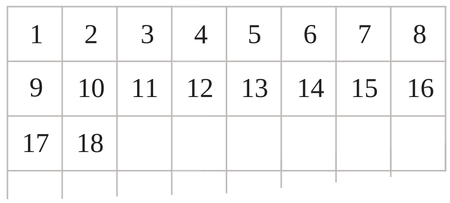
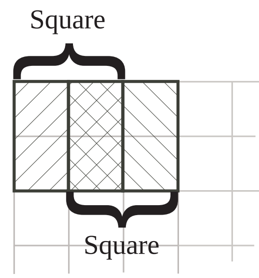
Problem

Bart and Lisa got tired of playing checkers because Lisa always won. They remembered one day in math class they counted the number of squares of all sizes on a checkerboard. They counted 64 squares 1 unit by 1 unit, and 49 overlapping squares 2 units by 2 units. They continued until they found the total of 204 squares.

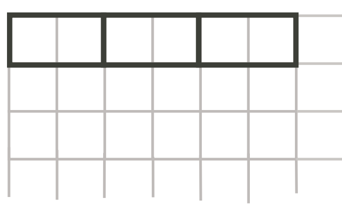


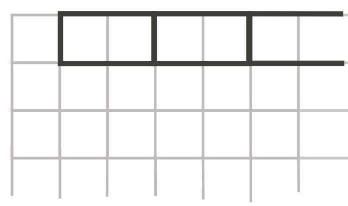
1 x 1 squares



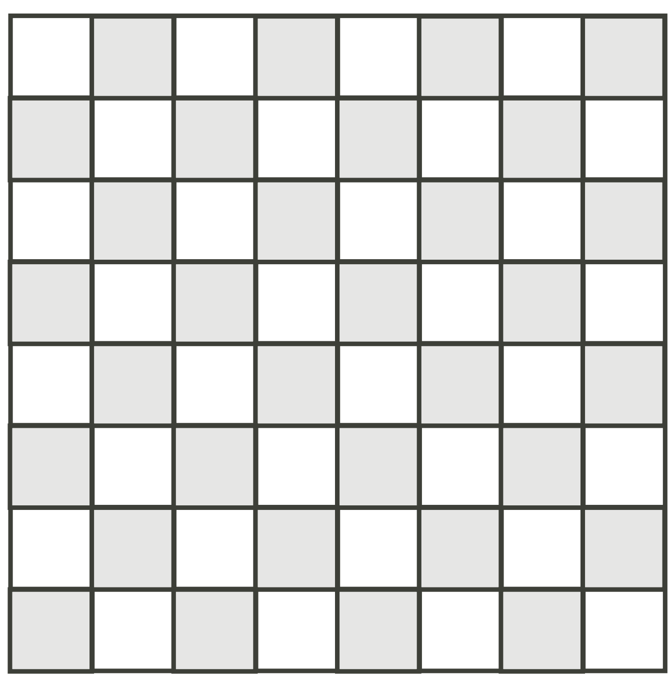
2 x 2 squares

They decided instead, to count the number of rectangles 1 unit by 2 units, such as those shown below. They were careful to count all the overlapping rectangles. If they counted correctly, what was their total?





Here is a checkerboard to work with.



Extension:
Draw diagrams to illustrate how Bart and Lisa counted the total number of squares of all sizes on the 8×8 checkerboard. Is their total correct?

Hints

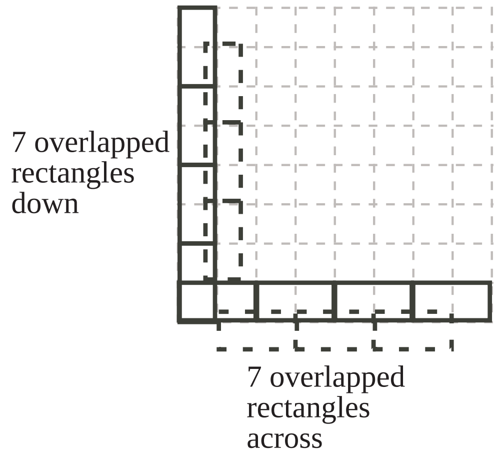
Hint 1 - Make a diagram for the case of a 3×3 square. How many 1×2 rectangles are there? How did you count them? Try a 4×4 square the same way.

Hint 2 - Is a 1×2 rectangle the same as a 2×1 rectangle (i.e., could the rectangle stand "on end"?)

Suggestion: If the class hasn't seen the problem of counting squares of all sizes on a checkerboard, you may wish to challenge them to try the Extension. Encourage students to look for patterns.

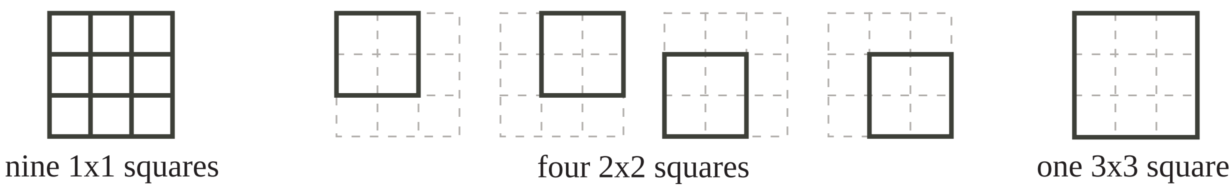
Solution

Each column contains 7 overlapping 1 unit $\times$ 2 units vertical rectangles, and similarly for each row there are 7 overlapping horizontal 2 units $\times$ 1 unit rectangles. (Think of shifting the dotted rectangles to the left to see the 7 overlapping vertical rectangles, or up to see the overlapping horizontal rectangles.)
There are 8 rows and 8 columns on the board, so there are 56 horizontal, and 56 vertical rectangles, in total. Thus there are 112 such rectangles altogether.



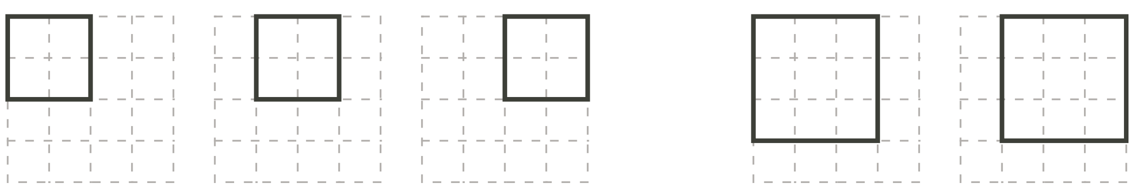
Extension:

a) Consider a 3×3 square as follows:



Thus the 3×3 square contains $1 + 4 + 9 = 14$ squares in total.

In a 4×4 square, there are sixteen 1×1 squares.



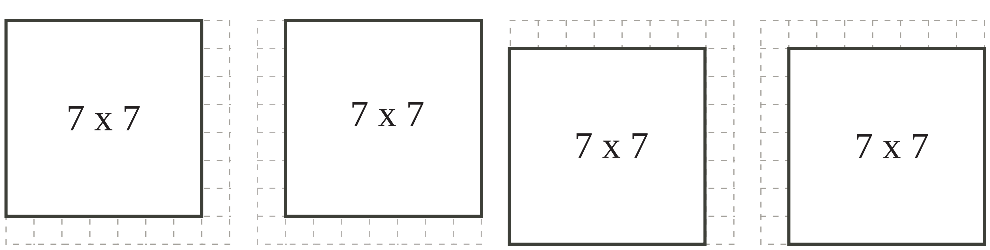
There are 3 rows of three 2x2 squares = nine 2x2 squares; (top row only is shown)

four 3x3 squares. (top 2 are shown)

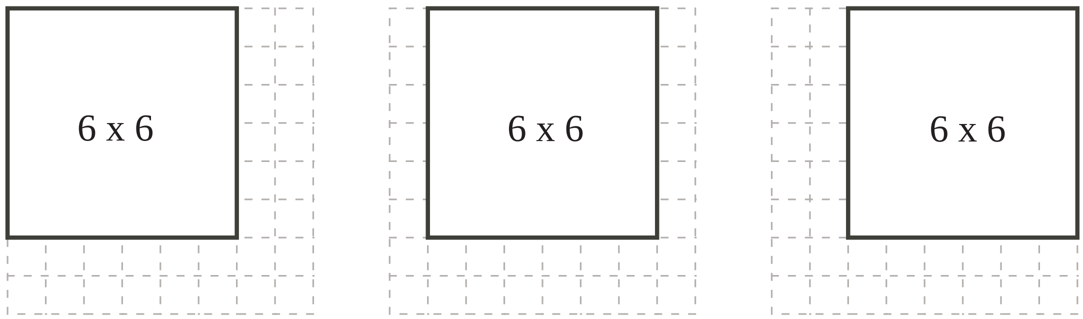
Thus, including the one 4×4 square, there are $1 + 4 + 9 + 16 = 30$ squares in total in the 4×4 square.

b) Examining the pattern in the 3×3 and 4×4 squares in a), we see that there will be one 8×8 square, and four 7×7 squares.

Picture them as:



Then there will be nine 6×6 squares, three of which are shown below.



Note that the pattern is the sum of the squares of all the whole numbers up to n^2, where n is the length of the side of the $n \times n$ square.

In conclusion, the total number of squares for all sizes in an 8×8 checkerboard is $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 = 204$ squares. So Bart and Lisa's count was correct.