baskets of peppers, and N baskets of zucchini. If the prices are as shown below, how much money, in dollars did the farm earn in total from these sales?

Basket of Tomatoes:	\$0.50
Basket of Peppers:	\$2.00
Basket of Zucchini:	\$1.00

Problem 3 ◆

Replace N with the number you receive.
Elise has N boxes, each containing x apples. She gives 12 apples to her sister. She then gives 20% of her remaining apples to her brother. After this, she has 120 apples left. What is the value of x?