Problem

Bart and Lisa got tired of playing checkers because Lisa always won. They remembered one day in math class they counted the number of squares of all sizes on a checkerboard. They counted 64 squares 1 unit by 1 unit, and 49 overlapping squares 2 units by 2 units. They continued until they found the total of 204 squares.

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1 x 1 squares

They decided instead, to count the number of rectangles 1 unit by 2 units, such as those shown below. They were careful to count all the overlapping rectangles. If they counted correctly, what was their total?

Here is a checkerboard to work with.

*Extension:*

Draw diagrams to illustrate how Bart and Lisa counted the total number of squares of all sizes on the 8 × 8 checkerboard. Is their total correct?
Hints

**Hint 1** - Make a diagram for the case of a $3 \times 3$ square. How many $1 \times 2$ rectangles are there? How did you count them? Try a $4 \times 4$ square the same way.

**Hint 2** - Is a $1 \times 2$ rectangle the same as a $2 \times 1$ rectangle (i.e., could the rectangle stand “on end”?)

*Suggestion:* If the class hasn’t seen the problem of counting squares of all sizes on a checkerboard, you may wish to challenge them to try the Extension. Encourage students to look for patterns.
Solution

Each column contains 7 overlapping 1 unit × 2 units vertical rectangles, and similarly for each row there are 7 overlapping horizontal 2 units × 1 unit rectangles. (Think of shifting the dotted rectangles to the left to see the 7 overlapping vertical rectangles, or up to see the overlapping horizontal rectangles.)

There are 8 rows and 8 columns on the board, so there are 56 horizontal, and 56 vertical rectangles, in total. Thus there are 112 such rectangles altogether.

Extension:

a) Consider a 3 × 3 square as follows:

nine 1x1 squares

four 2x2 squares

one 3x3 square

Thus the 3 × 3 square contains 1 + 4 + 9 = 14 squares in total.

In a 4 × 4 square, there are sixteen 1 × 1 squares.

There are 3 rows of three 2x2 squares = nine 2x2 squares; (top row only is shown)

four 3x3 squares. (top 2 are shown)

Thus, including the one 4 × 4 square, there are 1 + 4 + 9 + 16 = 30 squares in total in the 4 × 4 square.

b) Examining the pattern in the 3 × 3 and 4 × 4 squares in a), we see that there will be one 8 × 8 square, and four 7 × 7 squares.

Picture them as:
Then there will be nine $6 \times 6$ squares, three of which are shown below.

Note that the pattern is the sum of the squares of all the whole numbers up to $n^2$, where $n$ is the length of the side of the $n \times n$ square.

In conclusion, the total number of squares for all sizes in an $8 \times 8$ checkerboard is $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = 204$ squares. So Bart and Lisa’s count was correct.