Problem of the Month
Problem 6: March 2024

Hint

(a) The *Rational Root Theorem* could come in handy: If a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients and r is a rational root of $p(x)$, then it must be of the form $r = \frac{u}{v}$ where u and v are integers, u is a divisor of a_0, and v is a divisor of a_n.

For the polynomial in (a), this means the only possible rational roots are ± 1, ± 2, ± 4, and ± 8, the divisors of 8 (the leading coefficient is 1).

(b) This polynomial is a perfect square.

(c) This polynomial has no rational roots, but it does have a real root that is easy to find.

(d) A polynomial has a rational root if and only if it has a rational linear factor. If $p(x)$ and $q(x)$ are polynomials of degree m and n, then what is the degree of $p(x)q(x)$?

(e) To show that a number is algebraic, you need to find a rational polynomial with that number as a root. For $1 + \sqrt[3]{2}$, let $r = 1 + \sqrt[3]{2}$ so that $r - 1 = \sqrt[3]{2}$. Now cube both sides.

(f) If $p(x)$ has degree d and factors as the product of two polynomials of degree at least 1, then both of these polynomials have degree less than d. Read the definition of "degree" carefully.

For uniqueness, suppose that two polynomials, $p(x)$ and $q(x)$, have the described properties. What can be said about the degree of their difference?

(h) (i) Warm up by trying this with a polynomial of lower degree. It turns out that the polynomial $f_1(x)$ is the *derivative* of $f(x)$. This is not important for the problem, but it is interesting.

(ii) If you know some calculus, then there is a nice proof of this involving the product rule. Otherwise, if $f(x) = (x - r)^2 p(x)$, then $f(x + y) = [(x - r) + y]^2 p(x + y)$. Expand $p(x + y)$ and $f(x + y)$ as described in part (i) and compare "coefficients" of y.

(j) By definition, the shared root is algebraic and so has a minimal polynomial, $m(x)$. Show that each of $p(x)$ and $q(x)$ is a scalar multiple of $m(x)$.

Division Algorithm for Polynomials: For polynomials $f(x)$ and $g(x)$ with $g(x)$ not the zero polynomial, there exist unique polynomials $h(x)$ and $r(x)$ such that

$$f(x) = h(x)g(x) + r(x)$$

where the degree of $r(x)$ is less than the degree of $g(x)$. This is essentially the result of doing polynomial or synthetic division of $f(x)$ by $g(x)$.

(k) Convince yourself that every polynomial can be expressed as a product of irreducible polynomials. As well, as long as $f(x)$ has degree at least 1, the polynomial $f_1(x)$ is not the zero polynomial and has degree less than that of $f(x)$ (in fact, the degree is one less).