The problems in this booklet are organized into strands. A problem often appears in multiple strands. The problems are suitable for most students in Grade 9 or higher.
Data Management
&
Probability

TAKE ME TO THE COVER
Problem of the Week

Problem D

License to Ride

A Mathville license plate consists of exactly four digits chosen from the digits 0 to 9.

What is the probability that the sum of the digits on any of these license plates is 34 or higher?
Problem

A Mathville license plate consists of exactly four digits chosen from the digits 0 to 9. What is the probability that the sum of the digits on any of these license plates is 34 or higher?

Solution

First we must determine the total number of possible 4 digit number sequences. There are 10 choices for the first digit. For each of these choices there are 10 choices for the second digit. Therefore, there are \(10 \times 10 = 100\) choices for the first two digits. For each of these possibilities, there are 10 choices for the third digit. Therefore, there are \(100 \times 10 = 1000\) possibilities for the first three digits. And finally, for each of these 1000 choices for the first three digits there are 10 choices for the fourth digit. Therefore there are \(1000 \times 10 = 10000\) possibilities for license plates with four digits in Mathville.

Now we must determine the number of license plates with a digit sum of 34 or higher. We will consider cases.

1. **The license contains four nines.** If the digits are all nines, the sum is 36 which is acceptable. There is only one way to use all nines for digits.

2. **The license contains three nines and one other digit.** If three nines are used the sum is 27. In order to get 34 or higher the fourth digit must be 7 or 8. There are two choices for the fourth digit. Once the digit is chosen there are four places to put the digit. Once the digit is placed the remaining spots must be nines. Therefore there are \(2 \times 4 = 8\) license plates containing three nines. (It is possible to list them: 7999, 8999, 9799, 9899, 9979, 9989, 9997, 9998.)

3. **The license contains two nines.** If two nines are used the sum is 18. In order to get to 34 or higher, we need a sum of \(34 - 18 = 16\) or higher from the two remaining digits. The only way to do this, since we cannot use more nines, is to use two eights. The two eights can be placed in six ways and then the nines must go in the remaining spots. Therefore there are six license plates containing two nines. (It is possible to list them: 8899, 8989, 8998, 9889, 9898, 9988.)

4. **The license contains one nine.** If one nine is used the sum is 9. In order to get to 34 or higher, we need a sum of \(34 - 9 = 25\) or higher from the remaining three digits. But this sum would be made from three digits chosen from the digits 0 to 8. The maximum possible sum would be 24 if three eights were used. We need 25 or higher. Therefore no license plate containing only one 9 will produce a sum of 34 or higher. It should be noted that a license with no nines would not produce a sum of 34 or higher either.

Therefore, the number of license plates with a digit sum of 34 or higher is \(1 + 8 + 6 = 15\). To calculate the probability we divide the number of licenses with a digit sum of 34 or more by the number of possible license plates. The probability of getting a license plate in Mathville with a digit sum of 34 or higher is \(\frac{15}{10000} = \frac{3}{2000}\). Another way of looking at this result is out of every 2 000 cars with Mathville plates you could expect, on average, to find 3 with a digit sum of 34 or higher.
Problem of the Week
Problem D
An Average Bowl

In his latest game, Mark Striker bowled 199 and raised his average from 177 to 178. Mark would like to raise his average to 180 after bowling his next game. What would Mark need to bowl on his next game to accomplish his goal?
Problem of the Week
Problem D and Solution
An Average Bowl

Problem
In his latest game, Mark Striker bowled 199 and raised his average from 177 to 178. Mark would like to raise his average to 180 after bowling his next game.

What would Mark need to bowl on his next game to accomplish his goal?

Solution
Solution 1
Let \( n \) be the number of games bowled to achieve his 177 average. His total points scored in \( n \) games is his average times \( n \). Therefore, Mark has \( 177n \) total points in \( n \) games.

To compute his average after bowling the 199 game, we take his new total points and divide by \( n + 1 \), the new number of games.

\[
\text{Average} = \frac{\text{Total Points}}{\text{Games Played}}
\]

\[
178 = \frac{177n + 199}{n + 1}
\]

\[
178(n + 1) = 177n + 199
\]

\[
178n + 178 = 177n + 199
\]

\[
n = 21
\]

Prior to bowling the 199 game, Mark had bowled 21 games. So after bowling the 199 game, Mark has bowled 22 games. Mark wants to have a 180 average after bowling his 23rd game. The difference between his total points after 23 games with a 180 average and his total points after bowling 22 games with a 178 average must be his score on the 23rd game.

\[
\text{Score on 23rd Game} = 23 \times 180 - 22 \times 178 = 4140 - 3916 = 224
\]

Therefore Mr. Striker must bowl 224 on his next game to raise his average from 178 to 180.
Solution 2

Mark’s score of 199 is $199 - 177 = 22$ points above his previous average. Mark raised his average 1 point. Therefore, his latest game with the 199 score must have been his 22nd game.

To raise his average 2 points in his 23rd game he must bowl $2 \times 23 = 46$ points above his 178 average. He must bowl $178 + 46 = 224$ in his next game.

\[ \therefore \text{Mark must bowl 224 in his next game to move his average from 178 to 180.} \]

We can verify our results:

Average on 21 games is 177.

Average on 22 games is $\frac{21 \times 177 + 199}{22} = \frac{3916}{22} = 178$

Average on 23 games is $\frac{22 \times 178 + 224}{23} = \frac{4140}{23} = 180$
Problem of the Week
Problem D
Card Logic

Four playing cards are placed in a row, from left to right. One card is a club (♣), one card is a diamond (♦), one card is a heart (♥), and one card is a spade (♠), not necessarily in that order. The number on each card is different.

Using the following clues, determine the exact order of the cards, from left to right, including the suit and number.

1. The club and the spade are immediately beside each other, in that order, left to right.

2. The diamond is not the first (leftmost) card, and the heart is not the fourth (rightmost) card.

3. The 7 is somewhere to the left of the heart and the 3 is somewhere to the right of the heart.

4. The two cards whose values are still unknown add to nine.

5. From left to right, the cards are arranged from largest to smallest.
Problem

Four playing cards are placed in a row, from left to right. One card is a club (♣), one card is a diamond (♦), one card is a heart (♥), and one card is a spade (♠), not necessarily in that order. The number on each card is different. Using the following clues, determine the exact order of the cards, from left to right, including the suit and number.

1. The club and the spade are immediately beside each other, in that order, left to right.
2. The diamond is not the first (leftmost) card, and the heart is not the fourth (rightmost) card.
3. The 7 is somewhere to the left of the heart and the 3 is somewhere to the right of the heart.
4. The two cards whose values are still unknown add to nine.
5. From left to right, the cards are arranged from largest to smallest.

Solution

Let $C$ be the club, $D$ be the diamond, $H$ be the heart and $S$ be the spade.

Using the first clue, we can determine that there are only three possible placements for $C$ and $S$, namely \{$C,S,\Box,\Box$\}, \{\Box,$C,S,\Box$\} or \{\Box,\Box,$C,S$\}.

We will use the second clue with each of the three possible placements. From \{\Box,$C,S,\Box$\} we obtain \{$C,S,H,D$\} since $H$ cannot go last. From \{\Box,$C,S,\Box$\} we obtain \{H,$C,S,D$\} since $D$ cannot be first and $H$ cannot be last. From \{\Box,\Box,$C,S$\} we obtain \{H,$D,C,S$\} since $D$ cannot go first.

The third clue says that 7 is somewhere to the left of the heart. This means that the heart cannot be furthest to the left. Therefore, we can rule out two of the possibilities that we just found leaving \{\Box,$C,S,H,D$\} as the only possible ordering of the suits. Using the third clue further, we know that a 7 is somewhere to the left of the heart and a 3 is somewhere to the right of the heart. We can conclude from this that the rightmost card is the 3 of diamonds. At this point we know the following: it is either \{7,$C,S,H,3D$\} or \{C,$7S,H,3D$\}. We know the 7 is in either the first or second spot, the 3 of diamonds is in the fourth spot. We still do not know the other two numbers.

Using the fourth clue we know that the remaining pair of numbers add to 9. The numbers 7 and 3 are already used. Since numbers are arranged from highest to lowest and numbers are used only once, we can rule out the following pairs: 7 and 2, 6 and 3. That leaves only two possible pairs: 8 and 1, and 5 and 4. Using the fifth clue we can place the numbers to see if the solution is valid. Trying 8 and 1 in our possible solutions we obtain \{7C,8S,1H,3D\} or \{8C,7S,1H,3D\}. Neither of these are valid since the card values would not be arranged largest to smallest. Trying 5 and 4 in our possible solutions we obtain \{7C,5S,4H,3D\} or \{5C,7S,4H,3D\}. Only the first solution satisfies the condition that the numbers are arranged largest to smallest.

∴ the cards are the 7 of clubs (7♣), the five of spades (5♠), the four of hearts (4♥) and the three of diamonds (3♦).
Problem of the Week
Problem D
ABACUS Counting

We are going to count with the abacus. To be more precise, we are going to count certain six letter arrangements of the letters of the word ABACUS in which every letter is used exactly once.

The six letters, A, B, A, C, U, and S are arranged to form six letter “words”. When examining the “words”, how many of them have the vowels A, A and U appearing in alphabetical order and the consonants B, C and S not appearing in alphabetical order. The vowels may or may not be adjacent to each other and the consonants may or may not be adjacent to each other.

For example, each of CAAUSB and ASAUCB are valid arrangements but ABACUS, CUAAASB and AUABCS are invalid arrangements.
Problem of the Week
Problem D and Solution
ABACUS Counting

Problem

We are going to count with the abacus. To be more precise, we are going to count certain six letter arrangements of the letters of the word \textit{ABACUS} in which every letter is used exactly once. The six letters, \textit{A}, \textit{B}, \textit{A}, \textit{C}, \textit{U}, and \textit{S} are arranged to form six letter “words”. When examining the “words”, how many of the words have the vowels \textit{A}, \textit{A} and \textit{U} appearing in alphabetical order and the consonants \textit{B}, \textit{C} and \textit{S} not appearing in alphabetical order. The vowels may or may not be adjacent to each other and the consonants may or may not be adjacent to each other.

Solution

Solution 1

In this solution we will count all of the valid six letter arrangements of \textit{A}, \textit{A}, \textit{B}, \textit{C}, \textit{S}, and \textit{U} directly. First consider the number of ways to place the vowels alphabetically in the arrangement.

- If the first \textit{A} is in the first position, the \textit{U} and the second \textit{A} can be placed as follows:
  1. \textit{A A} \underline{---} and the \textit{U} can be placed in 4 ways after the second \textit{A}.
  2. \textit{A} \underline{A A} \underline{---} and the \textit{U} can be placed in 3 ways after the second \textit{A}.
  3. \underline{A A} \underline{---} \textit{A} and the \textit{U} can be placed in 2 ways after the second \textit{A}.
  4. \underline{A} \underline{---} \textit{A} \textit{A} and the \textit{U} can be placed in 1 way after the second \textit{A}.

There is a total of \(4 + 3 + 2 + 1 = 10\) ways to place the vowels in alphabetical order so that the first \textit{A} is in the first position.

- If the first \textit{A} is in the second position, the \textit{U} and the second \textit{A} can be placed as follows:
  1. \underline{---} \textit{A A} \underline{---} and the \textit{U} can be placed in 3 ways after the second \textit{A}.
  2. \underline{---} \textit{A} \underline{A A} \underline{---} and the \textit{U} can be placed in 2 ways after the second \textit{A}.
  3. \underline{---} \underline{A} \underline{A A} \underline{---} and the \textit{U} can be placed in 1 way after the second \textit{A}.

There is a total of \(3 + 2 + 1 = 6\) ways to place the vowels in alphabetical order so the first \textit{A} is in the second position.

- If the first \textit{A} is in the third position, the \textit{U} and the second \textit{A} can be placed as follows:
  1. \underline{---} \underline{---} \textit{A A} and the \textit{U} can be placed in 2 ways after the second \textit{A}.
  2. \underline{---} \underline{---} \textit{A} \underline{A A} and the \textit{U} can be placed in 1 way after the second \textit{A}.

There is a total of \(2 + 1 = 3\) ways to place the vowels in alphabetical order so the first \textit{A} is in the third position.

- If the first \textit{A} is in the fourth position, the \textit{U} and the second \textit{A} can be placed as follows:
  1. \underline{---} \underline{---} \textit{A} \underline{A A} and the \textit{U} can be placed in 1 way after the second \textit{A}.

There is a total of \(1\) way to place the vowels in alphabetical order so the first \textit{A} is in the fourth position.

Adding each of the above results, there is a total of \(10 + 6 + 3 + 1 = 20\) ways to place the vowels so that they are in alphabetical order.
For each of the 20 ways to place the vowels in alphabetical order, we find the number of valid ways to fill the remaining three positions with the consonants so that they are not in alphabetical order. It turns out that the consonants can be placed in the three remaining positions in six possible orders: BCS, BSC, CBS, CSB, SBC, and SCB. One of the arrangements of the consonants is in alphabetical order and five of the arrangements are not in alphabetical order.

So for each of the 20 arrangements in which the vowels are in alphabetical order there are 5 arrangements of the consonants so they are not in alphabetical order.

Therefore, there are $20 \times 5 = 100$ arrangements of the letters of the word ABACUS in which the vowels appear alphabetically and the consonants do not appear alphabetically.

Solution 2
In this solution we will first count the total number of arrangements of the letters in the word ABACUS.

Notice that there are two As. We need to be careful not to count arrangements twice. So we will place the distinct letters B, C, S, and U first. There are 6 places for the B. For each of these placements of B, there are 5 ways to place the C. This gives a total of $6 \times 5 = 30$ ways to place the B and C. For each of these placements of B and C, there are 4 ways to place the S. This gives a total of $30 \times 4 = 120$ ways to place the B, C and S. For each of these placements of B, C and S, there are 3 ways to place the U. This gives a total of $120 \times 3 = 360$ ways to place the B, C, S, and U. The two As must go in the remaining two spots and this can be done in only 1 way. Therefore, there is a total of 360 ways to arrange the six letters of the word ABACUS.

In examining the 360 arrangements, you would see the vowels appearing in one of three orders: AAU, AUA and UAA, only one of which is in alphabetical order. So in one-third of the 360 arrangements or 120 arrangements, the vowels will appear in alphabetical order.

In examining the 120 arrangements in which the vowels appear alphabetically, the consonants will appear in one of six different orders: BCS, BSC, CBS, CSB, SBC, and SCB. Five-sixths of these orderings have the consonants out of alphabetical order. So five-sixths of the 120 arrangements or 100 arrangements will be such that the vowels appear alphabetically and the consonants do not appear alphabetically.

Solution 3
This solution will be outlined only and is left as an exercise to be completed by the solver.

If we were to count the total number of possibilities as we did in the second solution, we could then subtract the arrangements which are invalid. The invalid arrangements include the number of arrangements in which the vowels and consonants are both not in alphabetical order, the number of arrangements in which the vowels and consonants are both in alphabetical order and the number of arrangements in which the vowels are not in alphabetical order but the consonants are in alphabetical order. Be careful not to count arrangements twice.
Problem of the Week
Problem D
Pick a Box, Any Box

A room contains 150 boxes. The boxes are identical except for colour. Each box contains a card which has one of three messages printed on it. In 78 of the boxes the card says “No Prize”. In 66 of the boxes the card says “Winner $20”. And in 6 of the boxes the card says “Winner $100”.

Contestants have been assigned a number corresponding to when they will make their selection. On a turn, each contestant gets to randomly choose 2 boxes. Once a box is chosen it is removed from the room.

You are the second contestant. What is the probability that you will select at least one of the boxes containing a $100 prize?
Problem of the Week
Problem D and Solution
Pick a Box, Any Box

Problem
A room contains 150 boxes. The boxes are identical except for colour. Each box contains a card which has one of three messages printed on it. In 78 of the boxes the card says “No Prize”. In 66 of the boxes the card says “Winner $20”. And in 6 of the boxes the card says “Winner $100”. Contestants have been assigned a number corresponding to when they will make their selection. On a turn, each contestant gets to randomly choose 2 boxes. Once a box is chosen it is removed from the room. You are the second contestant. What is the probability that you will select at least one of the boxes containing a $100 prize?

Solution
To start, we will count the total number of ways to select 4 boxes. There are 150 ways to select the first box. For each of these 150 possible selections, there are 149 ways to select the second box. There are then \(150 \times 149 = 22350\) ways to select the first two boxes. For each of these 22350 ways to select the first two boxes, there are 148 ways to select the third box. There are then \(22350 \times 148 = 3307800\) ways to select the first three boxes. And for each of these selections there are 147 ways to select the fourth box. There are then \(3307800 \times 147 = 486246600\) ways to select the first four boxes.

Six of the boxes contain a $100 card and \(150 - 6 = 144\) do not contain a $100 card. Once a $100 card is selected the number of available $100 cards decreases by 1. Once any card other than a $100 card is selected the number of such cards decreases by 1.

We can now approach the problem from one of two different directions. Either we determine the total number of ways that you, the second person, can select at least one $100 card and then calculate the associated probability. Or we could determine the number of ways that you, the second person, do not select any $100 card, calculate the related probability and then subtract that result from 1. The first approach is a direct approach. The second approach is an indirect approach.

We will present both approaches but start with the indirect approach first.

Indirect Approach
In how many ways can you not select a box containing a $100 card? What are the possibilities for Contestant 1?

Case 1 Contestant 1 selects two boxes, neither of which contain a $100 card.
Then you, Contestant 2, select two boxes, neither of which contain a $100 card. The number of possible selections is \(144 \times 143 \times 142 \times 141 = 412293024\).

Case 2 Contestant 1 selects two boxes, the first contains a $100 card and the second one does not.
Then you, Contestant 2, select two boxes, neither of which contain a $100 card. The number of possible selections is \(6 \times 144 \times 143 \times 142 = 17544384\).

Case 3 Contestant 1 selects two boxes, the first does not contain a $100 card and the second box does.
Then you, Contestant 2, select two boxes, neither of which contain a $100 card. The number of possible selections is \(144 \times 6 \times 143 \times 142 = 17544384\).
Case 4  Contestant 1 selects two boxes, both of which contain a $100 card. Then you, Contestant 2, select two boxes, neither of which contain a $100 card. The number of possible selections is \(6 \times 5 \times 144 \times 143 = 617760\).

The total number of ways in which you, the second contestant, do not select a box containing a $100 card is
\[412293024 + 17544384 + 17544384 + 617760 = 447999552\].

The probability you do not select a $100 card is the total number of ways for you to not select a $100 card divided by the total number of ways to select four boxes.

\[P(\text{do not select a$100 card as second contestant}) = \frac{447999552}{486246600} = \frac{3432}{3725}\]

The probability of you selecting at least one $100 card as the second contestant is 1 subtract the probability of you not selecting any $100 cards as the second contestant

\[P(\text{selecting at least one$100 card as second contestant}) = 1 - \frac{3432}{3725} = \frac{293}{3725} \approx 0.079\]

As the second contestant, the probability of selecting at least one $100 card is \(\frac{293}{3725}\). You would select at least one $100 card approximately 8% of the time.

Direct Approach

There are more possibilities to consider using this approach. We will present a simplified solution in a chart to determine the total number of ways for contestant 2 to select at least one $100 card. If for any selection a $100 card is selected, we will represent the selection with a \(G\). If for any selection a $100 card is not selected, we will represent the selection with an \(N\). So \(GGNG\) represents the possibility that the first contestant picked two $100 cards and the second contestant picked a non-$100 card then a $100 card. The possible cases are shown in the chart.

<table>
<thead>
<tr>
<th>Contestant 1 Selection</th>
<th>Contestant 1 Selection</th>
<th>Calculation</th>
<th>Number of Possibilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>(GG)</td>
<td>(GN)</td>
<td>(6 \times 5 \times 4 \times 144)</td>
<td>17280</td>
</tr>
<tr>
<td>(GG)</td>
<td>(NG)</td>
<td>(6 \times 5 \times 144 \times 4)</td>
<td>17280</td>
</tr>
<tr>
<td>(GG)</td>
<td>(GG)</td>
<td>(6 \times 5 \times 4 \times 3)</td>
<td>360</td>
</tr>
<tr>
<td>(GN)</td>
<td>(GN)</td>
<td>(6 \times 144 \times 5 \times 143)</td>
<td>617760</td>
</tr>
<tr>
<td>(GN)</td>
<td>(NG)</td>
<td>(6 \times 144 \times 143 \times 5)</td>
<td>617760</td>
</tr>
<tr>
<td>(GN)</td>
<td>(GG)</td>
<td>(6 \times 144 \times 5 \times 4)</td>
<td>17280</td>
</tr>
<tr>
<td>(NG)</td>
<td>(GN)</td>
<td>(144 \times 6 \times 5 \times 143)</td>
<td>617760</td>
</tr>
<tr>
<td>(NG)</td>
<td>(NG)</td>
<td>(144 \times 6 \times 143 \times 5)</td>
<td>617760</td>
</tr>
<tr>
<td>(NG)</td>
<td>(GG)</td>
<td>(144 \times 6 \times 5 \times 4)</td>
<td>17280</td>
</tr>
<tr>
<td>(NN)</td>
<td>(GN)</td>
<td>(144 \times 143 \times 6 \times 142)</td>
<td>17544384</td>
</tr>
<tr>
<td>(NN)</td>
<td>(NG)</td>
<td>(144 \times 143 \times 142 \times 6)</td>
<td>17544384</td>
</tr>
<tr>
<td>(NN)</td>
<td>(GG)</td>
<td>(144 \times 143 \times 6 \times 5)</td>
<td>617760</td>
</tr>
<tr>
<td>Total Possibilities</td>
<td></td>
<td></td>
<td>(38247048)</td>
</tr>
</tbody>
</table>

\[P(\text{selecting at least one$100 card as second contestant}) = \frac{38247048}{486246600} = \frac{293}{3725} \approx 0.079\]

As the second contestant, the probability of selecting at least one $100 card is \(\frac{293}{3725}\). You would select at least one $100 card approximately 8% of the time.
Measurement & Trigonometry
Problem of the Week

Problem D

Thinking Inside the Box

$Q$ is the point of intersection of the diagonals of one face of a cube whose edges have length 2 cm.

Determine the length of $QR$. 
Problem of the Week
Problem D and Solution
Thinking Inside the Box

Problem

$Q$ is the point of intersection of the diagonals of one face of a cube whose edges have length 2 cm. Determine the length of $QR$.

Solution

Label the corners $S$ and $T$ as shown.

The faces of a cube are squares. The diagonals of a square right bisect each other. It follows that $PQ = QT = \frac{1}{2}PT$. Since the face is a square, $\angle PST = 90^\circ$ and $\triangle PST$ is right angled. Using the Pythagorean Theorem, $PT^2 = PS^2 + ST^2 = 2^2 + 2^2 = 8$ and $PT = \sqrt{8}$. Then $PQ = \frac{1}{2}PT = \frac{\sqrt{8}}{2}$.

Because of the 3-dimensional nature of the problem it may not be obvious to all that $\angle RPQ = 90^\circ$. To help visualize this, note that $\angle RPS = 90^\circ$ because the face of the cube is a square. Rotate $PS$ counterclockwise about point $P$ on the side face of the cube so that the image of $PS$ lies along $PQ$. The corner angle will not change as a result of the rotation so $\angle RPQ = \angle RPS = 90^\circ$.

We can now use the Pythagorean Theorem in $\triangle RPQ$ to find the length $RQ$.

\[
RQ^2 = RP^2 + PQ^2 = 2^2 + \left(\frac{\sqrt{8}}{2}\right)^2 = 4 + \frac{8}{4} = 4 + 2 = 6
\]

\[
RQ = \sqrt{6} \text{ cm.}
\]

\[\therefore \text{ the length of } RQ \text{ is } \sqrt{6} \text{ cm.}\]

A couple of notes are in order at this point.

First, although the mathematics required to solve this problem was fairly straightforward some students would have found it difficult because of the three-dimensional nature of the problem.

Second, we could have simplified $PQ = \frac{1}{2}PT = \frac{\sqrt{8}}{2}$ to $\sqrt{2}$ as follows:

\[
\frac{\sqrt{8}}{2} = \frac{\sqrt{4} \times \sqrt{2}}{2} = \frac{\sqrt{4} \times \sqrt{2}}{2} = \frac{2 \sqrt{2}}{2} = \sqrt{2}.
\]

Often simplifying radicals is not a part of the curriculum at the grade 9 or 10 level. The calculation of $RQ$ would have been simpler using $PT = \sqrt{2}$.  

In $\triangle ABD$, $C$ is on $AB$ such that $CA = CB = CD$ and $\angle BCD = z^\circ$.

Determine the measure of $\angle ADB$.

If you need a hint, consider solving this week’s Problem of the Week Problem C, “What’s Your Angle Anyway I?”.

If you want an extension to this problem, consider solving this week’s Problem of the Week Problem E, “What’s Your Angle Anyway III?”. 
Problem of the Week
Problem D and Solution
What’s Your Angle Anyway II?

Problem

In $\triangle ABD$, $C$ is on $AB$ such that $CA = CB = CD$ and $\angle BCD = z^\circ$.
Determine the measure of $\angle ADB$.

Solution

Solution 1

Since $ACB$ is a straight line, $\angle ACD + \angle DCB = 180^\circ$ but $\angle BCD = z^\circ$ so $\angle ACD = 180^\circ - z^\circ$.

In $\triangle ACD$, since $CA = CD$, $\triangle ACD$ is isosceles and $\angle CAD = \angle CDA = x^\circ$.
The angles in a triangle sum to $180^\circ$ so in $\triangle ACD$

$$\angle CAD + \angle CDA + \angle ACD = 180^\circ$$
$$x^\circ + x^\circ + 180^\circ - z^\circ = 180^\circ$$
$$2x^\circ = z^\circ$$
$$x^\circ = \frac{z^\circ}{2}$$

Similarly, in $\triangle BCD$, since $CB = CD$, $\triangle CBD$ is isosceles and $\angle CBD = \angle CDB = y^\circ$.
The angles in a triangle sum to $180^\circ$ so in $\triangle CBD$

$$\angle CBD + \angle CDB + \angle BCD = 180^\circ$$
$$y^\circ + y^\circ + z^\circ = 180^\circ$$
$$2y^\circ = 180^\circ - z^\circ$$
$$y^\circ = \frac{180^\circ - z^\circ}{2}$$

Then $\angle ADB = \angle CDA + \angle CDB = x^\circ + y^\circ = \frac{z^\circ}{2} + \frac{180^\circ - z^\circ}{2} = \frac{180^\circ}{2} = 90^\circ$.

$\therefore$ the measure of $\angle ADB$ is $90^\circ$.

See Solution 2 for a more general approach to the solution.
It turns out that it is not necessary to find expressions for $x$ and $y$ in terms of $z$ in the problem.

![Diagram of a quadrilateral with angles labeled x, y, z, and z again.]

**Solution 2**

Here is a second solution to the problem.

In $\triangle CAD$, since $CA = CD$, $\triangle CAD$ is isosceles and $\angle CAD = \angle CDA = x^\circ$.

In $\triangle CBD$, since $CB = CD$, $\triangle CBD$ is isosceles and $\angle CBD = \angle CDB = y^\circ$.

The angles in a triangle sum to $180^\circ$ so in $\triangle ABD$

\[
\angle BAD + \angle ADB + \angle ABD = 180^\circ
\]

\[
x^\circ + (x^\circ + y^\circ) + y^\circ = 180^\circ
\]

\[
2x^\circ + 2y^\circ = 180^\circ
\]

\[
x^\circ + y^\circ = 90^\circ
\]

But $\angle ADB = \angle ADC + \angle CDB = x^\circ + y^\circ = 90^\circ$.

\[\therefore\] the measure of $\angle ADC$ is $90^\circ$. 
Problem of the Week

Problem D

The Case of the Missing Square

Rectangle $DEFG$ has square $ABCD$ removed leaving an area of 92 m$^2$. Side $AE = 4$ m and side $CG = 8$ m.

Determine the original area of rectangle $DEFG$. 

---

![Diagram of rectangle $DEFG$ with square $ABCD$ removed]
Problem of the Week
Problem D and Solution
The Case of the Missing Square

Problem
Rectangle $DEFG$ has square $ABCD$ removed leaving an area of 92 m$^2$. Side $AE = 4$ m and side $CG = 8$ m. Determine the original area of rectangle $DEFG$.

Solution
Let $x$ represent the side length of square $ABCD$. In the diagram, extend $CB$ to intersect $EF$ at $H$. This creates rectangle $AEHB$ and rectangle $CHFG$. Then $FG = EA + AD = (4 + x)$ m and $EH = DC = x$ m.

$$Area \ AEHB + Area \ CHFG = Remaining \ Area$$
$$AE \times EH + CG \times FG = 92$$
$$4x + 8(4 + x) = 92$$
$$4x + 32 + 8x = 92$$
$$12x + 32 = 92$$
$$12x = 60$$
$$x = 5 \ m$$

Since $x = 5$ m, $DG = 8 + x = 13$ m and $FG = 4 + x = 9$ m. The original area of rectangle $DEFG = DG \times FG = 13 \times 9 = 117 \ m^2$. 
Problem of the Week
Problem D
Tangled Triangles

In the diagram, $A(0, a)$ lies on the $y$-axis above the origin. If $\triangle ABD$ and $\triangle COB$ have the same area, determine the value of $a$. 
Problem of the Week
Problem D and Solution
Tangled Triangles

Problem
In the diagram, \( A(0, a) \) lies on the \( y \)-axis above the origin. If \( \triangle ABD \) and \( \triangle COB \) have the same area, determine the value of \( a \).

Solution
Solution 1
Draw rectangle \( EDFC \) with sides parallel to the \( x \) and \( y \)-axes so that \( O(0,0) \) is on \( ED \) and \( B(2, -1) \) is on \( DF \). Since \( EC \) is parallel to the \( x \)-axis and \( E \) is on the \( y \)-axis, \( E \) has coordinates \((0, 2)\). Since \( CF \) is parallel to the \( y \)-axis, \( F \) has the same \( x \)-coordinate as \( C \). Since \( FD \) is parallel to the \( x \)-axis, \( F \) has the same \( y \)-coordinate as \( D \) and \( B \). Therefore the coordinates of \( F \) are \((3, -1)\).

To find the area of \( \triangle COB \), subtract the areas of \( \triangle CEO \), \( \triangle ODB \), and \( \triangle BFC \) from the area of rectangle \( EDFC \).

In rectangle \( EDFC \), \( EC = 3 - 0 = 3 \) and \( ED = 2 - (-1) = 3 \). The area of rectangle \( EDFC = EC \times ED = 3 \times 3 = 9 \) units\(^2\).

In \( \triangle CEO \), \( EC = 3 \) and \( EO = 2 - 0 = 2 \). The area of \( \triangle CEO = \frac{EC \times EO}{2} = \frac{3 \times 2}{2} = 3 \) units\(^2\).

In \( \triangle ODB \), \( OD = 0 - (-1) = 1 \) and \( DB = 2 - 0 = 2 \). The area of \( \triangle ODB = \frac{OD \times DB}{2} = \frac{1 \times 2}{2} = 1 \) unit\(^2\).

In \( \triangle BFC \), \( BF = 3 - 2 = 1 \) and \( CF = 2 - (-1) = 3 \). The area of \( \triangle BFC = \frac{BF \times CF}{2} = \frac{1 \times 3}{2} = 1.5 \) units\(^2\).

\[
\text{Area } \triangle COB = \text{Area Rectangle } EDFC - \triangle CEO - \triangle ODB - \triangle BFC
\]
\[
= 9 - 3 - 1 - 1.5
\]
\[
= 3.5 \text{ units}^2
\]

But the area \( \triangle ABD = \triangle COB \) so the area of \( \triangle ABD = 3.5 \) units\(^2\).

In \( \triangle ABD \), \( AD = a - (-1) = a + 1 \) and \( DB = 2 - 0 = 2 \) so

\[
\text{Area } \triangle ABD = \frac{AD \times DB}{2}
\]
\[
3.5 = \frac{(a + 1) \times 2}{2}
\]
\[
3.5 = a + 1
\]
\[
2.5 = a
\]

\[\therefore \text{ the value of } a \text{ is } 2.5.\]
Solution 2

Determine the equation of the line containing $C(3, 2)$ and $B(2, -1)$.

The slope of the line is \( \frac{2 - (-1)}{3 - 2} = 3 \). The equation of the line is of the form \( y = 3x + b \). Substitute \( x = 3, \ y = 2 \) to determine the value of \( b \). \( 2 = 3(3) + b \) and \( b = -7 \) follows. Therefore the equation of the line containing \( C \) and \( B \) is \( y = 3x - 7 \).

Let \( P(p, 0) \) be the \( x \)-intercept of the line. Substituting into \( y = 3x - 7 \) we obtain \( 0 = 3p - 7 \) and \( p = \frac{7}{3} \) follows.

To determine the area of \( \triangle COB \) determine the sum of the areas of \( \triangle COP \) and \( \triangle BOP \).

In \( \triangle COP \), \( OP = \frac{7}{3} \) and the height is the perpendicular distance from the \( x \)-axis to \( C(3, 2) \), which is 2 units. The area of \( \triangle COP = \frac{\frac{7}{3} \times 2}{2} = \frac{7}{3} \) units\(^2 \).

In \( \triangle BOP \), \( OP = \frac{7}{3} \) and the height is the perpendicular distance from the \( x \)-axis to \( B(2, -1) \), which is 1 unit. The area of \( \triangle BOP = \frac{\frac{7}{3} \times 1}{2} = \frac{7}{6} \) units\(^2 \).

\[
\text{Area } \triangle COB = \text{Area } \triangle COP + \text{Area } \triangle BOP \\
= \frac{7}{3} + \frac{7}{6} \\
= \frac{14}{6} + \frac{7}{6} \\
= \frac{21}{6} \\
= \frac{7}{2} \text{ units}^2
\]

But the area \( \triangle ABD = \triangle COB \) so the area of \( \triangle ABD = \frac{7}{2} \) units\(^2 \).

In \( \triangle ABD \), \( AD = a - (-1) = a + 1 \) and \( DB = 2 - 0 = 2 \) so

\[
\text{Area } \triangle ABD = \frac{AD \times DB}{2} \\
\frac{7}{2} = \frac{(a + 1) \times 2}{2} \\
7 = 2a + 2 \\
5 = 2a \\
\therefore \ a = \frac{5}{2} = 2.5.
\]
A wooden cube is cut into $n$ cubes each of edge length 1 unit. The combined surface area of the $n$ cubes is eight times the surface area of the original uncut cube.

Determine the edge length of the original uncut cube.
Problem of the Week
Problem D and Solution
Creating Cubes

Problem
A wooden cube is cut into \( n \) cubes each of edge length 1 unit. The combined surface area of the \( n \) cubes is eight times the surface area of the original uncut cube. Determine the edge length of the original uncut cube.

Solution
Let the edge length of the uncut cube be \( x \) units, \( x > 0 \). After cutting the original cube into \( n \) cubes of edge length 1 unit, there will be \( x^3 \) cubes of edge length 1 unit. It follows that \( n = x^3 \).

Each of the 6 sides of the original cube has area \( x^2 \) so the total surface area of the original cube is \( 6x^2 \).

Consider one of the smaller cubes. The surface area of one the six faces is 1 \( \text{unit}^2 \). So the total surface area of one of these smaller cubes is 6 \( \text{units}^2 \). The total surface area of the \( n \) smaller cubes is \( (6n) \) \( \text{units}^2 \).

Since the total surface area of the \( n \) cubes is eight times the surface area of the original uncut cube,

\[
\text{new surface area} = 8 \times \text{original surface area}
\]

\[
6n = 8(6x^2)
\]

Dividing both sides by 6,

\[
n = 8x^2
\]

But \( n = x^3 \) so

\[
x^3 = 8x^2
\]

Dividing by \( x^2 \) since \( x > 0 \),

\[
x = 8
\]

Therefore, the edge length of the original uncut cube was 8 units.

Extension:
Note, that if the combined surface area of the \( n \) cubes of edge length 1 unit was \( Q \) times the surface area of the original uncut cube, then the edge length of the original uncut cube would have been \( Q \) units. Can you see why?
Problem of the Week
Problem D
Several Areas of Interest

The area of $\triangle ACD$ is twice the area of square $BCDE$. Square $BCDE$ has sides of length 12 cm. $AD$ intersects $BE$ at $F$.

Determine the area of quadrilateral $BCDF$. 

![Diagram of a triangle and a square with intersecting lines]
Problem of the Week
Problem D and Solution
Several Areas of Interest

Problem
The area of $\triangle ACD$ is twice the area of square $BCDE$. Square $BCDE$ has sides of length 12 cm. $AD$ intersects $BE$ at $F$. Determine the area of quadrilateral $BCDF$.

Solution
The area of square $BCDE = 12 \times 12 = 144$ cm$^2$. The area of $\triangle ACD$ equals twice the area of square $BCDE$. Therefore area $\triangle ACD = 288$ cm$^2$.

The area of a triangle is calculated using the formula $\text{base} \times \text{height} ÷ 2$. It follows that:

\[
\text{Area } \triangle ACD = \frac{(CD) \times (AC)}{2} = \frac{288}{12} AC ÷ 2 = \frac{288}{6} AC = 48 \text{ cm} = AC
\]

But $AC = AB + BC$ so $48 = AB + 12$ and it follows that $AB = 36$ cm.

From this point, we will present three different approaches to obtaining the required area.

Method 1
Let the area of $\triangle ABF$ be $p$, quadrilateral $BCDF$ be $q$ and $\triangle DEF$ be $r$. Let $FE = x$. Since $BE = 12$, $BF = 12 - x$.

Area $\triangle ACD = 288 = p + q$ (1) and area square $BCDE = 144 = r + q$ (2).

Subtracting (2) from (1), $p - r = 144$ and $p = r + 144$ follows.

Area $\triangle ABF = p = r + 144 = (AB)(BF) ÷ 2 = 36(12 - x) ÷ 2 = 18(12 - x)$

$\therefore r + 144 = 18(12 - x)$

$r = 216 - 18x - 144$

$r = 72 - 18x$ (3)

Area $\triangle DEF = r = (DE)(EF) ÷ 2 = 12x ÷ 2 = 6x$. $\therefore r = 6x$ (4)

Using (3) and (4), since $r = r$, $6x = 72 - 18x$. Solving, $x = 3$ and $r = 18$ follow.

The area of quadrilateral $BCDF = \text{area of square } BCDE - \text{area } \triangle DEF$

$= 144 - r$

$= 144 - 18$

$= 126$ cm$^2$

Therefore the area of quadrilateral $BCDF$ is 126 cm$^2$.

See alternative methods on the next page.
Method 2

Let \( x \) represent the length of \( BF \).

In \( \triangle ABF \) and \( \triangle ACD \), \( \angle A \) is common and \( \angle ABF = \angle ACD = 90^\circ \). It follows that \( \triangle ABF \) is similar to \( \triangle ACD \) and \( \frac{AB}{AC} = \frac{BF}{CD} \). Therefore \( \frac{36}{48} = \frac{x}{12} \) and \( x = 9 \).

Since \( BCDE \) is a square, \( BE \parallel CD \). Then quadrilateral \( BCDF \) is a trapezoid.

\[
\text{Area Trapezoid } BCDF = (BC)(BF + CD) \div 2 = 12(9 + 12) \div 2 = 6(21) = 126 \text{ cm}^2.
\]

Therefore the area of quadrilateral \( BCDF \) is 126 cm\(^2\).

Method 3

Position the diagram so that \( C \) is at the origin, \( A \) and \( B \) are on the positive \( y \)-axis and \( D \) is on the positive \( x \)-axis. \( C \) has coordinates \((0,0)\), \( B \) has coordinates \((0,12)\), \( A \) has coordinates \((0,48)\), and \( D \) has coordinates \((12,0)\).

Find the equation of the line containing \( AD \). The slope is \( \frac{-48}{12} = -4 \) and the line crosses the \( y \)-axis at \( A \) so the \( y \)-intercept is 48. Therefore the equation is \( y = -4x + 48 \).

The line containing \( BE \) is horizontal crossing the \( y \)-axis at \( B \). The equation of the line containing \( BE \) is \( y = 12 \).

\( F \) is the intersection of \( y = -4x + 48 \) and \( y = 12 \). Since \( y = y \) at the point of intersection, \(-4x + 48 = 12 \) and \( x = 9 \) follows. Therefore \( F \) has coordinates \((9,12)\) and \( BF = 9 \text{ cm} \).

Since \( BCDE \) is a square, \( BE \parallel CD \). Then quadrilateral \( BCDF \) is a trapezoid.

\[
\text{Area Trapezoid } BCDF = (BC)(BF + CD) \div 2 = 12(9 + 12) \div 2 = 6(21) = 126 \text{ cm}^2.
\]

Therefore the area of quadrilateral \( BCDF \) is 126 cm\(^2\).
A heart is constructed by attaching two white semi-circles to the hypotenuse of an isosceles right triangle whose equal sides measure $\sqrt{8}$ cm. The new figure is then mounted onto a rectangular piece of red construction paper as shown below. (The dashed line, the side measurement and the right angle symbol will not actually be on the finished card.)

You are going to write your valentine a message in red ink on the white region of the card.

Determine the total amount of area available for your special valentine greeting.

If the solver finds this problem too straight forward, try solving POTWE “Love is Blind Valentine” as well this week. This second problem may provide a greater challenge.
Problem of the Week
Problem D and Solution
Forever Mine Valentine

Problem
A heart is constructed by attaching two white semi-circles to the hypotenuse of an isosceles right triangle whose equal sides measure $\sqrt{8}$ cm. The new figure is then mounted onto a rectangular piece of red construction paper as shown below. (The dashed line, the side measurement and the right angle symbol will not actually be on the finished card.) You are going to write your valentine a message in red ink on the white region of the card. Determine the total amount of area available for your special valentine greeting.

Solution
Let $h$ represent the length of the hypotenuse. Let $r$ represent the radius of the semi-circles. Since the two semi-circles lie along the hypotenuse, $h = 4r$ or $r = \frac{h}{4}$.

Since the triangle is isosceles right, we can find $h$ using Pythagoras’ Theorem, $h^2 = (\sqrt{8})^2 + (\sqrt{8})^2 = 8 + 8 = 16$ and $h = 4$ cm follows.

Then $r = \frac{h}{4} = \frac{4}{4} = 1$ cm. Since there are two semi-circles of radius 1 cm, the total area of the two semi-circles is the same as the area of a full circle of radius 1 cm. The area of the two semi-circles is $\pi r^2 = \pi (1)^2 = \pi$ cm$^2$.

The triangle is isosceles right so we can use the lengths of the two equal sides as the base and height in the calculation of the area of the triangle. The area of the triangle is $\frac{1}{2}bh = \frac{1}{2}(\sqrt{8})(\sqrt{8}) = 4$ cm$^2$.

The total area for writing the message is $(\pi + 4)$ cm$^2$. This area is approximately 7.1 cm$^2$. Hopefully you can write that special message in a very limited space.

Happy Valentine’s Day.
Problem of the Week
Problem D
Walking is Good Exercise

Ali, Bill and Carl are lined up such that Ali is 100 m west of Bill and Carl is 160 m east of Bill. At noon, Carl begins to walk north at a constant rate of 41 \( \frac{m}{min} \) and Ali walks south at a constant rate of 20 \( \frac{m}{min} \). (Bill does not move.)

At what time will the distance between Carl and Bill be the twice the distance between Ali and Bill?
Problem of the Week
Problem D and Solution
Walking is Good Exercise

Problem
Ali, Bill and Carl are lined up such that Ali is 100 m west of Bill and Carl is 160 m east of Bill. At noon, Carl begins to walk north at a constant rate of \(41\ \text{m/min}\) and Ali walks south at a constant rate of \(20\ \text{m/min}\). (Bill does not move.) At what time will the distance between Carl and Bill be the twice the distance between Ali and Bill?

Solution
Solution 1
Let \(t\) represent the number of minutes until Carl’s distance to Bill is twice that of Ali’s distance to Bill.

In \(t\) minutes Ali will walk \(20t\) m and Carl will walk \(41t\) m. The following diagram contains the information showing Ali’s position at \(A\) and Carl’s position at \(C\) at time \(t\).

In the diagram both triangles are right triangles and we can use the Pythagorean Theorem to set up an equation.

\[
BC = 2AB \\
(BC)^2 = (2AB)^2 \\
(BC)^2 = 4(AB)^2 \\
(41t)^2 + (160)^2 = 4[(20t)^2 + (100)^2] \\
1681t^2 + 25600 = 4[400t^2 + 10000] \\
1681t^2 + 25600 = 1600t^2 + 40000 \\
81t^2 = 14400 \\
t^2 = \frac{14400}{81} \\
t = \frac{120}{9} \text{ since } t > 0 \\
t = \frac{40}{3} \text{ min}
\]

\[ \therefore \text{ in } 13\frac{1}{3} \text{ minutes (13 minutes 20 seconds), Carl’s distance to Bill will be twice that of Ali’s distance to Bill.} \]

In the second solution coordinate geometry will be used to solve the problem.
Solution 2

Represent Ali, Bill and Carl’s respective positions at noon as points on the x-axis so that Bill is positioned at the origin \( B(0, 0) \), Ali is positioned 100 units left of Bill at \( D(-100, 0) \) and Carl is positioned 160 units right of Bill at \( E(160, 0) \).

Let \( t \) represent the number of minutes until Carl’s distance to Bill is twice that of Ali’s distance to Bill.

In \( t \) minutes Ali will walk south \( 20t \) m to the point \( A(-100, -20t) \). In \( t \) minutes Carl will walk north \( 41t \) m to the point \( C(160, 41t) \).

The distance from a point \( P(x, y) \) to the origin can be found using the formula \( d = \sqrt{x^2 + y^2} \).

Then \( AB = \sqrt{(-100)^2 + (-20t)^2} = \sqrt{10000 + 400t^2} \) and \( CB = \sqrt{(160)^2 + (41t)^2} = \sqrt{25600 + 1681t^2} \).

\[
\begin{align*}
CB &= 2AB \\
\sqrt{25600 + 1681t^2} &= 2\sqrt{10000 + 400t^2} \\
25600 + 1681t^2 &= 4(10000 + 400t^2) \\
25600 + 1681t^2 &= 40000 + 1600t^2 \\
81t^2 &= 14400 \\
t^2 &= \frac{14400}{81} \\
t &= \frac{120}{9} \text{ since } t > 0 \\
t &= \frac{40}{3} \text{ min}
\end{align*}
\]

\( \therefore \) in \( 13\frac{1}{3} \) minutes (13 minutes 20 seconds), Carl’s distance to Bill will be twice that of Ali’s distance to Bill.
Problem of the Week
Problem D
Moats for Boats

A municipality has the opportunity to upgrade one of two square parks by building a circular moat for paddle boats in one of them. The surface of the moat is the area outside of a smaller circle and inside a second larger concentric circle. (From above, the moat looks somewhat like a donut.) West Park is 300 m by 300 m and East Park is 500 m by 500 m. Both parks are divided by horizontal and vertical grid lines spaced 100 m apart, as shown, creating nine and twenty five equal squares, respectively.

Both moat designs shown are based on municipal park rules:

- the outer edge of the moat must touch the midpoint of each of the four outer sides of the park; and
- the inner edge of the moat must pass through the four corners of the largest square totally inside the park created by the grid lines.

The city is interested in conserving water and will choose the plan which uses less water. Assuming that both moats will have an equal and constant depth, which Park will be chosen for the moat?
Problem

A municipality has the opportunity to upgrade one of two square parks by building a circular moat for paddle boats in one of them. The surface of the moat is the area outside of a smaller circle and inside a second larger concentric circle. (From above, the moat looks somewhat like a donut.) West Park is 300 m by 300 m and East Park is 500 m by 500 m. Both parks are divided by horizontal and vertical grid lines spaced 100 m apart, as shown, creating nine and twenty five equal squares, respectively. Both moat designs shown are based on municipal park rules:

- the outer edge of the moat must touch the midpoint of each of the four outer sides of the park;
- the inner edge of the moat must pass through the four corners of the largest square totally inside the park created by the grid lines.

The city is interested in conserving water and will choose the plan which uses less water. Assuming that both moats will have an equal and constant depth, which Park will be chosen for the moat?

Solution

To find the volume of each moat we need to find the area of each donut-like ring and multiply by the depth of the water. Since the depth of the water in each moat is the same and is constant, we need only compare the areas to determine which one is larger.

For the West Park moat, let the diameter of the inner circle be $d_1$, the diameter of the larger circle be $D_1$, the radius of the inner circle be $r_1$ and the radius of the larger circle be $R_1$.

For the East Park moat, let the diameter of the inner circle be $d_2$, the diameter of the larger circle be $D_2$, the radius of the inner circle be $r_2$ and the radius of the larger circle be $R_2$.

The calculations for West Park will be shown on the left and the calculations for East Park will be shown on the right.

**West Park**

The diameter of the inner circle is the length of the diagonal of the contained 100 m by 100 m square. Using the Pythagorean Theorem,

\[
d_1 = \sqrt{100^2 + 100^2} = \sqrt{20000} = 100\sqrt{2} \text{ m}
\]

\[
r_1 = \frac{1}{2}d_1 = 50\sqrt{2} \text{ m}
\]

**East Park**

The diameter of the inner circle is the length of the diagonal of the contained 300 m by 300 m square. Using the Pythagorean Theorem,

\[
d_2 = \sqrt{300^2 + 300^2} = \sqrt{180000} = 300\sqrt{2} \text{ m}
\]

\[
r_2 = \frac{1}{2}d_2 = 150\sqrt{2} \text{ m}
\]
West Park

The diameter of the outer circle is the width of the 300 m by 300 m square. It follows that

\[ D_1 = 300 \text{ m} \]
\[ R_1 = \frac{1}{2} D_1 = 150 \text{ m} \]

The diameter of the outer circle is the width of the 500 m by 500 m square. It follows that

\[ D_2 = 500 \text{ m} \]
\[ R_2 = \frac{1}{2} D_2 = 250 \text{ m} \]

The surface area of each moat can be determined by subtracting the area of the inner circle from the area of the outer circle in each case.

Let \( A_1 \) be the surface area of the West Park moat and \( A_2 \) be the surface area of the East Park moat.

<table>
<thead>
<tr>
<th>West Park</th>
<th>East Park</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 = \pi (R_1)^2 - \pi (r_1)^2 )</td>
<td>( A_1 = \pi (R_2)^2 - \pi (r_2)^2 )</td>
</tr>
<tr>
<td>[ = \pi (150)^2 - \pi (50\sqrt{2})^2 ]</td>
<td>[ = \pi (250)^2 - \pi (150\sqrt{2})^2 ]</td>
</tr>
<tr>
<td>[ = 22500\pi - 5000\pi ]</td>
<td>[ = 62500\pi - 45000\pi ]</td>
</tr>
<tr>
<td>[ = 17500\pi \text{ m}^2 ]</td>
<td>[ = 17500\pi \text{ m}^2 ]</td>
</tr>
</tbody>
</table>

This may be a surprising result. Both moats have equal surface areas. Since the depths of the moats are equal and uniform, the volume of water in each moat will be the same.

The municipality can choose either moat and base their decision on other things.

**For Further Thought:**

North Park is a 700 m by 700 m park. If a moat were constructed in a similar manner to the moat in either East Park or West Park, how would the volume of the North Park moat compare? Can you explain what is happening here?
The shaded region on the diagram is bounded by the lines whose equations are $5x + 2y = 30$, $x + 2y = 22$, $x = 0$, and $y = 0$.

Determine the area of the shaded region.
Problem of the Week
Problem D and Solution

A Shady Region

Problem

The shaded region on the diagram is bounded by the lines whose equations are $5x + 2y = 30$, $x + 2y = 22$, $x = 0$, and $y = 0$. Determine the area of the shaded region.

Solution

On the diagram, $l_1$ represents the line $5x + 2y = 30$ that crosses the $x$-axis at point $R$. $l_2$ represents the line $x + 2y = 22$ which crosses the $y$-axis at point $Q$.

Let $P(h, k)$ represent the point of intersection of $l_1$ and $l_2$. Then $h$ is the horizontal distance from the $y$-axis to $P$ and $k$ is the vertical distance from the $x$-axis to $P$. Let $O$ represent the origin.

To find the $x$-intercept of $l_1$ let $y = 0$ in $5x + 2y = 30$. Therefore the $x$-intercept is 6 and the coordinates of $R$ are $(6, 0)$.

To find the $y$-intercept of $l_2$ let $x = 0$ in $x + 2y = 22$. Therefore the $y$-intercept is 11 and the coordinates of $Q$ are $(0, 11)$.

To find the intersection of $l_1$ and $l_2$, we can use elimination.

\[
\begin{align*}
\text{l}_1 : & \quad 5x + 2y = 30 \\
\text{l}_2 : & \quad x + 2y = 22
\end{align*}
\]

Subtracting, we obtain,
\[
4x = 8
\]
\[
\therefore x = 2
\]

Substituting $x = 2$ in $l_1$, $10 + 2y = 30$ and $y = 10$. The coordinates of $P$, the point of intersection, are $(2, 10)$. Therefore, $h = 2$ and $k = 10$. To find the shaded area:

\[
\text{Area } PQOR = \text{Area } \triangle PQO + \text{Area } \triangle POR
\]
\[
= \frac{1}{2}h \times OQ + \frac{1}{2}k \times OR
\]
\[
= \frac{1}{2}(2)(11) + \frac{1}{2}(10)(6)
\]
\[
= 11 + 30
\]
\[
= 41
\]

Therefore the shaded area is 41 units$^2$. 
Problem of the Week
Problem D
Traversing the Triangle

A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

A triangle has an area of $72 \text{ cm}^2$. The length of one side is $12 \text{ cm}$ and the length of the median to this side is $13 \text{ cm}$.

Determine the length of the other two sides of the triangle.
Problem of the Week
Problem D and Solution
Traversing the Triangle

Problem
A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side. A triangle has an area of $72 \text{ cm}^2$. The length of one side is 12 cm and the length of the median to this side is 13 cm. Determine the length of the other two sides of the triangle.

Solution
Start with a diagram to represent the problem.

Let $AD$ be the 12 cm side and $EC$ be the 13 cm median drawn to that side. Draw the altitude from $E$ to side $AD$ meeting it at $B$. Let $EB$ be $h$.
Since $EC$ is a median, $AC = CD = \frac{1}{2}(AD) = \frac{1}{2}(12) = 6 \text{ cm}$.
Let $BC$, the distance along $AD$ from the altitude to the median, be $x$.
Then $AB = 6 - x$.
The area of the triangle is $72 \text{ cm}^2$ so $\frac{AD \times EB}{2} = 72$. Then $\frac{12h}{2} = 72$ and $h = 12 \text{ cm}$ follows.

$\triangle EBC$ is right angled so, using Pythagoras’ Theorem,
$x^2 = 13^2 - h^2 = 13^2 - 12^2 = 169 - 144 = 25$ and $x = 5 \text{ cm} \ (x > 0)$.
Then $BD = BC + CD = x + 6 = 11 \text{ cm}$ and $AB = 6 - x = 1 \text{ cm}$.

$\triangle EAB$ is right angled so, using Pythagoras’ Theorem, $EA^2 = EB^2 + AB^2 = 12^2 + 1^2 = 145$ and $EA = \sqrt{145} \text{ cm} \ (EA > 0)$

$\triangle EBD$ is right angled so, using Pythagoras’ Theorem,
$ED^2 = EB^2 + BD^2 = 12^2 + 11^2 = 144 + 121 = 265$ and $ED = \sqrt{265} \text{ cm} \ (ED > 0)$.

Therefore the lengths of the other two sides are $\sqrt{145} \text{ cm}$ and $\sqrt{265} \text{ cm}$. These lengths are approximately 12.0 cm and 16.3 cm.
Number Sense & Algebra
Problem of the Week

Problem D

Thinking Inside the Box

$Q$ is the point of intersection of the diagonals of one face of a cube whose edges have length 2 cm.

Determine the length of $QR$. 
Problem of the Week
Problem D and Solution
Thinking Inside the Box

Problem

Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 cm. Determine the length of QR.

Solution

Label the corners S and T as shown.

The faces of a cube are squares. The diagonals of a square right bisect each other. It follows that \( PQ = QT = \frac{1}{2} PT \). Since the face is a square, \( \angle PST = 90^\circ \) and \( \triangle PST \) is right angled. Using the Pythagorean Theorem, \( PT^2 = PS^2 + ST^2 = 2^2 + 2^2 = 8 \) and \( PT = \sqrt{8} \). Then \( PQ = \frac{1}{2} PT = \frac{\sqrt{8}}{2} \).

Because of the 3-dimensional nature of the problem it may not be obvious to all that \( \angle RPQ = 90^\circ \). To help visualize this, note that \( \angle RPS = 90^\circ \) because the face of the cube is a square. Rotate \( PS \) counterclockwise about point \( P \) on the side face of the cube so that the image of \( PS \) lies along \( PQ \). The corner angle will not change as a result of the rotation so \( \angle RPQ = \angle RPS = 90^\circ \).

We can now use the Pythagorean Theorem in \( \triangle RPQ \) to find the length \( RQ \).

\[
RQ^2 = RP^2 + PQ^2 = 2^2 + \left( \frac{\sqrt{8}}{2} \right)^2 = 4 + \frac{8}{4} = 4 + 2 = 6 \text{ and } RQ = \sqrt{6} \text{ cm.}
\]

\( \therefore \) the length of \( RQ \) is \( \sqrt{6} \) cm.

A couple of notes are in order at this point.

First, although the mathematics required to solve this problem was fairly straightforward some students would have found it difficult because of the three dimensional nature of the problem.

Second, we could have simplified \( PQ = \frac{1}{2} PT = \frac{\sqrt{8}}{2} \) to \( \sqrt{2} \) as follows:

\[
\frac{\sqrt{8}}{2} = \frac{\sqrt{4} \times \sqrt{2}}{2} = \frac{\sqrt{4} \times \sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.
\]

Often simplifying radicals is not a part of the curriculum at the grade 9 or 10 level. The calculation of \( RQ \) would have been simpler using \( PT = \sqrt{2} \).
Among grandfather’s papers a faded old piece of paper was found with the following written on it:

72 turkeys $ \square 6 7 . 9 \square$

The first and the last digits of the number that represented the total price of the turkeys have been replaced here with black squares as the digits had faded and were no longer legible. Each black square represents one digit.

Determine the missing digits.
Problem of the Week
Problem D and Solution
Turkey Time

Problem
Among grandfather's papers a faded old piece of paper was found with the following written on it:

72 turkeys $ \[ \text{■} \ 67 \ . \ 9 \text{■} \]

The first and the last digits of the number that represented the total price of the turkeys have been replaced here with black squares as the digits had faded and were no longer legible. Each black square represents one digit. Determine the missing digits.

Solution
Let the total price of 72 turkeys be $A\ 679B$ cents.

We know the total value of 72 turkeys. We could find the value of 1 turkey by dividing the total value by 72. Since the total value of the turkeys is divisible by 72, it is also divisible by the divisors of 72, namely 1,2,3,4,6,8,9,12,18,24,36 and 72.

If a number is divisible by 4, the last two digits of the number are divisible by 4. Therefore $9B$ is divisible by 4. The only two digit numbers beginning with 9 that are divisible by 4 are 92 and 96. So the only possible values for $B$ are 2 and 6. The value of the 72 turkeys is either $A\ 6792$ or $A\ 6796$. But the number must also be divisible by 8. To be divisible by 8, the last three digits of the number must be divisible by 8. Of the two possible numbers, 792 and 796, only 792 is divisible by 8. Therefore the last digit of the price is 2 and we now know that 72 turkeys cost $A\ 6792$ cents.

If a number is divisible by 9, the sum of the digits of the number is divisible by 9. So $A + 6 + 7 + 9 + 2 = A + 24$ must be divisible by 9. $A$ is a single digit from 0 to 9. The sum $A + 24$ is therefore an integer from 24 to 33. The only number in this range divisible by 9 is 27. It follows that $A + 24 = 27$ and $A = 3$.

∴ 72 turkeys cost $367.92 and the missing digits are 3 and 2. Each turkey costs $367.92 \div 72$ or $5.11$.

Note: This approach is very efficient but the solver must be careful. The number 4 divides 72 and any number that is divisible by 72. It does not follow that a number divisible by 4 is also divisible by 72. For example, 76 is divisible by 4 but not 72. In this solution we found a number divisible by both 8 and 9. Since 8 and 9 have no common factors, a number divisible by 8 and 9 is also divisible by 72.

It is possible to approach this question systematically using multiplication. In essence you can check possible values for $B$ by performing the multiplication. In this case, since $B = 2$, the solution can be found reasonably quickly. However, in general, this would not be an effective strategy.
Problem of the Week
Problem D
Left, Right, Left, Right, ...

Your friend writes down all of the integers starting from 0 in the following way:

Specifically, below every number there are two numbers: one on the left and one on the right. For example, below 3, the number 7 is on the left, and the number 8 is on the right. The numbers can be read in increasing order from top row to bottom row and from left-to-right within a row. Notice that we can get from 0 to 12 by going right (R), left (L) then right (R).

What number do you end at if you take the following path from 0:

L $\rightarrow$ L $\rightarrow$ R $\rightarrow$ L $\rightarrow$ L $\rightarrow$ R $\rightarrow$ L $\rightarrow$ R $\rightarrow$ L
Problem of the Week
Problem D and Solution
Left, Right, Left, Right, ...

Problem
Your friend writes down all of the integers starting from 0 as shown in the diagram to the right. Specifically, below every number there are two numbers: one on the left and one on the right. For example, below 3, the number 7 is on the left, and the number 8 is on the right. The numbers can be read in increasing order from top row to bottom row and from left-to-right within a row. Notice that we can get from 0 to 12 by going right (R), left (L) then right (R). What number do you end at if you take the following path from 0:

\[ \text{L} \rightarrow \text{L} \rightarrow \text{R} \rightarrow \text{L} \rightarrow \text{R} \rightarrow \text{L} \rightarrow \text{R} \rightarrow \text{L} \]

Solution
At first it may not be obvious how to proceed. We can see from the given diagram that L → L → R takes us from 0 to 1 to 3 to 8. But from there where do we go? We could write out more rows of the chart until we are able to make the required number of moves and then read off the final answer. This approach would work for a relatively small number of moves but would not be practical in general for “longer” strings of moves.

We will proceed by making an observation. When we perform a move to the left (L) from any number, we end up at an odd number. When we perform a move to the right (R) from any number, we end up at an even number. Is there a general formula which can be used when asked to move left (L)? Is there a general formula which can be used when asked to move right (R)?

The diagram to the right has two parts of the tree circled. Can we discover a pattern that takes us from each initial number to the odd and even numbers below? To get from 1 to 3 we could add 2 and to get from 1 to 4 we could add 3. But doing the same with 6 would not get us to 13 and 14. As we go down the chart, each new row has twice as many numbers as the row above. Let’s try multiplying the initial number by 2 and then seeing what is necessary to get to the odd and even number below. If we double 1 we get 2. Then we would need to add 1 to get to the odd number 3 below and add 2 to get to the even number 4 below. Does this work with the 6? If we double 6 and add 1, we get 13. It appears to work. If we double 6 and add 2, we get 14. It also appears to work.
So it would appear that if we make a move left (L) from any number \(a\) in the tree, the resulting number is one more than twice the value of \(a\). That is, a move left (L) from \(a\) takes us to the number \(2a + 1\) in the tree.

It would appear that if we make a move right (R) from any number \(a\) in the tree, the resulting number is two more than twice the value of \(a\). That is, a move right (R) from \(a\) takes us to the number \(2a + 2\) in the tree.

These results are true but unproven. This relationship has worked for all of the rows we have sampled but we have not proven it true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

The following table shows the result of performing the given sequence of moves

\[ L \rightarrow L \rightarrow R \rightarrow L \rightarrow L \rightarrow R \rightarrow L \rightarrow R \rightarrow L. \]

<table>
<thead>
<tr>
<th>Initial Number</th>
<th>Move</th>
<th>Calculation</th>
<th>Next Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>L</td>
<td>(2(0) + 1)</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>L</td>
<td>(2(1) + 1)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>R</td>
<td>(2(3) + 2)</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>L</td>
<td>(2(8) + 1)</td>
<td>17</td>
</tr>
<tr>
<td>17</td>
<td>L</td>
<td>(2(17) + 1)</td>
<td>35</td>
</tr>
<tr>
<td>35</td>
<td>R</td>
<td>(2(35) + 2)</td>
<td>72</td>
</tr>
<tr>
<td>72</td>
<td>L</td>
<td>(2(72) + 1)</td>
<td>145</td>
</tr>
<tr>
<td>145</td>
<td>R</td>
<td>(2(145) + 2)</td>
<td>292</td>
</tr>
<tr>
<td>292</td>
<td>L</td>
<td>(2(292) + 1)</td>
<td>585</td>
</tr>
<tr>
<td>585</td>
<td>R</td>
<td>(2(585) + 2)</td>
<td>1172</td>
</tr>
<tr>
<td>1172</td>
<td>L</td>
<td>(2(1172) + 1)</td>
<td>2345</td>
</tr>
</tbody>
</table>

Starting at 0 and making the moves \(L \rightarrow L \rightarrow R \rightarrow L \rightarrow L \rightarrow R \rightarrow L \rightarrow R \rightarrow L\), we end at the number 2345.

Once we determined the operations required to make a move left (L) and a move right (R), the problem was quite straightforward to solve. It would be possible to write a computer program which would accept any length sequence of Ls and Rs, and get the computer to determine the final position in this specific tree.
Problem of the Week
Problem D
Full of Beans

Canada celebrates its birthday every July 1st on Canada Day. The colours red and white, the colours on the Canadian flag, are seen everywhere.

In honour of Canada Day, a company produced a jumbo bag of candy containing only red and white jelly beans in the ratio 14:11 respectively.

Khan Di loves jelly beans and purchased one of the special packages of candy. He decided to make the candy last as long as possible by eating exactly four red jelly beans and three white jelly beans each day. Finally one day he ate the last four red jelly beans along with three white jelly beans leaving three white jelly beans in the bag.

How many candies were in the original bag of red and white jelly beans?
Problem of the Week
Problem D and Solution
Full of Beans

Problem
Canada celebrates its birthday every July 1st on Canada Day. The colours red and white, the colours on the Canadian flag, are seen everywhere. In honour of Canada Day, a company produced a jumbo bag of candy containing only red and white jelly beans in the ratio 14:11 respectively. Khan Di loves jelly beans and purchased one of the special packages of candy. He decided to make the candy last as long as possible by eating exactly four red jelly beans and three white jelly beans each day. Finally one day he ate the last four red jelly beans along with three white jelly beans leaving three white jelly beans in the bag. How many candies were in the original bag of red and white jelly beans?

Solution
Let \( d \) represent the number of days Khan ate four red jelly beans and three white jelly beans. Since he ate four red jelly beans each day for \( d \) days, he ate \( 4d \) red jelly beans. Since he ate three white jelly beans each day for \( d \) days, he ate \( 3d \) white jelly beans. But there were three white jelly beans left so Khan had a total of \( 3d + 3 \) white jelly beans.

The ratio of red jelly beans to white jelly beans was 14:11.

\[
\frac{4d}{3d + 3} = \frac{14}{11}
\]

“Cross-multiplying” to simplify we obtain:

\[
44d = 42d + 42
\]
\[
2d = 42
\]
\[
d = 21
\]

Therefore Khan was able to follow his pattern for 21 days.

Since \( d = 21 \), \( 4d = 4(21) = 84 \) and he had 84 red jelly beans. Also \( 3d + 3 = 3(21) + 3 = 66 \) so Khan had 66 white jelly beans. The bag contained \( 84 + 66 = 150 \) jelly beans.

\[\therefore \text{the special bag of candy contained 150 jelly beans. The number 150 is significant because that is how old Canada will be on July 1, 2017. The company is getting a jump start on Canada’s Sesquicentennial celebrations! That is, on July 1, 2017 Canada celebrates its 150}^{\text{th}} \text{ birthday.}\]
Problem of the Week

Problem D

Actress or Comedian - Dare to Compare

Holly Woods is a popular young actress and Joe King is an up and coming young comedian. Joe has an income which is five-eighths of Holly’s income. Joe’s expenses are one-half those of Holly, and Joe saves 40% of his income.

Determine the percentage of her income that Holly Woods saves.
Problem of the Week

Problem D and Solution

Actress or Comedian - Dare to Compare

Problem

Holly Woods is a popular young actress and Joe King is an up and coming young comedian. Joe has an income which is five-eighths of Holly’s income. Joe’s expenses are one-half those of Holly, and Joe saves 40% of his income. Determine the percentage of her income that Holly Woods saves.

Solution

Solution 1 Using only one variable

Let \( h \) represent Holly’s income. Then Joe’s income is \( \frac{5}{8}h \).

Since Joe saves 40% of his income, his expenses are 100% – 40% = 60% of his income. Therefore, his expenses are \( 60\% \times \frac{5}{8}h = \frac{60}{100} \times \frac{5}{8}h = \frac{3}{8}h \).

Joe’s expenses are one-half of Holly’s expenses so Holly’s expenses are twice Joe’s expenses. Therefore, Holly’s expenses are \( 2 \times \frac{3}{8}h = \frac{3}{4}h = 0.75h = 75\% \) of \( h \). Since Holly’s expenses are 75% of her income, she saves 100% – 75% = 25% of her income.

\( \therefore \) Holly Woods saves 25% of her income.

Solution 2 Using two variables

Let \( x \) represent Holly’s income and \( y \) represent her expenses. Then Joe’s income is \( \frac{5}{8}x \) and his expenses are \( \frac{1}{2}y \).

Since Joe saves 40% of his income, his expenses are 60% of his income.

\[
\begin{align*}
\frac{1}{2}y & = 0.60 \times \frac{5}{8}x \\
\frac{1}{2}y & = \frac{6}{10} \times \frac{5}{8}x \\
\frac{1}{2}y & = \frac{3}{8}x \\
y & = \frac{3}{4}x \\
\end{align*}
\]

Holly saves whatever is left of her income after expenses. Therefore Holly saves

\[
x - y = x - \frac{3}{4}x = \frac{1}{4}x = 0.25x = 25\% \text{ of } x.
\]

\( \therefore \) Holly Woods saves 25% of her income.
Solution 3 Using two variables a bit differently

Let $8x$ represent Holly’s income and $2y$ represent her expenses. Then Joe’s income is $\frac{5}{8}(8x) = 5x$ and his expenses are $\frac{1}{2}(2y) = y$.

Since Joe saves 40% of his income, his expenses are 60% of his income.

\[
y = 0.60 \times 5x
y = \frac{6}{10} \times 5x
y = 3x
\]

Holly earns $8x$ and her expenses are $2y$ so her savings are $8x - 2y$. We want the ratio of her savings to her income, \[
\frac{8x - 2y}{8x} = \frac{8x - 2(3x)}{8x} = \frac{2x}{8x} = \frac{1}{4} \text{ or } 25\%.
\]

\[\therefore\] Holly Woods saves 25% of her income.
Problem of the Week
Problem D
The Case of the Missing Square

Rectangle $DEFG$ has square $ABCD$ removed leaving an area of 92 m$^2$. Side $AE = 4$ m and side $CG = 8$ m.

Determine the original area of rectangle $DEFG$. 
Problem of the Week
Problem D and Solution
The Case of the Missing Square

Problem
Rectangle \( DEFG \) has square \( ABCD \) removed leaving an area of 92 m\(^2\). Side \( AE = 4 \) m and side \( CG = 8 \) m. Determine the original area of rectangle \( DEFG \).

Solution
Let \( x \) represent the side length of square \( ABCD \). In the diagram, extend \( CB \) to intersect \( EF \) at \( H \). This creates rectangle \( AEHB \) and rectangle \( CHFG \). Then \( FG = EA + AD = (4 + x) \) m and \( EH = DC = x \) m.

\[
\begin{align*}
\text{Area } AEHB + \text{Area } CHFG &= \text{Remaining Area} \\
AE \times EH + CG \times FG &= 92 \\
4x + 8(4 + x) &= 92 \\
4x + 32 + 8x &= 92 \\
12x + 32 &= 92 \\
12x &= 60 \\
x &= 5 \text{ m}
\end{align*}
\]

Since \( x = 5 \) m, \( DG = 8 + x = 13 \) m and \( FG = 4 + x = 9 \) m. The original area of rectangle \( DEFG = DG \times FG = 13 \times 9 = 117 \text{ m}^2 \).
Problem of the Week

Problem D

The Key Factor

A specific type of six-digit number is formed by repeating a three-digit number, for example, 265 265 or 325 325 or 143 143.

Determine the largest integer which will divide all such numbers.
Problem of the Week
Problem D and Solution
The Key Factor

Problem
A specific type of six-digit number is formed by repeating a three-digit number, for example, 265 265 or 325 325 or 143 143. Determine the largest integer which will divide all such numbers.

Solution
To get started look at the prime factorization of each of the given numbers.

\[
265 \times 5 \times 53 = 5 \times 7 \times 7 \times 759 = 5 \times 7 \times 11 \times 689 = 5 \times 7 \times 11 \times 13 \times 53
\]

\[
325 \times 5 \times 5 \times 65 = 5 \times 5 \times 13 \times 013 = 5 \times 5 \times 7 \times 1 \times 859 = 5 \times 5 \times 7 \times 11 \times 169 = 5 \times 5 \times 7 \times 11 \times 13 \times 13
\]

\[
143 \times 7 \times 20 \times 449 = 7 \times 11 \times 1 \times 859 = 7 \times 11 \times 11 \times 169 = 7 \times 11 \times 11 \times 13 \times 13
\]

All of the numbers are divisible by \(7 \times 11 \times 13 = 1001\). These are the only three factors common to all three numbers.

Pick another six-digit number formed by repeating a three-digit number and test to see if it is also divisible by 1 001. The number 246 246, for example, is \(1001 \times 246\). It would appear that our conjecture (guess) is correct but it has not been proven.

Let \(abc abc\) be any six digit number formed by repeating the three-digit number \(abc\).

\[
\text{Then } abc abc = \text{ abc000 + abc} = 1000 \times abc + abc = 1000 \times abc + 1 \times abc = 1 \times 001 \times abc
\]

Since \(abc abc = 1001 \times abc\), it is divisible by 1 001. A specific number \(abc abc\) may also have other divisors but 1 001 is the largest divisor common to all such numbers. In the first example 265 265 = \(1001 \times 5 \times 53\) and in the third example 143 143 = \(1001 \times 11 \times 13\). Both numbers have other factors but no other common factors. In some cases there will be other common factors but not in general.

This problem is not hard if you initially “get it”. The solution presented shows an approach that can be taken when you may not be certain where to begin. Try some specific examples and then attempt to generalize based on what you observe from the specific examples. Also note that discovering that 1 001 worked for the three given examples and the test example is not sufficient to make a general conclusion that 1 001 divides all such numbers.
Problem of the Week
Problem D
Tangled Triangles

In the diagram, \( A(0, a) \) lies on the \( y \)-axis above the origin. If \( \triangle ABD \) and \( \triangle COB \) have the same area, determine the value of \( a \).
Problem of the Week
Problem D and Solution
Tangled Triangles

Problem
In the diagram, \(A(0, a)\) lies on the \(y\)-axis above the origin. If \(\triangle ABD\) and \(\triangle COB\) have the same area, determine the value of \(a\).

Solution

Solution 1

Draw rectangle \(EDFC\) with sides parallel to the \(x\) and \(y\)-axes so that \(O(0, 0)\) is on \(ED\) and \(B(2, -1)\) is on \(DF\). Since \(EC\) is parallel to the \(x\)-axis and \(E\) is on the \(y\)-axis, \(E\) has coordinates \((0, 2)\). Since \(CF\) is parallel to the \(y\)-axis, \(F\) has the same \(x\)-coordinate as \(C\). Since \(FD\) is parallel to the \(x\)-axis, \(F\) has the same \(y\)-coordinate as \(D\) and \(B\). Therefore the coordinates of \(F\) are \((3, -1)\).

To find the area of \(\triangle COB\), subtract the areas of \(\triangle CEO\), \(\triangle ODB\), and \(\triangle BFC\) from the area of rectangle \(EDFC\).

In rectangle \(EDFC\), \(EC = 3 - 0 = 3\) and \(ED = 2 - (-1) = 3\). The area of rectangle \(EDFC = EC \times ED = 3 \times 3 = 9\) units\(^2\).

In \(\triangle CEO\), \(EC = 3\) and \(EO = 2 - 0 = 2\). The area of \(\triangle ECO = \frac{EC \times EO}{2} = \frac{3 \times 2}{2} = 3\) units\(^2\).

In \(\triangle ODB\), \(OD = 0 - (-1) = 1\) and \(DB = 2 - 0 = 2\). The area of \(\triangle ODB = \frac{OD \times DB}{2} = \frac{1 \times 2}{2} = 1\) unit\(^2\).

In \(\triangle BFC\), \(BF = 3 - 2 = 1\) and \(CF = 2 - (-1) = 3\). The area of \(\triangle BFC = \frac{BF \times CF}{2} = \frac{1 \times 3}{2} = 1.5\) units\(^2\).

\[
\text{Area } \triangle COB = \text{Area Rectangle } EDFC - \triangle CEO - \triangle ODB - \triangle BFC
\]
\[
= 9 - 3 - 1 - 1.5
\]
\[
= 3.5\text{ units}^2
\]

But the area \(\triangle ABD = \triangle COB\) so the area of \(\triangle ABD = 3.5\) units\(^2\).

In \(\triangle ABD\), \(AD = a - (-1) = a + 1\) and \(DB = 2 - 0 = 2\) so

\[
\text{Area } \triangle ABD = \frac{AD \times DB}{2}
\]
\[
3.5 = \frac{(a + 1) \times 2}{2}
\]
\[
3.5 = a + 1
\]
\[
2.5 = a
\]

\(\therefore\) the value of \(a\) is 2.5.
Solution 2

Determine the equation of the line containing $C(3, 2)$ and $B(2, -1)$.

The slope of the line is $\frac{2-(-1)}{3-2} = 3$. The equation of the line is of the form $y = 3x + b$. Substitute $x = 3$, $y = 2$ to determine the value of $b$. $2 = 3(3) + b$ and $b = -7$ follows. Therefore the equation of the line containing $C$ and $B$ is $y = 3x - 7$.

Let $P(p, 0)$ be the $x$-intercept of the line. Substituting into $y = 3x - 7$ we obtain $0 = 3p - 7$ and $p = \frac{7}{3}$ follows.

To determine the area of $\triangle COB$ determine the sum of the areas of $\triangle COP$ and $\triangle BOP$.

In $\triangle COP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the $x$-axis to $C(3, 2)$, which is 2 units. The area of $\triangle COP = \frac{\frac{7}{3} \times 2}{2} = \frac{7}{3}$ units$^2$.

In $\triangle BOP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the $x$-axis to $B(2, -1)$, which is 1 unit. The area of $\triangle BOP = \frac{\frac{7}{3} \times 1}{2} = \frac{7}{6}$ units$^2$.

$$ \text{Area } \triangle COB = \text{Area } \triangle COP + \text{Area } \triangle BOP $$

$$ = \frac{7}{3} + \frac{7}{6} $$

$$ = \frac{14}{6} + \frac{7}{6} $$

$$ = \frac{21}{6} $$

$$ = \frac{7}{2} \text{ units}^2 $$

But the area $\triangle ABD = \triangle COB$ so the area of $\triangle ABD = \frac{7}{2}$ units$^2$.

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$ so

$$ \text{Area } \triangle ABD = \frac{AD \times DB}{2} $$

$$ = \frac{(a + 1) \times 2}{2} $$

$$ = 2a + 2 $$

$$ = 5 $$

$$ \therefore a = \frac{5}{2} = 2.5. $$
Problem of the Week
Problem D
An Average Bowl

In his latest game, Mark Striker bowled 199 and raised his average from 177 to 178. Mark would like to raise his average to 180 after bowling his next game. What would Mark need to bowl on his next game to accomplish his goal?
Problem of the Week
Problem D and Solution
An Average Bowl

Problem
In his latest game, Mark Striker bowled 199 and raised his average from 177 to 178. Mark would like to raise his average to 180 after bowling his next game.

What would Mark need to bowl on his next game to accomplish his goal?

Solution
Solution 1
Let \( n \) be the number of games bowled to achieve his 177 average. His total points scored in \( n \) games is his average times \( n \). Therefore, Mark has \( 177n \) total points in \( n \) games.

To compute his average after bowling the 199 game, we take his new total points and divide by \( n + 1 \), the new number of games.

\[
\text{Average} = \frac{\text{Total Points}}{\text{Games Played}} = \frac{177n + 199}{n + 1}
\]

\[
178 = \frac{177n + 199}{n + 1}
\]

\[
178(n + 1) = 177n + 199
\]

\[
178n + 178 = 177n + 199
\]

\[
n = 21
\]

Prior to bowling the 199 game, Mark had bowled 21 games. So after bowling the 199 game, Mark has bowled 22 games. Mark wants to have a 180 average after bowling his 23rd game. The difference between his total points after 23 games with a 180 average and his total points after bowling 22 games with a 178 average must be his score on the 23rd game.

\[
\text{Score on 23rd Game} = 23 \times 180 - 22 \times 178 = 4140 - 3916 = 224
\]

Therefore Mr. Striker must bowl 224 on his next game to raise his average from 178 to 180.
Solution 2

Mark's score of 199 is $199 - 177 = 22$ points above his previous average. Mark raised his average 1 point. Therefore, his latest game with the 199 score must have been his 22nd game.

To raise his average 2 points in his 23rd game he must bowl $2 \times 23 = 46$ points above his 178 average. He must bowl $178 + 46 = 224$ in his next game.

\[ \therefore \text{Mark must bowl 224 in his next game to move his average from 178 to 180.} \]

We can verify our results:

Average on 21 games is 177.

Average on 22 games $= \frac{21 \times 177 + 199}{22} = \frac{3916}{22} = 178$

Average on 23 games $= \frac{22 \times 178 + 224}{23} = \frac{4140}{23} = 180$
Problem of the Week
Problem D
Creating Cubes

A wooden cube is cut into $n$ cubes each of edge length 1 unit. The combined surface area of the $n$ cubes is eight times the surface area of the original uncut cube.

Determine the edge length of the original uncut cube.
Problem of the Week
Problem D and Solution
Creating Cubes

Problem
A wooden cube is cut into \( n \) cubes each of edge length 1 unit. The combined surface area of the \( n \) cubes is eight times the surface area of the original uncut cube. Determine the edge length of the original uncut cube.

Solution
Let the edge length of the uncut cube be \( x \) units, \( x > 0 \). After cutting the original cube into \( n \) cubes of edge length 1 unit, there will be \( x^3 \) cubes of edge length 1 unit. It follows that \( n = x^3 \).

Each of the 6 sides of the original cube has area \( x^2 \) so the total surface area of the original cube is \( 6x^2 \).

Consider one of the smaller cubes. The surface area of one the six faces is \( 1 \) unit\(^2\). So the total surface area of one of these smaller cubes is \( 6 \) units\(^2\). The total surface area of the \( n \) smaller cubes is \( 6n \) units\(^2\).

Since the total surface area of the \( n \) cubes is eight times the surface area of the original uncut cube,

\[
\text{new surface area} = 8 \times \text{original surface area} \quad 6n = 8(6x^2)
\]

Dividing both sides by 6,

\[
n = 8x^2
\]

But \( n = x^3 \) so

\[
x^3 = 8x^2
\]

Dividing by \( x^2 \) since \( x > 0 \),

\[
x = 8
\]

Therefore, the edge length of the original uncut cube was 8 units.

Extension:
Note, that if the combined surface area of the \( n \) cubes of edge length 1 unit was \( Q \) times the surface area of the original uncut cube, then the edge length of the original uncut cube would have been \( Q \) units. Can you see why?
A perfect square is an integer that is the square of another integer. For example, $36$ is a perfect square since $36 = 6^2$.

Determine the smallest perfect square greater than $4000$ that is divisible by $392$. 

$1024$
Problem

A perfect square is an integer that is the square of another integer. For example, 36 is a perfect square since \(36 = 6^2\). Determine the smallest perfect square greater than 4 000 that is divisible by 392.

Solution

In order to understand the nature of perfect squares we should examine the prime factorization of a few perfect squares.

From the example, \(36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2\).
The number \(144 = 12^2 = (3 \times 4)^2 = 3^2 \times (2^2)^2 = 3^2 \times 2^4\).

From the above examples we note that, for each perfect square, the exponent on each of its prime factors is even. For some integer \(a\), if \(m\) is an even integer greater than or equal to zero then \(a^m\) is a perfect square. For example, \(2^6 \times 3^4 \times 6^2 = 64 \times 81 \times 36 = 186 \text{,}624 = 432^2\).

The number \(392 = 8 \times 49 = 2^3 \times 7^2\). This clearly is not a perfect square since the exponent of 2 is odd. We require another factor of 2 to obtain a multiple of 392 that is a perfect square, namely \(2 \times 392 = 784\). We now must multiply 784 by a perfect square so that the product is greater than 4 000. Since \(4 \text{,}000 \div 784 \approx 5.1\) we must multiply 784 by the smallest perfect square greater than 5.1 to create a perfect square greater than 4 000. In this case that perfect square is 9. So the smallest perfect square greater than 4 000 that is divisible by 392 is \(784 \times 9 = 7 \text{,}056 = 84^2\). We can also show that \(7 \text{,}056 \div 392 = 18\) to verify that 7056 is divisible by 392.

\(\therefore\) the smallest perfect square greater than 4 000 that is divisible by 392 is \(7 \text{,}056\).
Problem of the Week

Problem D

Several Areas of Interest

The area of \( \triangle ACD \) is twice the area of square \( BCDE \). Square \( BCDE \) has sides of length 12 cm. \( AD \) intersects \( BE \) at \( F \).

Determine the area of quadrilateral \( BCDF \).
Problem of the Week
Problem D and Solution
Several Areas of Interest

Problem
The area of $\triangle ACD$ is twice the area of square $BCDE$. Square $BCDE$ has sides of length 12 cm. $AD$ intersects $BE$ at $F$. Determine the area of quadrilateral $BCDF$.

Solution
The area of square $BCDE = 12 \times 12 = 144$ cm$^2$. The area of $\triangle ACD$ equals twice the area of square $BCDE$. Therefore area $\triangle ACD = 288$ cm$^2$.

The area of a triangle is calculated using the formula $\text{base} \times \text{height} \div 2$. It follows that:

\[
\text{Area } \triangle ACD = (CD) \times (AC) \div 2
\]

\[
288 = 12 AC \div 2
\]

\[
288 = 6 AC
\]

\[
48 \text{ cm} = AC
\]

But $AC = AB + BC$ so $48 = AB + 12$ and it follows that $AB = 36$ cm.

From this point, we will present three different approaches to obtaining the required area.

Method 1
Let the area of $\triangle ABF$ be $p$, quadrilateral $BCDF$ be $q$ and $\triangle DEF$ be $r$. Let $FE = x$. Since $BE = 12$, $BF = 12 - x$.

Area $\triangle ACD = 288 = p + q$ (1) and area square $BCDE = 144 = r + q$ (2).

Subtracting (2) from (1), $p - r = 144$ and $p = r + 144$ follows.

Area $\triangle ABF = p = r + 144 = (AB)(BF) \div 2 = 36(12 - x) \div 2 = 18(12 - x)$

\[
\therefore r + 144 = 18(12 - x)
\]

\[
r = 216 - 18x - 144
\]

\[
r = 72 - 18x \quad (3)
\]

Area $\triangle DEF = r = (DE)(EF) \div 2 = 12x \div 2 = 6x$. $\therefore r = 6x \quad (4)$

Using (3) and (4), since $r = r$, $6x = 72 - 18x$. Solving, $x = 3$ and $r = 18$ follow.

The area of quadrilateral $BCDF = \text{area of square } BCDE - \text{area } \triangle DEF$

\[
= 144 - r
\]

\[
= 144 - 18
\]

\[
= 126 \text{ cm}^2
\]

Therefore the area of quadrilateral $BCDF$ is $126 \text{ cm}^2$.

See alternative methods on the next page.
Method 2

Let $x$ represent the length of $BF$.

In $\triangle ABF$ and $\triangle ACD$, $\angle A$ is common and $\angle ABF = \angle ACD = 90^\circ$. It follows that $\triangle ABF$ is similar to $\triangle ACD$ and $\frac{AB}{AC} = \frac{BF}{CD}$. Therefore $\frac{36}{48} = \frac{x}{12}$ and $x = 9$.

Since $BCDE$ is a square, $BE \parallel CD$. Then quadrilateral $BCDF$ is a trapezoid.

\[
\text{Area Trapezoid } BCDF = (BC)(BF + CD) \div 2 = 12(9 + 12) \div 2 = 6(21) = 126 \text{ cm}^2
\]

Therefore the area of quadrilateral $BCDF$ is 126 cm$^2$.

Method 3

Position the diagram so that $C$ is at the origin, $A$ and $B$ are on the positive $y$-axis and $D$ is on the positive $x$-axis. $C$ has coordinates $(0,0)$, $B$ has coordinates $(0,12)$, $A$ has coordinates $(0,48)$, and $D$ has coordinates $(12,0)$.

Find the equation of the line containing $AD$. The slope is $-\frac{48}{12} = -4$ and the line crosses the $y$-axis at $A$ so the $y$-intercept is 48. Therefore the equation is $y = -4x + 48$.

The line containing $BE$ is horizontal crossing the $y$-axis at $B$. The equation of the line containing $BE$ is $y = 12$.

$F$ is the intersection of $y = -4x + 48$ and $y = 12$. Since $y = y$ at the point of intersection, $-4x + 48 = 12$ and $x = 9$ follows. Therefore $F$ has coordinates $(9,12)$ and $BF = 9$ cm.

Since $BCDE$ is a square, $BE \parallel CD$. Then quadrilateral $BCDF$ is a trapezoid.

\[
\text{Area Trapezoid } BCDF = (BC)(BF + CD) \div 2 = 12(9 + 12) \div 2 = 6(21) = 126 \text{ cm}^2
\]

Therefore the area of quadrilateral $BCDF$ is 126 cm$^2$. 
Problem of the Week
Problem D
A Sour Taste

A container is filled with a mixture of water and vinegar in the ratio 2:1, by volume. Another container, with twice the volume of the first container, is filled with a mixture of water and vinegar in the ratio 3:1, by volume. The contents of the two containers are emptied into a third container.

Determine the ratio of water to vinegar, by volume, in the third container.
Problem of the Week
Problem D and Solution
A Sour Taste

Problem
A container is filled with a mixture of water and vinegar in the ratio 2:1, by volume. Another container, with twice the volume of the first container, is filled with a mixture of water and vinegar in the ratio 3:1, by volume. The contents of the two containers are emptied into a third container. Determine the ratio of water to vinegar, by volume, in the third container.

Solution
Solution 1
Let $V$ represent the volume of the first container. Then $2V$ represents the volume of the second container.

Since the ratio of water to vinegar, by volume, in the first container is 2:1 then $\frac{2}{3}$ of the volume of the first container is water. That is, the volume of water in the first container is $\frac{2}{3}V$ and the volume of vinegar in the first container is $\frac{1}{3}V$.

Since the ratio of water to vinegar, by volume, in the second container is 3:1 then $\frac{3}{4}$ of the volume of the second container is water. That is, the volume of water in the second container is $\frac{3}{4}(2V) = \frac{3}{2}V$ and the volume of vinegar in the second container is $\frac{1}{4}(2V) = \frac{1}{2}V$.

When the contents of the two containers are combined into the third container, the volume of water is $\frac{2}{3}V + \frac{3}{2}V = \frac{4}{6}V + \frac{9}{6}V = \frac{13}{6}V$ and the volume of vinegar is $\frac{1}{3}V + \frac{1}{2}V = \frac{2}{6}V + \frac{3}{6}V = \frac{5}{6}V$.

The ratio of water to vinegar, by volume, in the third container is $\frac{13}{6} : \frac{5}{6} = 13 : 5$.

Solution 2
Let $\frac{1}{6}$ of the contents of the first container be a unit of volume. Since the ratio of water to vinegar, by volume, in the first container is 2:1, then 4 units of volume are water and 2 units of volume are vinegar, a total of 6 units of volume in the first container.

Since the second container has twice the volume of the first container, the second container has 12 units of volume. The ratio of water to vinegar, by volume, in the second container is 3:1 so 9 units of volume are water and 3 units of volume are vinegar.

When the two containers are combined there is a total of $6 + 12 = 18$ units of volume, $4 + 9 = 13$ of which are water and $2 + 3 = 5$ of which are vinegar.

$\therefore$ the ratio of water to vinegar, by volume, in the third container is $13 : 5$. 
Problem of the Week
Problem D
Check Please

Debit and credit cards contain account numbers which consist of many digits. Often, when purchasing items online you are asked to type in your account number. Because there are so many digits it is easy to type the number incorrectly. The last digit of the number is a “check digit” which can be used to quickly verify the validity of the number. A common algorithm used for verifying that numbers have been entered correctly is called the Luhn Algorithm. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.

**Example 1**
Number: 135792

Reversal: 297531

\[ A = 2 + 7 + 3 \]
\[ = 12 \]
\[ 2 \times 9 = 18 \]
\[ 2 \times 5 = 10 \]
\[ 2 \times 1 = 2 \]

\[ B = (1+8)+(1+0)+2 \]
\[ = 9 + 1 + 2 \]
\[ = 12 \]

\[ C = 12 + 12 \]
\[ = 24 \]

\[ C \] does not end in zero. The number is not valid.

**Example 2**
Number: 1357987

Reversal: 7897531

\[ A = 7 + 9 + 5 + 1 \]
\[ = 22 \]
\[ 2 \times 8 = 16 \]
\[ 2 \times 7 = 14 \]
\[ 2 \times 3 = 6 \]

\[ B = (1+6)+(1+4)+6 \]
\[ = 7 + 5 + 6 \]
\[ = 18 \]

\[ C = 22 + 18 \]
\[ = 40 \]

\[ C \] ends in zero. The number is valid.

The number 8664 R8R4 R6R9 0359 is a valid number when verified by the Luhn Algorithm. \( R \) is an integer from 0 to 9 occurring four times in the number. (It may also be one of the existing known digits.) Determine all possible values of \( R \).
Problem of the Week
Problem D and Solution
Check Please

Problem
Debit and credit cards contain account numbers which consist of many digits. Often, when purchasing items online you are asked to type in your account number. Because there are so many digits it is easy to type the number incorrectly. Most companies use a “check digit” to quickly verify the validity of the number. One commonly used algorithm for verifying that numbers have been entered correctly is called the Luhn Algorithm. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid. The steps performed in the Luhn Algorithm are illustrated in the flowchart to the right.

The number 8664 R84 R69 0359 is a valid number when verified by the Luhn Algorithm. R is an integer from 0 to 9 occurring four times in the number. (It may also be one of the existing known digits.) Determine all possible values of R.

Solution
Solution 1

When the digits of the number are reversed the resulting number is 9530 9R6R 4R8R 4668. The sum of the digits in the odd positions is

\[ A = 9 + 3 + 9 + 6 + 4 + 8 + 4 + 6 = 49 \]

When the digits in the remaining positions are doubled, the following products are obtained:

\[ 2 \times 5 = 10; \ 2 \times 0 = 0; \ 2 \times R = 2R; \ 2 \times R = 2R; \ 2 \times R = 2R; \ 2 \times R = 2R; 2 \times 6 = 12; \text{ and } 2 \times 8 = 16. \]

Let \( x \) represent the sum of the digits of \( 2R. \)

When the digit sums from each of the products are added, the sum is:

\[ B = (1 + 0) + 0 + x + x + x + x + (1 + 2) + (1 + 6) = 1 + 0 + 4x + 3 + 7 = 4x + 11 \]

\( C \) is the sum of \( A \) and \( B \), so \( C = 49 + 4x + 11 = 60 + 4x. \)

When an integer from 0 to 9 is doubled and the digits of the product are added together, what are the possible sums which can be obtained?

<table>
<thead>
<tr>
<th>Original Digit</th>
<th>R</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twice the Original Digit</td>
<td>2R</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>The Sum of the Digits of 2R</td>
<td></td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

Notice that the sum of the digits of twice the original digit can only be an integer from 0 to 9 inclusive. It follows that the only values for \( x \) are the integers from 0 to 9.
To satisfy the Luhn Algorithm, the units digit of $C$ must be zero. We want $60 + 4x$ to be an integer greater than or equal to 60 such that the units digit is 0.

Can $60 + 4x = 60$? When $4x = 0$, then $x = 0$, $60 + 4x = 0$ and $R = 0$. That is, when $R = 0$, $2R = 0$ and the sum of the digits of $2R$ is $x = 0$. This value of $R$ produces a valid number.

Can $60 + 4x = 70$? When $4x = 10$, then $x = 2.5$ and $60 + 4x = 70$. But $x$ in an integer value so this is not possible.

Can $60 + 4x = 80$? When $4x = 20$, then $x = 5$, $60 + 4x = 80$ and $R = 7$. That is, when $R = 7$, $2R = 14$ and the sum of the digits of $2R$ is $x = 5$. This value of $R$ produces a valid number.

Can $60 + 4x = 90$? When $4x = 30$, then $x = 7.5$ and $60 + 4x = 90$. But $x$ in an integer value so this is not possible.

Can $60 + 4x = 100$? When $4x = 40$, then $x = 10$ and $60 + 4x = 100$. But $x$ is an integer from 0 to 9 inclusive, so this is not possible.

Every integer ending in zero that is larger than 100 would produce a value for $x$ greater than 10. There are no more possible values for $x$ or $R$.

The two valid possibilities for $R$ are 0 and 7.

When $R = 0$, the valid number is 8664 0804 0609 0359.

When $R = 7$, the valid number is 8664 7874 7679 0359.

**Solution 2**

The second solution looks at each of the possible values of $R$ and applies the Luhn Algorithm to the resulting number. A computer program or spreadsheet could be developed to solve this problem efficiently.

Remember that $A$ is the sum of the digits in the odd positions of the reversal. Each of the digits in the even positions of the reversal are doubled and $B$ is the sum of the sum of the digits of each of these products. $C$ is the sum $A + B$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>Number</th>
<th>Reversal</th>
<th>$A$</th>
<th>Double Even Digits</th>
<th>$B$</th>
<th>$C$</th>
<th>Valid / Invalid</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8664 0804 0609 0359</td>
<td>9530 9060 4080 4668</td>
<td>49</td>
<td>10, 0, 0, 0, 0, 0, 12, 16</td>
<td>11</td>
<td>60</td>
<td>Valid</td>
</tr>
<tr>
<td>1</td>
<td>8664 1814 1619 0359</td>
<td>9530 9161 4181 4668</td>
<td>49</td>
<td>10, 0, 2, 2, 2, 12, 16</td>
<td>19</td>
<td>68</td>
<td>Invalid</td>
</tr>
<tr>
<td>2</td>
<td>8664 2824 2629 0359</td>
<td>9530 9262 4282 4668</td>
<td>49</td>
<td>10, 0, 4, 4, 4, 4, 12, 16</td>
<td>27</td>
<td>76</td>
<td>Invalid</td>
</tr>
<tr>
<td>3</td>
<td>8664 3834 3639 0359</td>
<td>9530 9363 4383 4668</td>
<td>49</td>
<td>10, 0, 6, 6, 6, 6, 12, 16</td>
<td>35</td>
<td>84</td>
<td>Invalid</td>
</tr>
<tr>
<td>4</td>
<td>8664 4844 4649 0359</td>
<td>9530 9464 4484 4668</td>
<td>49</td>
<td>10, 0, 8, 8, 8, 12, 16</td>
<td>43</td>
<td>92</td>
<td>Invalid</td>
</tr>
<tr>
<td>5</td>
<td>8664 5854 5659 0359</td>
<td>9530 9565 4585 4668</td>
<td>49</td>
<td>10, 0, 10, 10, 10, 12, 16</td>
<td>15</td>
<td>64</td>
<td>Invalid</td>
</tr>
<tr>
<td>6</td>
<td>8664 6864 6669 0359</td>
<td>9530 9666 4686 4668</td>
<td>49</td>
<td>10, 0, 12, 12, 12, 12, 16</td>
<td>23</td>
<td>72</td>
<td>Invalid</td>
</tr>
<tr>
<td>7</td>
<td>8664 7874 7679 0359</td>
<td>9530 9767 4787 4668</td>
<td>49</td>
<td>10, 0, 14, 14, 14, 12, 16</td>
<td>31</td>
<td>80</td>
<td>Valid</td>
</tr>
<tr>
<td>8</td>
<td>8664 8884 8689 0359</td>
<td>9530 9868 4888 4668</td>
<td>49</td>
<td>10, 0, 16, 16, 16, 12, 16</td>
<td>39</td>
<td>88</td>
<td>Invalid</td>
</tr>
<tr>
<td>9</td>
<td>8664 9894 9699 0359</td>
<td>9530 9969 4989 4668</td>
<td>49</td>
<td>10, 0, 18, 18, 18, 12, 16</td>
<td>47</td>
<td>96</td>
<td>Invalid</td>
</tr>
</tbody>
</table>

The two valid possibilities for $R$ are 0 and 7.
Problem of the Week

Problem D

Forever Mine Valentine

A heart is constructed by attaching two white semi-circles to the hypotenuse of an isosceles right triangle whose equal sides measure $\sqrt{8}$ cm. The new figure is then mounted onto a rectangular piece of red construction paper as shown below. (The dashed line, the side measurement and the right angle symbol will not actually be on the finished card.)

You are going to write your valentine a message in red ink on the white region of the card.

Determine the total amount of area available for your special valentine greeting.

If the solver finds this problem too straight forward, try solving POTWE “Love is Blind Valentine” as well this week. This second problem may provide a greater challenge.
Problem of the Week
Problem D and Solution
Forever Mine Valentine

Problem
A heart is constructed by attaching two white semi-circles to the hypotenuse of an isosceles right triangle whose equal sides measure $\sqrt{8}$ cm. The new figure is then mounted onto a rectangular piece of red construction paper as shown below. (The dashed line, the side measurement and the right angle symbol will not actually be on the finished card.) You are going to write your valentine a message in red ink on the white region of the card. Determine the total amount of area available for your special valentine greeting.

Solution
Let $h$ represent the length of the hypotenuse. Let $r$ represent the radius of the semi-circles. Since the two semi-circles lie along the hypotenuse, $h = 4r$ or $r = \frac{h}{4}$. Since the triangle is isosceles right, we can find $h$ using Pythagoras’ Theorem, $h^2 = (\sqrt{8})^2 + (\sqrt{8})^2 = 8 + 8 = 16$ and $h = 4$ cm follows.

Then $r = \frac{h}{4} = \frac{4}{4} = 1$ cm. Since there are two semi-circles of radius 1 cm, the total area of the two semi-circles is the same as the area of a full circle of radius 1 cm. The area of the two semi-circles is $\pi r^2 = \pi (1)^2 = \pi$ cm$^2$.

The triangle is isosceles right so we can use the lengths of the two equal sides as the base and height in the calculation of the area of the triangle. The area of the triangle is $\frac{1}{2}bh = \frac{1}{2}(\sqrt{8})(\sqrt{8}) = 4$ cm$^2$.

The total area for writing the message is $(\pi + 4)$ cm$^2$. This area is approximately 7.1 cm$^2$. Hopefully you can write that special message in a very limited space. Happy Valentine’s Day.
Grandmother has four grandchildren, two boys and two girls. When the ages of the grandchildren are multiplied together, the product of their ages is $67\,184$. Only one of the grandchildren is a teenager and the oldest grandchild is under 40. The difference in ages between the oldest and youngest grandchild is thirty years.

Determine the ages of Grandmother’s grandchildren.
Problem of the Week
Problem D and Solution
A Problem for the Ages

Problem

Grandmother has four grandchildren, two boys and two girls. When the ages of the grandchildren are multiplied together, the product of their ages is 67 184. Only one of the grandchildren is a teenager and the oldest grandchild is under 40. The difference in ages between the oldest and youngest grandchild is thirty years. Determine the ages of Grandmother’s grandchildren.

Solution

The first task is to factor 67 184. Since the number is even, we can divide out powers of 2 such that $67\,184 = 2^4 \times 4199$. After some trial we discover that $67\,184 = 1 \times 2^4 \times 13 \times 17 \times 19$. Notice the inclusion of the number 1 as one of the factors. (One of the grandchildren could be 1.)

We can rule out 1 as a possible age since no matter how we combine the other factors to create the three remaining ages, at least one of the ages will be greater than 40.

It should also be noted that multiplying any of the factors 13, 17, 19 together, in any combination, will produce a number greater than 40. Since all of the grandchildren are under 40, we can eliminate these possibilities.

We are left with combining various powers of 2 with the other factors to form the four ages.

If $2^4$ is one of the ages, the only possibility for the ages is \{13,16,17,19\}. This is not a valid combination because we know only one grandchild is a teenager.

If $2^3$ is one of the ages, we can multiply the fourth factor of 2 by one of the other ages to produce three possible sets of ages: \{8,26,17,19\}, \{8,13,34,19\}, and \{8,13,17,38\}. In each of these cases there are two teenagers so none of these sets produces valid ages.

If $2^2$ is one of the ages, we could multiply one of the ages by 4 or multiply two out of three of the other ages by 2. Multiplying either 13, 17 or 19 by 4 produces a product greater than 40 so this is not a possibility. Multiplying two out of three ages 13, 17, 19 by 2 produces the following results: \{4,26,34,19\}, \{4,26,17,38\}, and \{4,13,34,38\}. In each of these cases exactly one of the grandchildren is a teenager. But only the first possibility also satisfies the condition that the difference in ages between the oldest and the youngest is 30. Therefore, \{4,26,34,19\} is a possibility for the ages.

There is one final possibility to consider. If one of the grandchildren is 2, we could double each of the remaining ages to create the set \{2,26,34,38\}. This possibility is quickly dismissed since none of the ages is in the range 13 to 19. That is, there are no teenagers in this list.

Therefore the only possible ages for Grandmother’s grandchildren are 4, 19, 26 and 34.
Problem of the Week
Problem D
Walking is Good Exercise

Ali, Bill and Carl are lined up such that Ali is 100 m west of Bill and Carl is 160 m east of Bill. At noon, Carl begins to walk north at a constant rate of 41 \(\frac{\text{m}}{\text{min}}\) and Ali walks south at a constant rate of 20 \(\frac{\text{m}}{\text{min}}\). (Bill does not move.)

At what time will the distance between Carl and Bill be the twice the distance between Ali and Bill?
Problem of the Week
Problem D and Solution
Walking is Good Exercise

Problem
Ali, Bill and Carl are lined up such that Ali is 100 m west of Bill and Carl is 160 m east of Bill. At noon, Carl begins to walk north at a constant rate of $41 \, \text{m/min}$ and Ali walks south at a constant rate of $20 \, \text{m/min}$. (Bill does not move.) At what time will the distance between Carl and Bill be the twice the distance between Ali and Bill?

Solution
Solution 1
Let $t$ represent the number of minutes until Carl’s distance to Bill is twice that of Ali’s distance to Bill.

In $t$ minutes Ali will walk $20t$ m and Carl will walk $41t$ m. The following diagram contains the information showing Ali’s position at $A$ and Carl’s position at $C$ at time $t$.

In the diagram both triangles are right triangles and we can use the Pythagorean Theorem to set up an equation.

\[
BC = 2AB \\
(BC)^2 = (2AB)^2 \\
(BC)^2 = 4(AB)^2 \\
(41t)^2 + (160)^2 = 4 \left[ (20t)^2 + (100)^2 \right] \\
1681t^2 + 25600 = 4 \left[ 400t^2 + 10000 \right] \\
1681t^2 + 25600 = 1600t^2 + 40000 \\
81t^2 = 14400 \\
t^2 = \frac{14400}{81} \\
t = \frac{120}{9} \, \text{min} \quad \text{since} \quad t > 0 \\
t = \frac{40}{3} \, \text{min}
\]

\therefore in 13\frac{1}{3} \, \text{minutes} (13 \text{ minutes} 20 \text{ seconds}), Carl’s distance to Bill will be twice that of Ali’s distance to Bill.

In the second solution coordinate geometry will be used to solve the problem.
Solution 2

Represent Ali, Bill and Carl’s respective positions at noon as points on the x-axis so that Bill is positioned at the origin $B(0, 0)$, Ali is positioned 100 units left of Bill at $D(-100, 0)$ and Carl is positioned 160 units right of Bill at $E(160, 0)$.

Let $t$ represent the number of minutes until Carl’s distance to Bill is twice that of Ali’s distance to Bill. In $t$ minutes Ali will walk south $20t$ m to the point $A(-100, -20t)$. In $t$ minutes Carl will walk north $41t$ m to the point $C(160, 41t)$.

The distance from a point $P(x, y)$ to the origin can be found using the formula $d = \sqrt{x^2 + y^2}$.

Then $AB = \sqrt{(-100)^2 + (-20t)^2} = \sqrt{10000 + 400t^2}$ and $CB = \sqrt{(160)^2 + (41t)^2} = \sqrt{25600 + 1681t^2}$.

\[
\begin{align*}
CB &= 2AB \\
\sqrt{25600 + 1681t^2} &= 2\sqrt{10000 + 400t^2} \\
25600 + 1681t^2 &= 4(10000 + 400t^2) \\
25600 + 1681t^2 &= 40000 + 1600t^2 \\
81t^2 &= 14400 \\
t^2 &= \frac{14400}{81} \\
t &= \frac{120}{9} \text{ since } t > 0 \\
t &= \frac{40}{3} \text{ min}
\end{align*}
\]

∴ in $13 \frac{1}{3}$ minutes (13 minutes 20 seconds), Carl’s distance to Bill will be twice that of Ali’s distance to Bill.
People from the town of Formidable like to pose problems involving fractions. Here is one of their problems.

There are some positive integers \( a \) and \( c \) such that
\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = 18.
\]

For example, when \( a = 36 \) and \( c = 5 \), the value of the numerator is
\[
\frac{a}{c} + \frac{a}{2} + 1 = \frac{36}{5} + \frac{36}{2} + 1 = \frac{72}{10} + \frac{180}{10} + \frac{10}{10} = \frac{262}{10} = \frac{131}{5},
\]
the value of the denominator is
\[
\frac{2}{a} + \frac{2}{c} + 1 = \frac{2}{36} + \frac{2}{5} + 1 = \frac{1}{18} + \frac{2}{5} + 1 = \frac{5}{90} + \frac{36}{90} + \frac{90}{90} = \frac{131}{90},
\]
and the left side of the equation simplifies to
\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = \frac{131}{5} \div \frac{131}{90} = \frac{131}{5} \times \frac{90}{131} = 18.
\]
Since 18 is also the value of the right side of the equation, then the ordered pair \( (a, c) = (36, 5) \) satisfies the given equation.

Many ordered pairs satisfy the given equation so we will add a restriction.

Determine the total number of ordered pairs \((a, c)\) that satisfy the equation such that \( a + 3c \leq 99 \).
Problem of the Week
Problem D and Solution
Formidable Fractions

Problem

There are some positive integers \(a\) and \(c\) such that \(\left(\frac{a}{c} + \frac{a}{2} + 1\right) \left(\frac{2}{a} + \frac{2}{c} + 1\right) = 18\). Determine the total number of ordered pairs \((a, c)\) that satisfy the equation such that \(a + 3c \leq 99\).

Solution

Solution 1

\[
\left(\frac{a}{c} + \frac{a}{2} + 1\right) \left(\frac{2}{a} + \frac{2}{c} + 1\right) = 18
\]

Find common denominators:

\[
\left(\frac{2a + ac + 2c}{2c} + \frac{2}{2c} \cdot \frac{2c}{2c} + \frac{2c}{2c}\right) = 18
\]

Simplifying:

\[
\left(\frac{2a + ac + 2c}{2c}\right) = 18
\]

Multiplying by the reciprocal:

\[
\frac{(2a + ac + 2c)}{2c} \times \frac{ac}{(2c + 2a + ac)} = 18
\]

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to \(\frac{ac}{2c} = 18\). Since \(c \neq 0\), the expression further simplifies to \(\frac{a}{2} = 18\) or \(a = 18(2) = 36\). Substituting \(a = 36\) into \(a + 3c \leq 99\) we obtain \(36 + 3c \leq 99\) which simplifies to \(3c \leq 63\) and \(c \leq 21\) follows.

But \(c \geq 1\) and \(c\) is an integer so \(1 \leq c \leq 21\). The value of \(a\) is 36 for each of the 21 possible values of \(c\).

\(\therefore\) there are 21 ordered pairs \((a, c)\) that satisfy the problem.
Solution 2

\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = 18
\]

Multiply numerator and denominator by \(2ac\):

\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} \times \frac{2ac}{2ac} = 18
\]

Simplify:

\[
\frac{2a^2 + a^2c + 2ac}{4c + 4a + 2ac} = 18
\]

Factoring:

\[
\frac{a(2a + ac + 2c)}{2(2c + 2a + ac)} = 18
\]

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to \(\frac{a}{2} = 18\) and \(a = 36\) follows.

Substituting \(a = 36\) into \(a + 3c \leq 99\) we obtain \(36 + 3c \leq 99\) which simplifies to \(3c \leq 63\) and \(c \leq 21\). But \(c \geq 1\) and \(c\) is an integer so \(1 \leq c \leq 21\). The value of \(a\) is 36 for each of the 21 possible values of \(c\).

\(\therefore\) there are 21 ordered pairs \((a, c)\) that satisfy the problem.
Problem of the Week
Problem D
A Pane - Full Experience

A contractor is building new houses. Each house has a large window with width 2 feet and height 8 feet to be made up of eight identical 1 foot by 2 feet glass panes.

The contractor wants each house to look slightly different. The window size is fixed, but he plans on arranging the glass panes in the windows so that no two windows have the same configuration. Two possible arrangements of the glass panes are given below.

Assuming that the glass panes cannot be cut, how many different arrangements can be made?
Problem

A contractor is building new houses. Each house has a large window with width 2 feet and height 8 feet to be made up of eight identical 1 foot by 2 feet glass panes. The contractor wants each house to look slightly different. The window size is fixed, but he plans on arranging the glass panes in the windows so that no two windows have the same configuration. Two possible arrangements of the glass panes are given below. Assuming that the glass panes cannot be cut, how many different arrangements can be made?

Solution

Let’s consider the ways that the glass panes can be arranged. First, notice that there must always be an even number of window panes that have a vertical orientation (standing on end).

- **All glass panes are horizontal, none are vertical**
  This can only be done one way.

- **Six glass panes are horizontal, two are vertical**
  There could be 0, 1, 2, 3, 4, 5, or 6 horizontal panes below the two vertical panes. So there are 7 ways that six glass panes are horizontal and two are vertical.

- **Four glass panes are horizontal, four are vertical**
  We need to consider sub cases:
  
  - **Case 1: There are no horizontal panes between the vertical panes.**
    There could be 0, 1, 2, 3, or 4 horizontal panes below the four vertical panes. So there are 5 ways that four glass panes are horizontal, four are vertical and there are no horizontal panes between the vertical panes.
  
  - **Case 2: There is one horizontal pane between the vertical panes.**
    There could be 0, 1, 2, or 3 horizontal panes below the bottom two vertical panes. So there are 4 ways that four glass panes are horizontal, four are vertical and there is one horizontal pane between the vertical panes.
  
  - **Case 3: There are two horizontal panes between the vertical panes.**
    There could be 0, 1 or 2 horizontal panes below the bottom two vertical panes. So there are 3 ways that four glass panes are horizontal, four are vertical and there are two horizontal panes between the vertical panes.
Case 4: There are three horizontal panes between the vertical panes. There could be 0 or 1 horizontal pane below the bottom two vertical panes. So there are 2 ways that four glass panes are horizontal, four are vertical and there are three horizontal panes between the vertical panes.

Case 5: There are four horizontal panes between the vertical panes. Since there are no more glass panes left, there can only be 0 horizontal panes below the bottom two vertical panes. So there is 1 way that four glass panes are horizontal, four are vertical and there are four horizontal panes between the vertical panes.

Two glass panes are horizontal, six are vertical

Case 1: There are no vertical panes between the horizontal panes. There could be 0, 2, 4 or 6 vertical panes below the bottom horizontal pane. So there are 4 ways that two glass panes are horizontal, six are vertical and there are no vertical panes between the horizontal panes.

Case 2: There are two vertical panes between the horizontal panes. There could be 0, 2 or 4 vertical panes below the bottom horizontal pane. So there are 3 ways that two glass panes are horizontal, six are vertical and there are two vertical panes between the horizontal panes.

Case 3: There are four vertical panes between the horizontal panes. There could be 0 or 2 vertical panes below the bottom horizontal pane. So there are 2 ways that two glass panes are horizontal, six are vertical and there are four vertical panes between the horizontal panes.

Case 4: There are six vertical panes between the horizontal panes. Since there are no more glass panes left, there can only be 0 vertical panes below the bottom horizontal pane. So there is 1 way that two glass panes are horizontal, six are vertical and there are six vertical panes between the horizontal panes.

All glass panes are vertical, none are horizontal

This can only be done one way.

Therefore, the total number of different configurations of the glass panes is

\[ 1 + 7 + (5 + 4 + 3 + 2 + 1) + (4 + 3 + 2 + 1) + 1 = 34. \]

So the contractor can build 34 houses before he has to start duplicating window patterns.
Problem of the Week
Problem D
Moats for Boats

A municipality has the opportunity to upgrade one of two square parks by building a circular moat for paddle boats in one of them. The surface of the moat is the area outside of a smaller circle and inside a second larger concentric circle. (From above, the moat looks somewhat like a donut.) West Park is 300 m by 300 m and East Park is 500 m by 500 m. Both parks are divided by horizontal and vertical grid lines spaced 100 m apart, as shown, creating nine and twenty five equal squares, respectively.

Both moat designs shown are based on municipal park rules:

- the outer edge of the moat must touch the midpoint of each of the four outer sides of the park; and

- the inner edge of the moat must pass through the four corners of the largest square totally inside the park created by the grid lines.

The city is interested in conserving water and will choose the plan which uses less water. Assuming that both moats will have an equal and constant depth, which Park will be chosen for the moat?
Problem

A municipality has the opportunity to upgrade one of two square parks by building a circular moat for paddle boats in one of them. The surface of the moat is the area outside of a smaller circle and inside a second larger concentric circle. (From above, the moat looks somewhat like a donut.) West Park is 300 m by 300 m and East Park is 500 m by 500 m. Both parks are divided by horizontal and vertical grid lines spaced 100 m apart, as shown, creating nine and twenty five equal squares, respectively. Both moat designs shown are based on municipal park rules:

- the outer edge of the moat must touch the midpoint of each of the four outer sides of the park; and
- the inner edge of the moat must pass through the four corners of the largest square totally inside the park created by the grid lines.

The city is interested in conserving water and will choose the plan which uses less water. Assuming that both moats will have an equal and constant depth, which Park will be chosen for the moat?

Solution

To find the volume of each moat we need to find the area of each donut-like ring and multiply by the depth of the water. Since the depth of the water in each moat is the same and is constant, we need only compare the areas to determine which one is larger.

For the West Park moat, let the diameter of the inner circle be $d_1$, the diameter of the larger circle be $D_1$, the radius of the inner circle be $r_1$ and the radius of the larger circle be $R_1$.

For the East Park moat, let the diameter of the inner circle be $d_2$, the diameter of the larger circle be $D_2$, the radius of the inner circle be $r_2$ and the radius of the larger circle be $R_2$.

The calculations for West Park will be shown on the left and the calculations for East Park will be shown on the right.

**West Park**

The diameter of the inner circle is the length of the diagonal of the contained 100 m by 100 m square. Using the Pythagorean Theorem,

\[
\begin{align*}
d_1 &= \sqrt{100^2 + 100^2} \\
&= \sqrt{20000} \\
&= 100\sqrt{2} \text{ m}
\end{align*}
\]

\[
\begin{align*}
r_1 &= \frac{1}{2}d_1 \\
&= 50\sqrt{2} \text{ m}
\end{align*}
\]

**East Park**

The diameter of the inner circle is the length of the diagonal of the contained 300 m by 300 m square. Using the Pythagorean Theorem,

\[
\begin{align*}
d_2 &= \sqrt{300^2 + 300^2} \\
&= \sqrt{180000} \\
&= 300\sqrt{2} \text{ m}
\end{align*}
\]

\[
\begin{align*}
r_2 &= \frac{1}{2}d_2 \\
&= 150\sqrt{2} \text{ m}
\end{align*}
\]
West Park

The diameter of the outer circle is the width of the 300 m by 300 m square. It follows that

\[ D_1 = 300 \text{ m} \]
\[ R_1 = \frac{1}{2} D_1 \]
\[ = 150 \text{ m} \]

East Park

The diameter of the outer circle is the width of the 500 m by 500 m square. It follows that

\[ D_2 = 500 \text{ m} \]
\[ R_2 = \frac{1}{2} D_2 \]
\[ = 250 \text{ m} \]

The surface area of each moat can be determined by subtracting the area of the inner circle from the area of the outer circle in each case.

Let \( A_1 \) be the surface area of the West Park moat and \( A_2 \) be the surface area of the East Park moat.

West Park

\[ A_1 = \pi (R_1)^2 - \pi (r_1)^2 \]
\[ = \pi (150)^2 - \pi (50\sqrt{2})^2 \]
\[ = 22500\pi - 5000\pi \]
\[ = 17500\pi \text{ m}^2 \]

East Park

\[ A_1 = \pi (R_2)^2 - \pi (r_2)^2 \]
\[ = \pi (250)^2 - \pi (150\sqrt{2})^2 \]
\[ = 62500\pi - 45000\pi \]
\[ = 17500\pi \text{ m}^2 \]

This may be a surprising result. Both moats have equal surface areas. Since the depths of the moats are equal and uniform, the volume of water in each moat will be the same.

The municipality can choose either moat and base their decision on other things.

For Further Thought:

North Park is a 700 m by 700 m park. If a moat were constructed in a similar manner to the moat in either East Park or West Park, how would the volume of the North Park moat compare? Can you explain what is happening here?
Problem of the Week
Problem D
Tiddlywinks Anyone?

The game of Tiddlywinks dates back to 1955. Rumour has it that the game has made a comeback with some students in recent years. Last year, our local TPL, Tiddlywink Premier League, launched its first season. Each team in the league played each of the other teams in the league the same number of times.

In its second season the TPL has grown. The total number of matches played this season will be twice the number of matches played in the first season and there will be 40% more teams than in season one. Each team in the league will still play each of the other teams in the league the same number of times as in season one.

How many teams were in the TPL in season one?
Problem of the Week

Problem D and Solution

Tiddlywinks Anyone?

Problem

The game of Tiddlywinks dates back to 1955. Rumour has it that the game has made a comeback with students in recent years. Last year, our local TPL, Tiddlywink Premier League, launched its first season. Each team in the league played each of the other teams in the league the same number of times. In its second season the TPL has grown. The total number of matches played this season will be twice the number of matches played in the first season and there will be 40% more teams than in season one. The number of games that each team plays each other team stays the same from season to season. How many teams were in the TPL in season one?

Solution

Let $t$ represent the number of teams in the league in season one. Then, the number of teams in the league in the second season is $1.4t$. Note that both $t$ and $1.4t$ must be positive integers.

We need to first establish how many games are played. If there were 4 teams, $A$, $B$, $C$, $D$, and each team played every other team once, then there would be 6 games played: $AB$, $AC$, $AD$, $BC$, $BD$, $CD$. Often, when counting something like this we “double-count”. That is, there are 4 teams and each plays the 3 other teams so there are $4 \times 3 = 12$ games. But each game is counted twice. We need to divide the result by 2. So in a 4 team league, with each team playing each other team once, there are $\frac{4 \times 3}{2} = 6$ games played.

In general, if there are $t$ teams and each team plays every other team once, there would be $\frac{t(t-1)}{2}$ games played. If each of $t$ teams plays every other team $n$ times, there would be $n \left( \frac{t(t-1)}{2} \right)$ games played. If each of $1.4t$ teams plays every other team $n$ times, there would be $n \left( \frac{1.4t(1.4t-1)}{2} \right)$ games played.

We know that the number of games played in season two is twice the number of games played in season one. So,

$$n \left( \frac{1.4t(1.4t - 1)}{2} \right) = 2 \left[ n \left( \frac{t(t-1)}{2} \right) \right]$$

Dividing both sides by $\frac{n}{2}$, $n \neq 0$, this simplifies to

$$1.4t(1.4t - 1) = 2t(t - 1)$$

Dividing both sides by $t$, $t \neq 0$, this simplifies to

$$1.4(1.4t - 1) = 2(t - 1)$$

$$1.96t - 1.4 = 2t - 2$$

$$0.6 = 0.04t$$

$$15 = t$$

Therefore, in season one there were 15 teams in the TPL.
When fifty consecutive *even* integers are added together, their sum is 3 250. Determine the value of the largest number.

The formula for the sum of the first $n$ positive integers is \( \frac{n(n+1)}{2} \). This formula *may* be helpful in the solution of the above problem.
Problem of the Week
Problem D and Solution
Sum Time Soon

Problem

When fifty consecutive even integers are added together, their sum is 3250. Determine the value of the largest number. (The formula for the sum of the first \( n \) positive integers is \( \frac{n(n+1)}{2} \). This formula may be helpful in the solution of the above problem.)

Solution

Solution 1

In this solution we will solve using patterns.

Let \( a \) represent the smallest number. Since the numbers are even, they increase by 2. So the second number is \( (a + 2) \), the third is \( (a + 4) \), the fourth is \( (a + 6) \), and so on. What does the fiftieth number look like?

A closer look at the numbers reveals that the second number is \( (a + 1(2)) \), the third is \( (a + 2(2)) \), the fourth is \( (a + 3(2)) \), and so on. Following the pattern, the fiftieth number is \( (a + 49(2)) = a + 98 \). Then

\[
\begin{align*}
a + (a + 2) + (a + 4) + (a + 6) + \cdots + (a + 98) &= 3250 \\
50a + 2 + 4 + 6 + \cdots + 98 &= 3250 \\
50a + 2(1 + 2 + 3 + \cdots + 49) &= 3250 \\
50a + 2 \left( \frac{49 \times 50}{2} \right) &= 3250, \text{ using the helpful formula.} \\
50a + 2450 &= 3250 \\
50a &= 800 \\
a &= 16 \\
\therefore a + 98 &= 114
\end{align*}
\]

\( \therefore \) the largest number is 114.
Solution 2

In this solution we will use averages to solve the problem

Let \( A \) represent the average of the fifty numbers. The average times the number of integers equals the sum of the integers. Since the sum of the fifty integers is 3 250, then \( 50A = 3 250 \) and \( A = 65 \).

Now the numbers in the sequence are consecutive even integers. The average is odd. It follows that 25 numbers are below the average and 25 numbers are above. We are looking for the 25\textsuperscript{th} even number after the average. In fact we want the 25\textsuperscript{th} even number after the even number 64, the first even number below the average. This number is easily found, \( 64 + 25(2) = 64 + 50 = 114 \).

\[ \therefore \text{the largest number is 114}. \]

Solution 3

In this solution we will use arithmetic sequences. This solution is presented last since many students in grade 9 or 10 have not encountered arithmetic sequences yet.

In this solution we will use arithmetic sequence formulas to solve the problem. An arithmetic sequence is a sequence in which each term after the first is obtained from the previous term by adding a constant. The general term, \( t_n \), of an arithmetic sequence is \( t_n = a + (n - 1)d \) where \( a \) is the first term, \( d \) is the difference between consecutive terms and \( n \) is the term number. The sum of the first \( n \) terms of an arithmetic sequence, \( S_n \), can be found using the formula \( S_n = \frac{n}{2}(2a + (n - 1)d) \). \( a \), \( d \), and \( n \) are the same variables used in the general term formula.

Let \( a \) represent the first term in the sequence. Since the numbers in the sequence are consecutive even integers, the numbers go up by two. Therefore \( d = 2 \). Since there are 50 terms in the sequence, \( n = 50 \). The sum of the fifty numbers in the sequence is 3250 so \( S_{50} = 3250 \).

\[
S_n = \frac{n}{2}(2a + (n - 1)d) \\
3250 = \frac{50}{2}(2a + (50 - 1)(2)) \\
3250 = 25(2a + 49(2)) \\
\text{Dividing by 25,} \quad 130 = 2a + 98 \\
32 = 2a \\
16 = a \\
\]

Since we want the largest number in the sequence, we are looking for the fiftieth term.

Using \( t_n = a + (n - 1)d \), with \( a = 16 \), \( d = 2 \), \( n = 50 \)

\[
t_{50} = 16 + 49(2) \\
t_{50} = 114 \\
\]

\[ \therefore \text{the largest number is 114}. \]
The shaded region on the diagram is bounded by the lines whose equations are $5x + 2y = 30$, $x + 2y = 22$, $x = 0$, and $y = 0$.

Determine the area of the shaded region.
Problem of the Week

Problem D and Solution

A Shady Region

Problem

The shaded region on the diagram is bounded by the lines whose equations are \(5x + 2y = 30\), \(x + 2y = 22\), \(x = 0\), and \(y = 0\). Determine the area of the shaded region.

Solution

On the diagram, \(l_1\) represents the line \(5x + 2y = 30\) that crosses the \(x\)-axis at point \(R\). \(l_2\) represents the line \(x + 2y = 22\) which crosses the \(y\)-axis at point \(Q\).

Let \(P(h, k)\) represent the point of intersection of \(l_1\) and \(l_2\). Then \(h\) is the horizontal distance from the \(y\)-axis to \(P\) and \(k\) is the vertical distance from the \(x\)-axis to \(P\). Let \(O\) represent the origin.

To find the \(x\)-intercept of \(l_1\) let \(y = 0\) in \(5x + 2y = 30\). Therefore the \(x\)-intercept is 6 and the coordinates of \(R\) are \((6, 0)\).

To find the \(y\)-intercept of \(l_2\) let \(x = 0\) in \(x + 2y = 22\). Therefore the \(y\)-intercept is 11 and the coordinates of \(Q\) are \((0, 11)\).

To find the intersection of \(l_1\) and \(l_2\), we can use elimination.

\[
\begin{align*}
l_1 : & \quad 5x + 2y = 30 \\
l_2 : & \quad x + 2y = 22
\end{align*}
\]

Subtracting, we obtain,

\[
4x = 8 \quad \therefore x = 2
\]

Substituting \(x = 2\) in \(l_1\), \(10 + 2y = 30\) and \(y = 10\). The coordinates of \(P\), the point of intersection, are \((2, 10)\). Therefore, \(h = 2\) and \(k = 10\). To find the shaded area:

\[
\text{Area } PQOR = \text{Area } \triangle PQO + \text{Area } \triangle POR
\]

\[
= \frac{1}{2}h \times OQ + \frac{1}{2}k \times OR
\]

\[
= \frac{1}{2}(2)(11) + \frac{1}{2}(10)(6)
\]

\[
= 11 + 30
\]

\[
= 41
\]

Therefore the shaded area is 41 units\(^2\).
Problem of the Week
Problem D
This Difference is Some Sum

When the expression $10^{2016} - 2016$ is evaluated, the result is a very large number. You probably do not have enough time to perform the calculation. So, in an effort to save you some time and paper, instead of evaluating the expression $10^{2016} - 2016$, determine the sum of the digits in the difference.
\[ 10^{2016} - 2016 \]

Problem D and Solution

This Difference is Some Sum

Problem

Determine the sum of the digits in the difference when \( 10^{2016} - 2016 \) is evaluated.

Solution

Solution 1

When the number \( 10^{2016} \) is written out there is a one followed by 2016 zeroes, a total of 2017 digits. Let’s look at what happens in our effort to subtract.

\[
\begin{array}{c}
100000 \cdots 0000000 \\
\end{array}
\]

\[
- \begin{array}{c}
2016 \\
\end{array}
\]

Using the standard subtraction algorithm, we start with the rightmost digits. In this case we need to borrow. But the borrowing creates a chain reaction. The result after the borrowing is complete is shown below.

\[
\begin{array}{c}
10909909 \cdots 090909091 \\
\end{array}
\]

\[
- \begin{array}{c}
2016 \\
\end{array}
\]

\[
9999 \cdots 997984 
\]

The four rightmost digits in the difference are 7, 9, 8 and 4. To the left of these digits every digit is a 9. But how many nines are there? The difference has one less digit than \( 10^{2016} \) and therefore has 2016 digits. We have accounted for the four rightmost digits. So to the left of 7984 there are \( 2016 - 4 = 2012 \) nines.

The digit sum is now straightforward to calculate. The digit sum is

\[
2012 \times 9 + (7 + 9 + 8 + 4) = 18\,108 + 28 = 18\,136.
\]

Note: If you were able to solve this problem, consider attempting level E problem.
Solution 2

The expression $10^{2016} - 2016$ has the same value as $(10^{2016} - 1) - (2016 - 1)$.

As mentioned in Solution 1, when $10^{2016}$ is written out, there is a one followed by 2016 zeroes, a total of 2017 digits. The number $(10^{2016} - 1)$ is one less than $10^{2016}$ and therefore is the positive whole number made up of exactly 2016 nines. When 1 is subtracted from 2016, the difference is 2015. The following is the equivalent subtraction question:

\[
\begin{array}{c}
9999 \cdots 999999 \\
\hline
-2015 \\
9999 \cdots 997984
\end{array}
\]

The four rightmost digits in the difference are 7, 9, 8 and 4. To the left of these digits every digit is a 9. But how many nines are there? The difference has one less digit than $10^{2016}$ and therefore has 2016 digits. We have accounted for the four rightmost digits. So to the left of 7984 there are $2016 - 4 = 2012$ nines.

The digit sum is now straightforward to calculate. The digit sum is

\[2012 \times 9 + (7 + 9 + 8 + 4) = 18108 + 28 = 18136.\]

Note: If you were able to solve this problem, consider attempting level E problem.
Relations & Systems

TAKEME TO THE COVER
Problem of the Week
Problem D
Left, Right, Left, Right, ...

Your friend writes down all of the integers starting from 0 in the following way:

Specifically, below every number there are two numbers: one on the left and one on the right. For example, below 3, the number 7 is on the left, and the number 8 is on the right. The numbers can be read in increasing order from top row to bottom row and from left-to-right within a row. Notice that we can get from 0 to 12 by going right (R), left (L) then right (R).

What number do you end at if you take the following path from 0:

\[ L \rightarrow L \rightarrow R \rightarrow L \rightarrow L \rightarrow R \rightarrow L \rightarrow R \rightarrow L \rightarrow R \rightarrow L \]
Problem of the Week
Problem D and Solution
Left, Right, Left, Right, ...

Problem
Your friend writes down all of the integers starting from 0 as shown in the diagram to the right. Specifically, below every number there are two numbers: one on the left and one on the right. For example, below 3, the number 7 is on the left, and the number 8 is on the right. The numbers can be read in increasing order from top row to bottom row and from left-to-right within a row. Notice that we can get from 0 to 12 by going right (R), left (L) then right (R). What number do you end at if you take the following path from 0:

L → L → R → L → L → R → L → R → L → L → L

Solution
At first it may not be obvious how to proceed. We can see from the given diagram that L → L → R takes us from 0 to 1 to 3 to 8. But from there where do we go? We could write out more rows of the chart until we are able to make the required number of moves and then read off the final answer. This approach would work for a relatively small number of moves but would not be practical in general for “longer” strings of moves.

We will proceed by making an observation. When we perform a move to the left (L) from any number, we end up at an odd number. When we perform a move to the right (R) from any number, we end up at an even number. Is there a general formula which can be used when asked to move left (L)? Is there a general formula which can be used when asked to move right (R)?

The diagram to the right has two parts of the tree circled. Can we discover a pattern that takes us from each initial number to the odd and even numbers below? To get from 1 to 3 we could add 2 and to get from 1 to 4 we could add 3. But doing the same with 6 would not get us to 13 and 14. As we go down the chart, each new row has twice as many numbers as the row above. Let’s try multiplying the initial number by 2 and then seeing what is necessary to get to the odd and even number below. If we double 1 we get 2. Then we would need to add 1 to get to the odd number 3 below and add 2 to get to the even number 4 below. Does this work with the 6? If we double 6 and add 1, we get 13. It appears to work. If we double 6 and add 2, we get 14. It also appears to work.
So it would appear that if we make a move left (L) from any number $a$ in the tree, the resulting number is one more than twice the value of $a$. That is, a move left (L) from $a$ takes us to the number $2a + 1$ in the tree.

It would appear that if we make a move right (R) from any number $a$ in the tree, the resulting number is two more than twice the value of $a$. That is, a move right (R) from $a$ takes us to the number $2a + 2$ in the tree.

These results are true but unproven. This relationship has worked for all of the rows we have sampled but we have not proven it true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

The following table shows the result of performing the given sequence of moves

$L \rightarrow L \rightarrow R \rightarrow L \rightarrow L \rightarrow R \rightarrow L \rightarrow R \rightarrow L$.

<table>
<thead>
<tr>
<th>Initial Number</th>
<th>Move</th>
<th>Calculation</th>
<th>Next Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>L</td>
<td>$2(0) + 1$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>L</td>
<td>$2(1) + 1$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>R</td>
<td>$2(3) + 2$</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>L</td>
<td>$2(8) + 1$</td>
<td>17</td>
</tr>
<tr>
<td>17</td>
<td>L</td>
<td>$2(17) + 1$</td>
<td>35</td>
</tr>
<tr>
<td>35</td>
<td>R</td>
<td>$2(35) + 2$</td>
<td>72</td>
</tr>
<tr>
<td>72</td>
<td>L</td>
<td>$2(72) + 1$</td>
<td>145</td>
</tr>
<tr>
<td>145</td>
<td>R</td>
<td>$2(145) + 2$</td>
<td>292</td>
</tr>
<tr>
<td>292</td>
<td>L</td>
<td>$2(292) + 1$</td>
<td>585</td>
</tr>
<tr>
<td>585</td>
<td>R</td>
<td>$2(585) + 2$</td>
<td>1172</td>
</tr>
<tr>
<td>1172</td>
<td>L</td>
<td>$2(1172) + 1$</td>
<td>2345</td>
</tr>
</tbody>
</table>

Starting at 0 and making the moves $L \rightarrow L \rightarrow R \rightarrow L \rightarrow L \rightarrow R \rightarrow L \rightarrow R \rightarrow L$, we end at the number 2345.

Once we determined the operations required to make a move left (L) and a move right (R), the problem was quite straightforward to solve. It would be possible to write a computer program which would accept any length sequence of Ls and Rs, and get the computer to determine the final position in this specific tree.
Problem of the Week
Problem D
Actress or Comedian - Dare to Compare

Holly Woods is a popular young actress and Joe King is an up and coming young comedian. Joe has an income which is five-eighths of Holly’s income. Joe’s expenses are one-half those of Holly, and Joe saves 40% of his income.

Determine the percentage of her income that Holly Woods saves.
Problem of the Week
Problem D and Solution
Actress or Comedian - Dare to Compare

Problem
Holly Woods is a popular young actress and Joe King is an up and coming young comedian. Joe has an income which is five-eighths of Holly’s income. Joe’s expenses are one-half those of Holly, and Joe saves 40% of his income. Determine the percentage of her income that Holly Woods saves.

Solution
Solution 1 Using only one variable
Let \( h \) represent Holly’s income. Then Joe’s income is \( \frac{5}{8} h \).

Since Joe saves 40% of his income, his expenses are 100% - 40% = 60% of his income. Therefore, his expenses are \( 60\% \times \frac{5}{8} h = \frac{60}{100} \times \frac{5}{8} h = \frac{3}{8} h \).

Joe’s expenses are one-half of Holly’s expenses so Holly’s expenses are twice Joe’s expenses. Therefore, Holly’s expenses are \( 2 \times \frac{3}{8} h = \frac{3}{4} h = 0.75h = 75\% \) of \( h \). Since Holly’s expenses are 75% of her income, she saves 100% - 75% = 25% of her income.

\( \therefore \) Holly Woods saves 25% of her income.

Solution 2 Using two variables
Let \( x \) represent Holly’s income and \( y \) represent her expenses. Then Joe’s income is \( \frac{5}{8} x \) and his expenses are \( \frac{1}{2} y \).

Since Joe saves 40% of his income, his expenses are 60% of his income.

\[
\begin{align*}
\frac{1}{2} y &= 0.60 \times \frac{5}{8} x \\
\frac{1}{2} y &= \frac{6}{10} \times \frac{5}{8} x \\
\frac{1}{2} y &= \frac{3}{8} x \\
y &= \frac{3}{4} x \\
\end{align*}
\]

Holly saves whatever is left of her income after expenses. Therefore Holly saves

\[
x - y = x - \frac{3}{4} x = \frac{1}{4} x = 0.25x = 25\% \text{ of } x.
\]

\( \therefore \) Holly Woods saves 25% of her income.
Solution 3 Using two variables a bit differently

Let $8x$ represent Holly’s income and $2y$ represent her expenses. Then Joe’s income is $\frac{5}{8}(8x) = 5x$ and his expenses are $\frac{1}{2}(2y) = y$.

Since Joe saves 40% of his income, his expenses are 60% of his income.

\[
y = 0.60 \times 5x \\
y = \frac{6}{10} \times 5x \\
y = 3x
\]

Holly earns $8x$ and her expenses are $2y$ so her savings are $8x - 2y$. We want the ratio of her savings to her income, $\frac{8x - 2y}{8x} = \frac{8x - 2(3x)}{8x} = \frac{2x}{8x} = \frac{1}{4}$ or 25%.

\[\therefore\] Holly Woods saves 25% of her income.
Problem of the Week
Problem D
Tangled Triangles

In the diagram, $A(0, a)$ lies on the $y$-axis above the origin. If $\triangle ABD$ and $\triangle COB$ have the same area, determine the value of $a$. 
Problem of the Week
Problem D and Solution
Tangled Triangles

Problem
In the diagram, \(A(0, a)\) lies on the \(y\)-axis above the origin. If \(\triangle ABD\) and \(\triangle COB\) have the same area, determine the value of \(a\).

Solution
Solution 1
Draw rectangle \(EDFC\) with sides parallel to the \(x\) and \(y\)-axes so that \(O(0,0)\) is on \(ED\) and \(B(2,-1)\) is on \(DF\). Since \(EC\) is parallel to the \(x\)-axis and \(E\) is on the \(y\)-axis, \(E\) has coordinates \((0,2)\). Since \(CF\) is parallel to the \(y\)-axis, \(F\) has the same \(x\)-coordinate as \(C\). Since \(FD\) is parallel to the \(x\)-axis, \(F\) has the same \(y\)-coordinate as \(D\) and \(B\). Therefore the coordinates of \(F\) are \((3,-1)\).

To find the area of \(\triangle COB\), subtract the areas of \(\triangle CEO\), \(\triangle ODB\), and \(\triangle BFC\) from the area of rectangle \(EDFC\).

In rectangle \(EDFC\), \(EC = 3 - 0 = 3\) and \(ED = 2 - (-1) = 3\). The area of rectangle \(EDFC = EC \times ED = 3 \times 3 = 9\) units\(^2\).

In \(\triangle CEO\), \(EC = 3\) and \(EO = 2 - 0 = 2\). The area of \(\triangle CEO = \frac{EC \times EO}{2} = \frac{3 \times 2}{2} = 3\) units\(^2\).

In \(\triangle ODB\), \(OD = 0 - (-1) = 1\) and \(DB = 2 - 0 = 2\). The area of \(\triangle ODB = \frac{OD \times DB}{2} = \frac{1 \times 2}{2} = 1\) unit\(^2\).

In \(\triangle BFC\), \(BF = 3 - 2 = 1\) and \(CF = 2 - (-1) = 3\). The area of \(\triangle BFC = \frac{BF \times CF}{2} = \frac{1 \times 3}{2} = 1.5\) units\(^2\).

\[
\text{Area } \triangle COB = \text{ Area Rectangle } EDFC - \triangle CEO - \triangle ODB - \triangle BFC
\]
\[
= 9 - 3 - 1 - 1.5
\]
\[
= 3.5 \text{ units}^2
\]

But the area \(\triangle ABD = \triangle COB\) so the area of \(\triangle ABD = 3.5\) units\(^2\).

In \(\triangle ABD\), \(AD = a - (-1) = a + 1\) and \(DB = 2 - 0 = 2\) so

\[
\text{Area } \triangle ABD = \frac{AD \times DB}{2}
\]
\[
3.5 = \frac{(a + 1) \times 2}{2}
\]
\[
3.5 = a + 1
\]
\[
2.5 = a
\]

\[
\therefore \text{ the value of } a \text{ is } 2.5.
\]
Solution 2

Determine the equation of the line containing $C(3, 2)$ and $B(2, -1)$.

The slope of the line is $\frac{2-(-1)}{3-2} = 3$. The equation of the line is of the form $y = 3x + b$. Substitute $x = 3$, $y = 2$ to determine the value of $b$. $2 = 3(3) + b$ and $b = -7$ follows. Therefore the equation of the line containing $C$ and $B$ is $y = 3x - 7$.

Let $P(p, 0)$ be the $x$-intercept of the line. Substituting into $y = 3x - 7$ we obtain $0 = 3p - 7$ and $p = \frac{7}{3}$ follows.

To determine the area of $\triangle COB$ determine the sum of the areas of $\triangle COP$ and $\triangle BOP$.

In $\triangle COP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the $x$-axis to $C(3, 2)$, which is 2 units. The area of $\triangle COP = \frac{\frac{7}{3} \times 2}{2} = \frac{7}{3}$ units$^2$.

In $\triangle BOP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the $x$-axis to $B(2, -1)$, which is 1 unit. The area of $\triangle BOP = \frac{\frac{7}{3} \times 1}{2} = \frac{7}{6}$ units$^2$.

Area $\triangle COB = \text{Area } \triangle COP + \text{Area } \triangle BOP$

$= \frac{7}{3} + \frac{7}{6}$

$= \frac{14}{6} + \frac{7}{6}$

$= \frac{21}{6}$

$= \frac{7}{2}$ units$^2$

But the area $\triangle ABD = \triangle COB$ so the area of $\triangle ABD = \frac{7}{2}$ units$^2$.

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$ so

Area $\triangle ABD = \frac{AD \times DB}{2}$

$\frac{7}{2} = \frac{(a + 1) \times 2}{2}$

$7 = 2a + 2$

$5 = 2a$

$\therefore a = \frac{5}{2} = 2.5$.
Problem of the Week
Problem D
Several Areas of Interest

The area of $\triangle ACD$ is twice the area of square $BCDE$. Square $BCDE$ has sides of length 12 cm. $AD$ intersects $BE$ at $F$.

Determine the area of quadrilateral $BCDF$. 

![Diagram with points A, B, C, D, E, F and line segments AD, BE, and CD.]
Problem of the Week
Problem D and Solution
Several Areas of Interest

Problem
The area of $\triangle ACD$ is twice the area of square $BCDE$. Square $BCDE$ has sides of length 12 cm. $AD$ intersects $BE$ at $F$. Determine the area of quadrilateral $BCDF$.

Solution
The area of square $BCDE = 12 \times 12 = 144 \text{ cm}^2$. The area of $\triangle ACD$ equals twice the area of square $BCDE$. Therefore area $\triangle ACD = 288 \text{ cm}^2$.

The area of a triangle is calculated using the formula $\text{base} \times \text{height} \div 2$. It follows that:

Area $\triangle ACD = (CD) \times (AC) \div 2$

$288 = 12 AC \div 2$

$288 = 6 AC$

$48 \text{ cm} = AC$

But $AC = AB + BC$ so $48 = AB + 12$ and it follows that $AB = 36 \text{ cm}$.

From this point, we will present three different approaches to obtaining the required area.

Method 1
Let the area of $\triangle ABF$ be $p$, quadrilateral $BCDF$ be $q$ and $\triangle DEF$ be $r$. Let $FE = x$. Since $BE = 12$, $BF = 12 - x$.

Area $\triangle ACD = 288 = p + q$ (1) and area square $BCDE = 144 = r + q$ (2).

Subtracting (2) from (1), $p - r = 144$ and $p = r + 144$ follows.

Area $\triangle ABF = p = r + 144 = (AB)(BF) \div 2 = 36(12 - x) \div 2 = 18(12 - x)$

$\therefore r + 144 = 18(12 - x)$

$r = 216 - 18x - 144$

$r = 72 - 18x$ (3)

Area $\triangle DEF = r = (DE)(EF) \div 2 = 12x \div 2 = 6x$. $\therefore r = 6x$ (4)

Using (3) and (4), since $r = r$, $6x = 72 - 18x$. Solving, $x = 3$ and $r = 18$ follow.

The area of quadrilateral $BCDF = \text{area of square } BCDE - \text{area } \triangle DEF$

$= 144 - r$

$= 144 - 18$

$= 126 \text{ cm}^2$

Therefore the area of quadrilateral $BCDF$ is 126 cm$^2$.

See alternative methods on the next page.
Method 2

Let \( x \) represent the length of \( BF \).

In \( \triangle ABF \) and \( \triangle ACD \), \( \angle A \) is common and \( \angle ABF = \angle ACD = 90^\circ \). It follows that \( \triangle ABF \) is similar to \( \triangle ACD \) and \( \frac{AB}{AC} = \frac{BF}{CD} \). Therefore \( \frac{36}{48} = \frac{x}{12} \) and \( x = 9 \).

Since \( BCDE \) is a square, \( BE \parallel CD \). Then quadrilateral \( BCDF \) is a trapezoid.

\[
\text{Area Trapezoid } BCDF = (BC)(BF + CD) \div 2 = 12(9 + 12) \div 2 = 6(21) = 126 \text{ cm}^2
\]

Therefore the area of quadrilateral \( BCDF \) is 126 cm\(^2\).

Method 3

Position the diagram so that \( C \) is at the origin, \( A \) and \( B \) are on the positive \( y \)-axis and \( D \) is on the positive \( x \)-axis. \( C \) has coordinates \((0,0)\), \( B \) has coordinates \((0,12)\), \( A \) has coordinates \((0,48)\), and \( D \) has coordinates \((12,0)\).

Find the equation of the line containing \( AD \). The slope is \( -\frac{48}{12} = -4 \) and the line crosses the \( y \)-axis at \( A \) so the \( y \)-intercept is 48. Therefore the equation is \( y = -4x + 48 \).

The line containing \( BE \) is horizontal crossing the \( y \)-axis at \( B \). The equation of the line containing \( BE \) is \( y = 12 \).

\( F \) is the intersection of \( y = -4x + 48 \) and \( y = 12 \). Since \( y = y \) at the point of intersection, \( -4x + 48 = 12 \) and \( x = 9 \) follows. Therefore \( F \) has coordinates \((9,12)\) and \( BF = 9 \) cm.

Since \( BCDE \) is a square, \( BE \parallel CD \). Then quadrilateral \( BCDF \) is a trapezoid.

\[
\text{Area Trapezoid } BCDF = (BC)(BF + CD) \div 2 = 12(9 + 12) \div 2 = 6(21) = 126 \text{ cm}^2
\]

Therefore the area of quadrilateral \( BCDF \) is 126 cm\(^2\).
Problem of the Week
Problem D
Formidable Fractions

People from the town of Formidable like to pose problems involving fractions. Here is one of their problems.

There are some positive integers \(a\) and \(c\) such that

\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = 18.
\]

For example, when \(a = 36\) and \(c = 5\), the value of the numerator is

\[
\frac{a}{c} + \frac{a}{2} + 1 = \frac{36}{5} + \frac{36}{2} + 1 = \frac{36}{10} + 18 + \frac{10}{10} = \frac{262}{10} = \frac{131}{5},
\]

the value of the denominator is

\[
\frac{2}{a} + \frac{2}{c} + 1 = \frac{2}{36} + \frac{2}{5} + 1 = \frac{1}{18} + \frac{2}{5} + 1 = \frac{5}{90} + \frac{36}{90} + \frac{90}{90} = \frac{131}{90},
\]

and the left side of the equation simplifies to

\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = \frac{131}{5} \div \frac{131}{90} = \frac{131}{5} \times \frac{90}{131} = 18.
\]

Since 18 is also the value of the right side of the equation, then the ordered pair \((a, c) = (36, 5)\) satisfies the given equation.

Many ordered pairs satisfy the given equation so we will add a restriction.

Determine the total number of ordered pairs \((a, c)\) that satisfy the equation such that \(a + 3c \leq 99\).
Problem of the Week
Problem D and Solution
Formidable Fractions

Problem
There are some positive integers \(a\) and \(c\) such that
\[
\left( \frac{\frac{a}{c} + \frac{a}{2} + 1}{\frac{2}{a} + \frac{2}{c} + 1} \right) = 18.
\]

Determine the total number of ordered pairs \((a, c)\) that satisfy the equation such that \(a + 3c \leq 99\).

Solution
Solution 1
\[
\left( \frac{\frac{a}{c} + \frac{a}{2} + 1}{\frac{2}{a} + \frac{2}{c} + 1} \right) = 18
\]

Find common denominators:
\[
\left( \frac{2a + ac + 2c}{2c + 2a + ac} \right) = 18
\]

Simplifying:
\[
\left( \frac{2c + 2a + ac}{ac} \right) = 18
\]

Multiplying by the reciprocal:
\[
\frac{(2a + ac + 2c)}{2c} \times \frac{ac}{(2c + 2a + ac)} = 18
\]

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to \(\frac{ac}{2c} = 18\). Since \(c \neq 0\), the expression further simplifies to \(\frac{a}{2} = 18\) or \(a = 18(2) = 36\). Substituting \(a = 36\) into \(a + 3c \leq 99\) we obtain \(36 + 3c \leq 99\) which simplifies to \(3c \leq 63\) and \(c \leq 21\) follows.

But \(c \geq 1\) and \(c\) is an integer so \(1 \leq c \leq 21\). The value of \(a\) is 36 for each of the 21 possible values of \(c\).

\(\therefore\) there are 21 ordered pairs \((a, c)\) that satisfy the problem.
Solution 2

\[
\left(\frac{a}{c} + \frac{1}{2} + 1\right) \times \frac{2ac}{2a + \frac{2}{c} + 1} = 18
\]

Multiply numerator and denominator by \(2ac\):

\[
\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} \times \frac{2ac}{2ac} = 18
\]

Simplify:

\[
\frac{2a^2 + a^2c + 2ac}{4c + 4a + 2ac} = 18
\]

Factoring:

\[
\frac{a(2a + ac + 2c)}{2(2c + 2a + ac)} = 18
\]

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to \(\frac{a}{2} = 18\) and \(a = 36\) follows.

Substituting \(a = 36\) into \(a + 3c \leq 99\) we obtain \(36 + 3c \leq 99\) which simplifies to \(3c \leq 63\) and \(c \leq 21\). But \(c \geq 1\) and \(c\) is an integer so \(1 \leq c \leq 21\). The value of \(a\) is 36 for each of the 21 possible values of \(c\).

\(\therefore\) there are 21 ordered pairs \((a, c)\) that satisfy the problem.
The game of Tiddlywinks dates back to 1955. Rumour has it that the game has made a comeback with some students in recent years. Last year, our local TPL, Tiddlywink Premier League, launched its first season. Each team in the league played each of the other teams in the league the same number of times.

In its second season the TPL has grown. The total number of matches played this season will be twice the number of matches played in the first season and there will be 40% more teams than in season one. Each team in the league will still play each of the other teams in the league the same number of times as in season one.

How many teams were in the TPL in season one?
Problem of the Week
Problem D and Solution
Tiddlywinks Anyone?

Problem
The game of Tiddlywinks dates back to 1955. Rumour has it that the game has made a comeback with students in recent years. Last year, our local TPL, Tiddlywink Premier League, launched its first season. Each team in the league played each of the other teams in the league the same number of times. In its second season the TPL has grown. The total number of matches played this season will be twice the number of matches played in the first season and there will be 40% more teams than in season one. The number of games that each team plays each other team stays the same from season to season. How many teams were in the TPL in season one?

Solution
Let $t$ represent the number of teams in the league in season one.

Then, the number of teams in the league in the second season is $1.4t$.

Note that both $t$ and $1.4t$ must be positive integers.

We need to first establish how many games are played. If there were 4 teams, $A$, $B$, $C$, $D$, and each team played every other team once, then there would be 6 games played: $AB$, $AC$, $AD$, $BC$, $BD$, $CD$. Often, when counting something like this we “double-count”.

That is, there are 4 teams and each plays the 3 other teams so there are $4 \times 3 = 12$ games. But each game is counted twice. We need to divide the result by 2. So in a 4 team league, with each team playing each other team once, there are $\frac{4 \times 3}{2} = 6$ games played.

In general, if there are $t$ teams and each team plays every other team once, there would be $\frac{t(t-1)}{2}$ games played. If each of $t$ teams plays every other team $n$ times, there would be $n \left( \frac{t(t-1)}{2} \right)$ games played. If each of $1.4t$ teams plays every other team $n$ times, there would be $n \left( \frac{1.4t(1.4t-1)}{2} \right)$ games played.

We know that the number of games played in season two is twice the number of games played in season one. So,

$$n \left( \frac{1.4t(1.4t - 1)}{2} \right) = 2 \left[ n \left( \frac{t(t - 1)}{2} \right) \right]$$

Dividing both sides by $\frac{n}{2}$, $n \neq 0$, this simplifies to

$$1.4t(1.4t - 1) = 2t(t - 1)$$

Dividing both sides by $t$, $t \neq 0$, this simplifies to

$$1.4(1.4t - 1) = 2(t - 1)$$

$$1.96t - 1.4 = 2t - 2$$

$$0.96t = 0.6$$

$$t = 15$$

Therefore, in season one there were 15 teams in the TPL.
Problem of the Week
Problem D
A Shady Region

The shaded region on the diagram is bounded by the lines whose equations are $5x + 2y = 30$, $x + 2y = 22$, $x = 0$, and $y = 0$.

Determine the area of the shaded region.
Problem of the Week
Problem D and Solution
A Shady Region

Problem
The shaded region on the diagram is bounded by the lines whose equations are $5x + 2y = 30$, $x + 2y = 22$, $x = 0$, and $y = 0$. Determine the area of the shaded region.

Solution
On the diagram, $l_1$ represents the line $5x + 2y = 30$ that crosses the $x$-axis at point $R$. $l_2$ represents the line $x + 2y = 22$ which crosses the $y$-axis at point $Q$.

Let $P(h, k)$ represent the point of intersection of $l_1$ and $l_2$. Then $h$ is the horizontal distance from the $y$-axis to $P$ and $k$ is the vertical distance from the $x$-axis to $P$. Let $O$ represent the origin.

To find the $x$-intercept of $l_1$ let $y = 0$ in $5x + 2y = 30$. Therefore the $x$-intercept is 6 and the coordinates of $R$ are $(6,0)$.

To find the $y$-intercept of $l_2$ let $x = 0$ in $x + 2y = 22$. Therefore the $y$-intercept is 11 and the coordinates of $Q$ are $(0,11)$.

To find the intersection of $l_1$ and $l_2$, we can use elimination.

\[
\begin{align*}
l_1 : & \quad 5x + 2y = 30 \\
l_2 : & \quad x + 2y = 22
\end{align*}
\]

Subtracting, we obtain,

\[4x = 8\]
\[\therefore x = 2\]

Substituting $x = 2$ in $l_1$, $10 + 2y = 30$ and $y = 10$. The coordinates of $P$, the point of intersection, are $(2,10)$. Therefore, $h = 2$ and $k = 10$. To find the shaded area:

\[
\begin{align*}
\text{Area } PQOR &= \text{Area } \triangle PQO + \text{Area } \triangle POR \\
&= \frac{1}{2}h \times OQ + \frac{1}{2}k \times OR \\
&= \frac{1}{2}(2)(11) + \frac{1}{2}(10)(6) \\
&= 11 + 30 \\
&= 41
\end{align*}
\]

Therefore the shaded area is 41 units$^2$. 