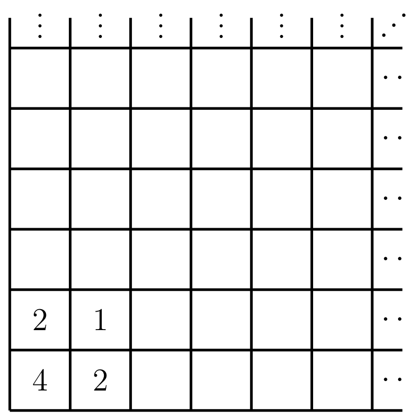
Problem of the Month

Problem 0: September 2020

A rectangular array extends up and to the right with infinitely many rows and infinitely many columns. Integers are placed in the four "bottom-left" cells as shown with 4 in the bottom-left corner, 2 in each of the cells sharing a side with the cell containing 4, and 1 in the cell immediately to the right and above the 4.



When referring to rows and columns, we start the enumeration from the bottom and the left. For example, the "third row" refers to the third row from the bottom.

Integers are placed in the remaining cells *recursively* as follows:

- In the first and second rows, each remaining cell contains the sum of the integer in the cell immediately to its left and twice the integer two cells to its left. For example, the third cell in the first row contains the integer $2 + 2(4) = 10$.
- Cells in or above the third row contain the sum of the integer in the cell immediately below and twice the integer in the cell two below. For example, the second cell in the third row contains the integer $1 + 2(2) = 5$.

We will denote by $f(m, n)$ the integer in the m^{th} row and the n^{th} column.

(a) Show that every cell other than those in the first two columns contains the sum of the integer in the cell immediately to its left and twice the integer in the cell two to its left. That is, show that $f(m, n) = f(m, n-1) + 2f(m, n-2)$ for all integers $m \geq 1$ and $n \geq 3$.

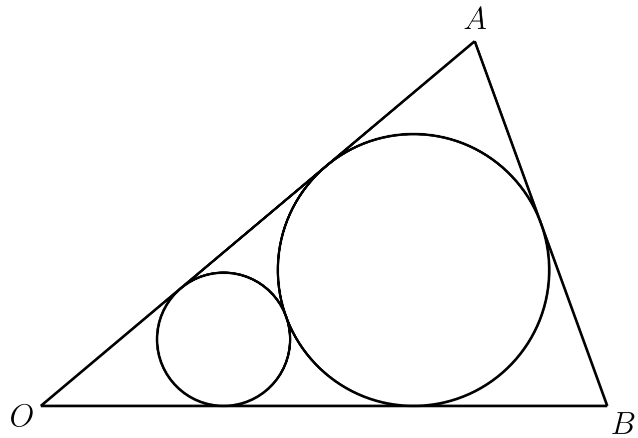
(b) Prove that $f(m, m)$ is a perfect square for every integer $m \geq 1$. In other words, prove that all of the cells on the diagonal contain perfect squares.

(c) Determine the value of $f(456, 789)$.

Problem of the Month

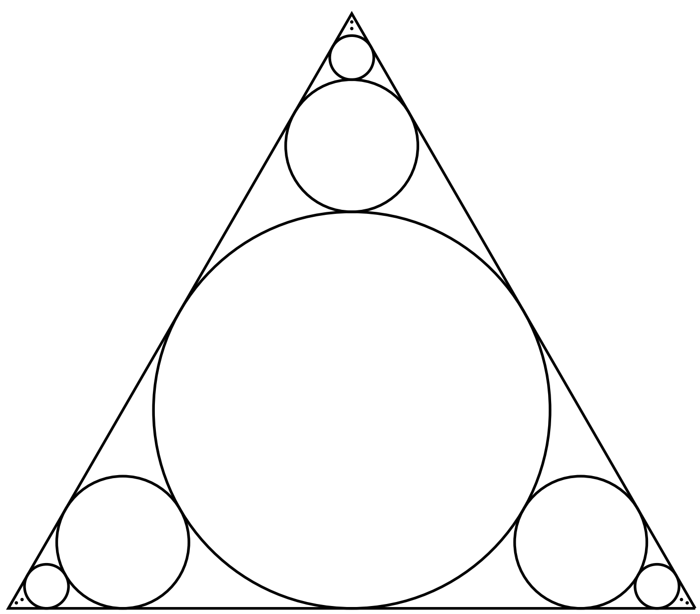
Problem 1: October 2020

(a) Let θ be an angle with $0 < \theta < 45°$. In the diagram, points A and B are configured so that $\angle AOB = 2\theta$ and $\triangle AOB$ is isosceles with $AO = BO$.



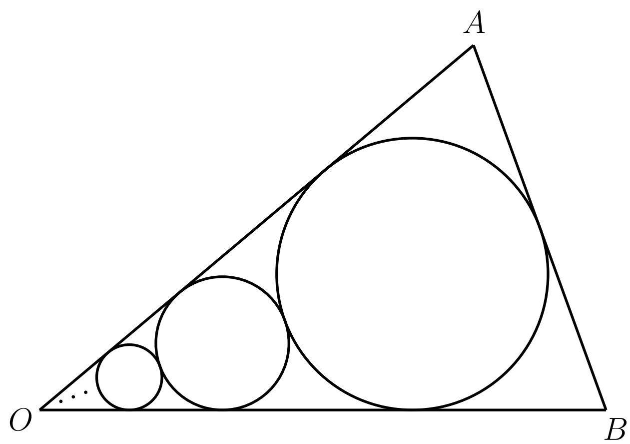
A circle is inscribed in $\triangle AOB$ and another circle is drawn so that it is tangent to the larger circle as well as OA and OB. In terms of θ, find the ratio of the radius of the larger circle to the radius of the smaller circle.

(b) Similar to part (a), an equilateral triangle has a circle inscribed in it. Three circles are then drawn, each tangent to two of the sides of the triangle as well as the larger circle. Another three circles are then drawn, each tangent to two of the three sides of the triangle as well as one of the circles drawn in the previous step.



If this process is continued indefinitely, what fraction of the area of the triangle is covered by circles?

(c) Suppose $\triangle AOB$ and θ are as they were defined in part (a). The process of drawing a circle tangent to OA, OB, and the smallest circle is repeated forever. What fraction of the area of $\triangle AOB$ is covered by circles? Your answer should be in terms of θ.



The result of part (c) can be applied to solve part (b). Can you see how?

Problem of the Month
Problem 2: November 2020

Suppose n is a positive integer and that you have n pairs of socks. Within each pair of socks, the two socks are the same colour. Every pair has a unique colour. After doing laundry, all of the $2n$ socks are in a laundry basket. You begin to remove socks one at a time (always choosing randomly and never replacing the socks) until you find a pair. That is, you remove socks until you remove a sock that matches a sock that has already been removed.

For positive integers n and k with $k < 2n$, we define $P(n, k)$ to be the probability that the first k socks removed are all different colours and there is a pair among the first $k + 1$ socks that are removed. That is, $P(n, k)$ is the probability that the first pair is found upon removing the $(k+1)^{\text{st}}$ sock.

(a) Compute $P(4, k)$ for each k from 1 through 7. Some of these probabilities should equal 0.

(b) Find a general formula for $P(n, k)$ when $k \leq n$. It might be useful later if you can find a formula that only uses addition, subtraction, multiplication, division, exponentiation, as well as factorials and binomial coefficients. Notice that the question does not ask about $P(n, k)$ for $k > n$. You might want to think about what happens in this case.

(c) For a positive integer i, define T_i to be the sum of the first i positive integers. For example, $T_1 = 1$, $T_2 = 1 + 2 = 3$, and $T_3 = 1 + 2 + 3 = 6$. The numbers T_1, T_2, T_3, T_4, and so on, are often called the *triangular numbers*. You may already know that $T_i = \dfrac{i(i+1)}{2}$ for every positive integer i. If not, think about why!

   (i) Suppose $n = T_i$ for some $i > 1$. Show that the largest value of $P(n, k)$ is achieved when $k = i$ and when $k = i + 1$.

   (ii) Suppose n is a positive integer that is not a triangular number. This means n is strictly between T_i and T_{i+1} for some i. Show that the largest value of $P(n, k)$ is achieved when

$$k = \left\lfloor \frac{1 + \sqrt{8n+1}}{2} \right\rfloor .$$

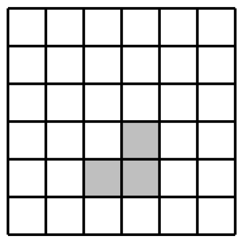
(d) For a positive integer n, we call k a *peak* for n if $P(n, k) \geq P(n, \ell)$ for all integers $1 \leq \ell \leq n$. Part (c) suggests that there are two peaks for n when $n > 1$ is a triangular number and that there is a unique peak when n is not a triangular number. Find a positive integer k for which there are exactly 2020 positive integers n with the property that k is a peak for n.

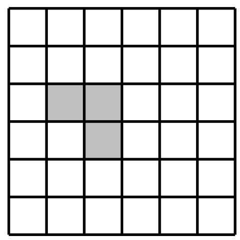
Problem of the Month

Problem 3: December 2020

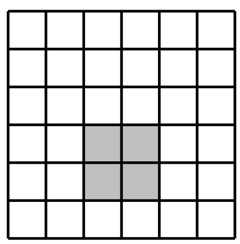
You and a friend are playing games involving a 6×6 grid with a coin in each cell. In each game, your friend arranges the coins so that each coin shows either a head or a tail. An arrangement of coins is called *winnable* if it is possible to perform a sequence of legal moves that results in all 36 coins showing a head. Each game has a different set of legal moves.

(a) In the first game, a legal move consists of flipping over exactly three of the four coins in a 2×2 subgrid. Each grid below has three cells highlighted. In each of these two grids, flipping over the coins in the highlighted cells is an example of a legal move.

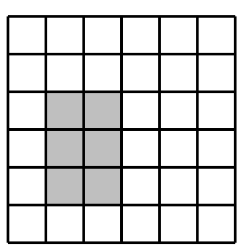


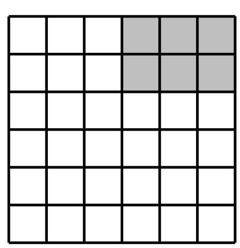


(b) In the second game, a legal move consists of flipping over all four of the coins in a 2×2 subgrid. In the grid below, flipping over the four coins in the highlighted cells is an example of a legal move.



(c) In the third game, a legal move consists of flipping over all 6 of the coins in a 3×2 or 2×3 subgrid. In each of the two grids below, flipping over the coins in the highlighted cells is an example of a legal move.





For each of the three games, determine how many of the 2^{36} arrangements are winnable. In all three games, subgrids must be "connected". For example, the four corners of the 6×6 grid is *not* a 2×2 subgrid.

Problem of the Month
Problem 4: January 2021

In this problem, we will explore when a quadratic polynomial of the form $x^2 + ux + v$ can be decomposed as the sum of the squares of two other polynomials. Keep in mind that a constant function is a polynomial. All polynomials in the problem statements below are assumed to have real coefficients, though they may not have real roots.

(a) Find at least three pairs $\big(p(x), q(x)\big)$ of polynomials such that $(p(x))^2 + (q(x))^2 = x^2 + 2x + 2$.

(b) Suppose $f(x) = x^2 + ux + v$ has the property that $f(x) \geq 0$ for all real numbers x. Prove that there are polynomials $p(x)$ and $q(x)$ such that $x^2 + ux + v = (p(x))^2 + (q(x))^2$.

In the remaining parts of this problem, we will say that the pair of polynomials $\big(p(x), q(x)\big)$ is *special* for the polynomial $x^2 + ux + v$ if

- the coefficients of $p(x)$ and $q(x)$ are all rational, and
- $x^2 + ux + v = (p(x))^2 + (q(x))^2$.

(c) Prove that there are no special pairs for $x^2 + x + 1$.

(d) Prove that if there is a special pair for $x^2 + ux + v$, then u and v are both rational and there is a rational number r such that $4v - u^2 = r^2$.

(e) Prove that if there is a special pair for $x^2 + ux + v$, then there are infinitely many special pairs for $x^2 + ux + v$.

Problem of the Month
Problem 5: February 2021

For an integer $n \geq 3$, we define T_n to be the triangle with side lengths $n-1$, n, and $n+1$, and define A_n to be the area of T_n. We will say that an integer $n \geq 3$ is *remarkable* if A_n is an integer.

(a) Determine all integers n for which T_n is right-angled.

(b) Suppose n is a remarkable integer. Prove that

  (i) $\dfrac{n^2-4}{3}$ is a perfect square,

  (ii) n is not a multiple of 3, and

  (iii) n is even.

(c) There are three remarkable integers less than or equal to 100. Determine these three integers.

(d) The only remarkable integers between 100 and 10 000 are $n = 194$, $n = 724$, and $n = 2702$. Find a polynomial function $f(n)$ of degree greater than 1 with the property that if n is a remarkable integer, then $f(n)$ is also a remarkable integer. Use this polynomial to deduce that there are infinitely many remarkable integers.

(e) Explain how to find all remarkable integers. This should involve somehow describing an infinite set of remarkable integers as well as justification that your set is complete. Keep in mind that the infinite set from part (d) may not include *all* remarkable integers.

Problem of the Month

Problem 6: March 2021

Here is a simple activity that leads to an interesting math problem.

- For a positive integer $n > 1$, draw n dots on a piece of paper. Draw a line to connect each pair of dots. The lines do not need to be straight, but should be drawn so that they do not pass through any dots other than the two they connect. If two lines intersect, the intersection does **not** define a new dot.
- Colour each dot either red or blue in any way that you like.
- Colour each line as follows: If the line connects two dots of the same colour, colour the line red. Otherwise, colour the line blue.

Call a colouring of the dots *balanced* if it leads to the lines being coloured so that there is the same number of blue lines as red lines.

(a) Show that there is no balanced colouring when $n = 5$.

(b) Show that there is a balanced colouring when $n = 9$. Find all possibilities for the number of red dots in a balanced colouring when $n = 9$.

(c) Determine all n for which there is a balanced colouring. For each such n, determine all possibilities for the number of red dots in a balanced colouring.

For part (d), the dots can now be coloured red, blue, or green. The table below describes how the lines should be coloured once the dots are coloured. For example, the letter R is in the cell corresponding to the row for B and the column for G. This means that if a line connects one blue dot and one green dot, then it is to be coloured red.

	R	G	B
R	R	G	B
G	G	B	R
B	B	R	G

For part (d), we redefine a *balanced* colouring of the dots to mean a colouring leads to equal numbers of red, blue, and green lines.

(d) Describe all n for which there is a balanced colouring.

Problem of the Month

Problem 7: April 2021

For an integer $n \geq 3$, n^2 points form an $n \times n$ square grid.

Define $P(n)$ to be the probability that three distinct points randomly selected from the grid are the vertices of a triangle with positive area. Also define $f(n)$ to be the number of sets of three distinct points from the grid that lie on a common line. We can think of $f(n)$ as the number of sets of three distinct points from the grid that are the vertices of a triangle with area 0.

For instance, with $n = 3$, it can be shown that there are 84 possible ways to select three distinct points, that 8 of the sets of three points lie on a line, and that 76 of the sets of three points form the vertices of a triangle with positive area. Thus, $f(3) = 8$ and $P(3) = \dfrac{76}{84} = \dfrac{19}{21}$.

The goal of this problem is to estimate $P(n)$ for large n. The approach outlined will be to estimate $f(n)$ and use it to estimate $P(n)$.

(a) When $n = 3$, $f(n) = 8$ and $P(3) = \dfrac{19}{21}$. Compute $f(n)$ and $P(n)$ for $n = 4$ and $n = 5$.

(b) For $n \geq 3$, prove that $f(n+1) < f(n)+5n^4+5n^3+5n^2+5n$. This will allow us to understand how quickly $f(n)$ grows which will help to estimate $P(n)$.

(c) Using part (b), prove that $f(n) < n^5$ for all $n \geq 3$.

(d) Prove that there is a constant c with the property that $P(n) > 1 - \dfrac{c}{n}$ for all $n \geq 3$. Use this to explain why the following statement makes sense: "For very large n, it is nearly certain that three points selected randomly from an $n \times n$ grid will be the vertices of a triangle with positive area."

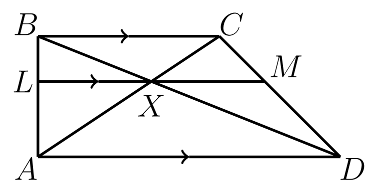
As indicated in part (d), this problem is meant to examine what happens to $P(n)$ as n gets large. Since it seems very difficult to calculate $f(n)$ (and hence, $P(n)$) directly for large n, we instead estimate its value. As long as we carefully keep track of how good/bad the estimates can be, we can say something meaningful about $P(n)$ for large n without actually computing it directly. Very frequently, mathematicians use estimates like these when exact answers are difficult or impossible to obtain. These estimates are often as useful as exact answers.

Problem of the Month

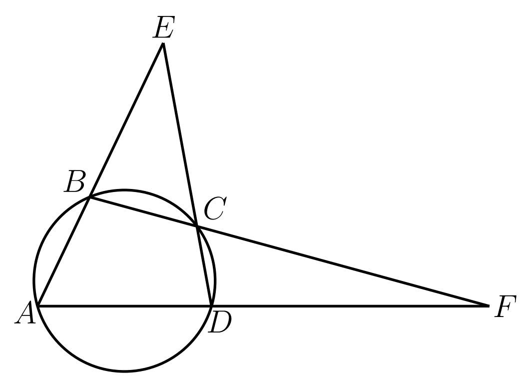
Problem 8: May 2021

Try these three geometry problems! Problems (a), (b), and (c) are not intended to be related to each other. In each part, a diagram is provided to give an example of a figure satisfying the conditions to be explored in that part.

(a) In trapezoid $ABCD$, $\angle BAD = 90^\circ$ and BC is parallel to AD with $BC < AD$. The diagonals AC and BD intersect at point X. A line parallel to AD is drawn through X and intersects AB at L and CD and M. Determine the length of LM in terms of the lengths of BC and AD.



(b) Suppose quadrilateral $ABCD$ has no pair of parallel sides and is inscribed in a circle. AB and DC are extended to meet at point E and AD and BC are extended to meet at point F. The degree measures of $\angle AFB$, $\angle AED$, and $\angle EAF$ form an increasing arithmetic sequence in that order. The degree measure of each of these three angles is an integer. Find all possible values of $\angle AFB$.



(c) Rectangle $DBCA$ has E on BC and F on AC so that $\triangle DEF$ is equilateral. Find all possible values of $\dfrac{BD}{AD}$.