The problems in this booklet are organized into strands. A problem often appears in multiple strands. The problems are suitable for most students in Grade 7 or higher.
Data Management &
Probability
Problem of the Week
Problem C
Totally Unusual

The dice shown below are unusual. A usual six-sided die would have the numbers 1, 2, 3, 4, 5 and 6 on the sides. These dice, however, are unusual because the numbers on the six sides are 1, 2, 3, 5, 7 and 9.

Two of these unusual dice, one red and one blue, are rolled and the numbers on the upper faces are added together. A winning roll occurs when the sum is either a perfect square or a prime number.

Determine the probability that you win on any particular roll.

A prime number is an integer greater than 1 that has only two positive divisors, 1 and itself. For example, the number 17 is prime.

A perfect square is an integer that is the product of some integer and itself. For example, 9 is a perfect square since $3 \times 3 = 9$. 

Strand Data Management and Probability
Problem

The dice shown above are unusual. A usual six-sided die would have the numbers 1, 2, 3, 4, 5 and 6 on the sides. These dice, however, are unusual because the numbers on the six sides are 1, 2, 3, 5, 7 and 9. Two of these unusual dice, one red and one blue, are rolled and the numbers on the upper faces are added together. A winning roll occurs when the sum is either a perfect square or a prime number. Determine the probability that you win on any particular roll.

A prime number is an integer greater than 1 that has only two positive divisors, 1 and itself. For example, the number 17 is prime. A perfect square is an integer that is the product of some integer and itself. For example, 9 is a perfect square since \(3 \times 3 = 9\).

Solution

To solve this problem we will create a chart showing all of the possible rolls and the corresponding sums.

<table>
<thead>
<tr>
<th>Upper Face of Blue Die</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>Upper Face of Red Die</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
</tr>
</tbody>
</table>

From the table, we see that there are 36 possible outcomes. We also see that the perfect squares 4, 9 and 16 appear in the table seven times.

The lowest number in the table is 2 and the highest number in the table is 18. The prime numbers appearing in the table in this range of numbers are 2, 3, 5, 7, and 11. These numbers appear in the table a total of nine times.

Since a number cannot be both a prime number and a perfect square, we can be certain that we have not counted a desirable outcome more than once. The total number of prime number sums and perfect square sums is \(7 + 9 = 16\).

To determine the probability of a specific outcome, we divide the number of times the specific outcome occurs by the total number of possible outcomes. The probability of winning on a particular roll is \(16 \div 36 = \frac{4}{9}\). You have approximately a 44% chance of winning. A game is considered “fair” if you have a 50% chance of winning.
Problem of the Week
Problem C
Probably Even

Four distinct integers are to be chosen from the integers 1, 2, 3, 4, 5, 6, and 7. How many different selections are possible so the sum of the four integers is even?

1 + 2 + 3 + 4   Even Sum
1 + 2 + 3 + 5   Odd Sum

Strands Data Management and Probability, Number Sense and Numeration
Problem of the Week
Problem C and Solution
Probably Even

Problem
Four distinct integers are to be chosen from the integers 1, 2, 3, 4, 5, 6, and 7. How many different selections are possible so the sum of the four integers is even?

Solution
We could look at every possible selection of four distinct numbers from the list, determine the sum of each selection and then count the number of selections for which the sum is even. There are 35 different selections to examine. A justification of this number is provided on the second page of this solution. This would not be an efficient approach!

We will make two simple observations. First, when even numbers are added together the sum is always even. And second, in order to produce an even sum using odd numbers, an even number of odd numbers is required in the sum. We will use these observations to break the problem into cases in which the sum is even. There are three cases to consider.

1. **No Odd Numbers are Selected**
   Since there are only three even numbers, namely 2, 4, and 6, it is not possible to select only even numbers. Therefore, there are no selections in which there are no odd numbers.

2. **Exactly Two Odd Numbers are Selected**
   There are four choices for the first odd number. For each of these four choices, there are three choices for the second number producing $4 \times 3 = 12$ choices for two odd numbers. However, each choice is counted twice. For example, 1 could be selected first and 3 could be chosen second or 3 could be selected first and 1 could be chosen second. Therefore, there are only $12 \div 2 = 6$ selections of two odd numbers. They are $\{1,3\}, \{1,5\}, \{1,7\}, \{3,5\}, \{3,7\}, \text{ and } \{5,7\}$. For each of the 6 possible selections of two distinct odd numbers, we need to select two even numbers from the three even numbers in the list. We could use a similar argument to the selection of the two odd numbers or simply list the (three) possibilities: $\{2,4\}, \{2,6\}, \text{ and } \{4,6\}$. Therefore, there are $6 \times 3 = 18$ selections of four distinct numbers in which exactly two of the numbers are odd.

3. **Exactly Four Odd Numbers are Selected**
   Since there are only four odd numbers in the list to choose from, there is only one way to select four distinct odd numbers from the list.

We have considered every possible case in which the selection produces an even sum. Therefore, there are $0 + 18 + 1 = 19$ selections of four distinct numbers from the list such that the sum is even.
Why are there 35 ways to select four different numbers from the list?

In the solution on the previous page we counted the selections in which the sum was even. There were 19 possibilities. The remaining selections must produce an odd sum. There are two possibilities: either there is 1 odd number and 3 even numbers, or there are 3 odd numbers and 1 even number.

If there is 1 odd number and 3 even numbers, there are only four possible selections, namely, \{1, 2, 4, 6\}, \{3, 2, 4, 6\}, \{5, 2, 4, 6\}, and \{7, 2, 4, 6\}. Once the odd number is selected, the 3 even numbers, \{2, 4, 6\}, must be selected.

If there are 3 odd numbers and 1 even number, there are twelve possible selections. The 3 odd numbers can be selected in four ways, namely, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 5, 7\}, and \{3, 5, 7\}. For each of these 4 selections of three odd numbers, the even number can be selected in 3 ways producing \(4 \times 3 = 12\) possible selections of four distinct numbers in which three of the numbers are odd and the other is even.

We have considered all possible ways in which four distinct numbers can be selected from the list. The total number of selections is \(19 + 4 + 12 = 35\).

We can arrive at this number in a different way.

There are 7 choices for the first number. For each of these choices for the first number, there are 6 choices for the second number, or \(7 \times 6 = 42\) choices for the first two numbers. For each of these 42 choices for the first two numbers, there are 5 choices for the third number, or \(42 \times 5 = 210\) choices for the first three numbers. For each of these 210 choices of the first three numbers, there are 4 choices for the final number, or \(210 \times 4 = 840\) selections of the four numbers. This is considerably higher than the 35 choices shown above!

Our 840 selections assume that the order of selection is important. Each selection has been counted 24 times. To justify this, we will look at the number of ways a specific four number selection can be arranged. Without loss of generality, we will consider the selection \{1, 2, 3, 4\}. The 1 could be placed in four spots. For each of these four placements of the 1, the 2 could be placed in three spots producing \(4 \times 3 = 12\) ways of placing the 1 and 2. For each of these twelve placements of the 1 and 2, the 3 could be placed in two spots producing \(12 \times 2 = 24\) ways of placing the 1, 2 and 3. Once the numbers 1, 2, and 3 are placed, the 4 must be placed in the remaining spot. There are 24 ways of arranging the four numbers. We have to divide 840 by 24 since we have counted each selection 24 times.

Therefore, there are \(840 \div 24 = 35\) ways to select four different numbers from the list of seven numbers.
Problem of the Week
Problem C
That’s Not Fair Or Is It?

Anna and Elle are twins. Everything they do together must be fair. For a school mathematics project they created a game that uses a specially made pair of six-sided dice. One die has the even numbers 2, 4, 6, 8, 10, and 12 on its faces and the other die has the odd numbers 1, 3, 5, 7, 9, and 11 on its faces.

A turn consists of rolling the dice and using the two numbers that appear on the top faces. Anna and Elle take turns rolling the dice.

Anna performs the following steps after each roll to determine whether or not she gets a point.

1. Anna determines the sum, $S$, of the numbers on the top faces. On the roll shown below, $S = 9$.

2. Using $S$, Anna determines, $D$, the digit sum. If $S$ is a single digit number, then $D$ is the same as $S$. If $S$ is a two digit number, then $D$ is the sum of the two digits. (If the roll is a 6 and a 3 like below, then the digit sum and the sum are both 9. If the roll is a 5 and 10, then the sum is 15 and the digit sum is $1 + 5 = 6$. If the roll is a 9 and 10, then the sum is 19 and the digit sum is $1 + 9 = 10$.)

Anna gets a point if the digit sum $D$ is a multiple of 4.

Elle gets a point if one of the numbers on the top face is a multiple of the number on the other top face. With the dice roll shown below, Elle would get a point since 6 is a multiple of 3.

Is this game fair? That is, do Anna and Elle have the same probability of getting a point on any roll? Justify your answer.

Strand Data Management and Probability
Problem of the Week
Problem C and Solution
That’s Not Fair Or Is It?

Problem
One die has the even numbers 2, 4, 6, 8, 10, and 12 on its faces and the other die has the odd numbers 1, 3, 5, 7, 9, and 11 on its faces. A turn consists of rolling the dice and using the two numbers that appear on the top faces. Anna and Elle take turns rolling the dice.

Anna performs the following steps after each roll to determine whether or not she gets a point. First, Anna determines the sum, $S$, of the numbers on the top faces. On the roll shown above, $S = 9$. Then, using $S$, Anna determines, $D$, the digit sum. If $S$ is a single digit number, then $D$ is the same as $S$. If $S$ is a two digit number, then $D$ is the sum of the two digits. (If the roll is a 6 and a 3 like above, then the digit sum and the sum are both 9. If the roll is a 5 and 10, then the sum is 15 and the digit sum is $1 + 5 = 6$. If the roll is a 9 and 10, then the sum is 19 and the digit sum is $1 + 9 = 10$.) Anna gets a point if the digit sum $D$ is a multiple of 4. Elle gets a point if one of the numbers on the top face is a multiple of the number on the other top face. With the dice roll shown above, Elle would get a point since 6 is a multiple of 3. Is this game fair? That is, do Anna and Elle have the same probability of getting a point on any roll? Justify your answer.

Solution
First, determine the number of possible rolls. For the even numbered die, there are six possible numbers that could appear on the top face. For each of these six possibilities, there are six possible numbers that could appear on the top face of the odd die. There are a total of $6 \times 6 = 36$ possible rolls.

For Anna to get a point she must have a roll that produces a digit sum that is a multiple of 4.

<table>
<thead>
<tr>
<th>Even Die Roll</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>Odd Die Roll</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>Roll</td>
</tr>
<tr>
<td>11</td>
</tr>
</tbody>
</table>

The next table calculates the digit sums from the roll sums. Remember that the digit sum and roll sum are the same for single digit sums.
If the digit sum is a multiple of 4, then Anna gets a point. From the table we see that there are 2 digit sums which are a multiple of 4. These digit sums are 4 and 8. The digit sum 4 occurs six times in the above table and the digit sum 8 occurs four times in the above table. This totals ten possible outcomes for Anna and her probability of scoring a point on any roll is \( \frac{10}{36} \).

Elle has far less work to do to determine when she gets a point. None of the odd numbers are multiples of the even numbers. All multiples of even numbers are even and hence will never be odd.

Whenever a 1 is rolled on the die containing only odd numbers, Elle will score a point. That is, each of the 6 even numbers is a multiple of 1.

When a 3 is rolled on the die containing only odd numbers, Elle will score a point if the number on the top face of the even die is a 6 or 12. That is, only 2 of the even numbers are multiples of 3.

When a 5 is rolled on the die containing only odd numbers, Elle will score a point if the number on the top face of the even die is a 10. That is, only 1 of the even numbers is a multiple of 5.

None of the numbers on the die containing only even numbers is a multiple of 7, 9 or 11.

So Elle will score a point on \( 6 + 2 + 1 = 9 \) of the 36 possible rolls. Therefore, Elle’s probability of scoring a point on any roll is \( \frac{9}{36} \).

The game is not fair since Anna’s probability of scoring a point on any roll is greater than Elle’s probability of scoring a point on any roll. It should be noted that the probabilities are close to being the same so one might say that this game is almost fair.
Geometry & Spatial Sense
Problem of the Week
Problem C
Curvy Contest

Two paths are built from $A$ to $F$ as shown.

The distance from $A$ to $F$ in a straight line is 100 m. Points $B$, $C$, $D$, and $E$ lie along $AF$ such that $AB = BC = CD = DE = EF$.

The upper path, shown with a dashed line, is a semi-circle with diameter $AF$. The lower path, shown with a solid line, consists of five semi-circles with diameters $AB$, $BC$, $CD$, $DE$, and $EF$.

Starting at the same time, Bev and Mike ride their tricycles along these paths from $A$ to $F$. Bev rides along the upper path from $A$ to $F$ while Mike rides along the lower path from $A$ to $F$. If they ride at the same speed, who will get to $F$ first?

Strands Geometry and Spatial Sense, Measurement
Problem of the Week
Problem C and Solution
Curvy Contest

Problem
Two paths are built from $A$ to $F$ as shown on the diagram above. The distance from $A$ to $F$ in a straight line is 100 m. Points $B$, $C$, $D$, and $E$ lie along $AF$ such that $AB = BC = CD = DE = EF$. The upper path, shown with a dashed line, is a semi-circle with diameter $AF$. The lower path, shown with a solid line, consists of five semi-circles with diameters $AB$, $BC$, $CD$, $DE$, and $EF$. Starting at the same time, Bev and Mike ride their tricycles along these paths from $A$ to $F$. Bev rides along the upper path from $A$ to $F$ while Mike rides along the lower path from $A$ to $F$. If they ride at the same speed, who will get to $F$ first?

Solution
The circumference of a circle is found by multiplying its diameter by $\pi$. To find the circumference of a semi-circle, divide its circumference by 2.

The length of the upper path is equal to half the circumference of a circle with diameter 100 m. The length of the upper path equals $\pi \times 100 \div 2 = 50\pi$ m. (This is approximately 157.1 m.)

Each of the semi-circles along the lower path have the same diameter. The diameter of each of these semi-circles is $100 \div 5 = 20$ m. The length of the lower path is equal to half the circumference of five circles, each with diameter 20 m. The distance along the lower path equals $5 \times (\pi \times 20 \div 2) = 5 \times (10\pi) = 50\pi$ m.

Since both Bev and Mike ride at the same speed and both travel the same distance, they will arrive at point $F$ at the same time. Neither wins the race since both arrive at the same time. The answer to the problem may surprise you. Most people, at first glance, would think that the upper path is longer.

If you were to extend the problem so that Bev travels the same route but Mike travels along a lower path made up of 100 semi-circles of equal diameter from $A$ to $F$, they would still both travel exactly the same distance, $50\pi$ m. Check it out!
Problem of the Week
Problem C
Around We Go

A circle with centre $O$ has a point $A$ on the circumference. Radius $OA$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OB$. Point $A$ is then connected to point $B$. Radius $OB$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OC$. Point $B$ is then connected to point $C$.

The process of clockwise rotations continues until some radius rotates back onto $OA$. Every point on the circumference is connected to the points immediately adjacent to it as a result of the process. A polygon is created.

![Construction of the polygon](image)

a) Determine the number of sides of the polygon.

b) Determine the sum of the angles in the polygon. That is, determine the sum of the angles at each of the vertices of the polygon.
Problem of the Week
Problem C and Solution
Around We Go

Problem
A circle with centre \( O \) has a point \( A \) on the circumference. Radius \( OA \) is rotated \( 20^\circ \) clockwise about the centre, resulting in the image \( OB \). Point \( A \) is then connected to point \( B \). Radius \( OB \) is rotated \( 20^\circ \) clockwise about the centre, resulting in the image \( OC \). Point \( B \) is then connected to point \( C \).

The process of clockwise rotations continues until some radius rotates back onto \( OA \). Every point on the circumference is connected to the points immediately adjacent to it as a result of the process. A polygon is created.

a) Determine the number of sides of the polygon.
b) Determine the sum of the angles in the polygon. That is, determine the sum of the angles at each of the vertices of the polygon.

Solution
Each time the process is repeated, another congruent triangle is created. Each of these triangles has a \( 20^\circ \) angle at \( O \), the centre of the circle. But a complete rotation at the centre is \( 360^\circ \). Since each angle in the triangles at the centre of the circle is \( 20^\circ \) and the total measure at the centre is \( 360^\circ \), then there are \( 360 \div 20 = 18 \) triangles formed. This means that there are 18 distinct points on the circumference of the circle and the polygon has 18 sides. An 18-sided polygon is called an octadecagon, from octa meaning 8 and deca meaning 10.

The other two angles in the each of the congruent triangles are equal. (Two sides of the triangle are radii of the circle. The triangles are therefore isosceles.) The angles in a triangle sum to \( 180^\circ \) so after the \( 20^\circ \) angle is removed, there is \( 160^\circ \) remaining for the other two angles. It follows that each of the other two angles in each triangle measures \( 160^\circ \div 2 = 80^\circ \). The following diagram illustrates this information for the two adjacent triangles \( AOB \) and \( BOC \).

Each angle in the polygon is formed by an \( 80^\circ \) angle from one triangle and the adjacent \( 80^\circ \) angle from the next triangle. For example, \( \angle ABC = \angle ABO + \angle OBC = 80^\circ + 80^\circ = 160^\circ \). There are 18 vertices in the octadecagon and the angle at each vertex is \( 160^\circ \). Therefore the sum of the angles in the octadecagon is \( 18 \times 160^\circ = 2880^\circ \).

Diagrams are provided on the next page to further support the solution.
Angle at the Centre of a Circle

Completing the Construction

The Octadecagon - 18 sided Polygon

Notice the vertices of the octadecagon are labelled $A$ to $S$, but the letter $O$ is missing since it was used in the original construction as the centre of the circle.
Problem of the Week
Problem C
The Inner Square

$ABCD$ is a square with area $64 \text{ m}^2$. $E, F, G,$ and $H$ are points on sides $AB, BC, CD,$ and $DA$, respectively, such that $AE = BF = CG = DH = 2 \text{ m}$. $E, F, G,$ and $H$ are connected to form square $EFGH$.

Determine the area of $EFGH$. 

**Strands**  Geometry and Spatial Sense, Measurement
Problem of the Week
Problem C and Solution
The Inner Square

Problem

$ABCD$ is a square with area $64 \text{ m}^2$. $E, F, G,$ and $H$ are points on sides $AB, BC, CD,$ and $DA$, respectively, such that $AE = BF = CG = DH = 2 \text{ m}$. $E, F, G,$ and $H$ are connected to form square $EFGH$. Determine the area of $EFGH$.

Solution

The area of square $ABCD$ is $64 \text{ m}^2$. Therefore the side lengths are $8 \text{ m}$ since $8 \times 8 = 64$ and the area is calculated by multiplying the length and the width.

Each of the smaller parts of the sides of square $ABCD$ are $2 \text{ m}$ so the longer parts of the sides are $8 - 2 = 6 \text{ m}$.

Approach #1 to finding the area of square $EFGH$

In right $\triangle HAE$, $AE = 2$ and $AH = 6$. We can use one side as the base and the other as the height in the calculation of the area of the triangle since they are perpendicular to each other. Therefore the area of $\triangle HAE = \frac{AE \times AH}{2} = \frac{2 \times 6}{2} = 6 \text{ m}^2$. Since each of the triangles has the same base length and height, their areas are equal and the total area of the four triangles is $4 \times 6 = 24 \text{ m}^2$.

The area of square $EFGH$ can be determined by subtracting the area of the four triangles from the area of square $ABCD$. Therefore the area of square $EFGH = 64 - 24 = 40 \text{ m}^2$.

Approach #2 to finding the area of square $EFGH$

Some students may be familiar with the Pythagorean Theorem. This theorem states that in a right triangle, the square of the length of the hypotenuse (the longest side) is equal to the sum of the squares of the other two sides. The longest side is located opposite the right angle.

In right $\triangle HAE$, $AE = 2$, $AH = 6$ and $HE$ is the hypotenuse. Therefore,

$$HE^2 = AE^2 + AH^2$$
$$= 2^2 + 6^2$$
$$= 4 + 36$$

$$\therefore HE^2 = 40 \quad \text{(See the note below.)}$$

Taking the square root, $HE = \sqrt{40} \text{ m}$

But $EFGH$ is a square so all of its side lengths equal $\sqrt{40}$. The area is calculated by multiplying the length and the width. The area of $EFGH = \sqrt{40} \times \sqrt{40} = 40 \text{ m}^2$.

Therefore, the area of square $EFGH$ is $40 \text{ m}^2$.

Note: The area of the square is $HE^2 = 40 \text{ m}^2$. We found this when we used the Pythagorean Theorem above.
Problem of the Week
Problem C
What’s Your Angle?

$JKLM$ is a square. Points $P$ and $Q$ are outside the square such that both $\triangle JMP$ and $\triangle MLQ$ are equilateral.

Determine the measure, in degrees, of $\angle MPQ$. 
Problem

$JKLM$ is a square. Points $P$ and $Q$ are outside the square such that both $\triangle JMP$ and $\triangle MLQ$ are equilateral. Determine the measure, in degrees, of $\angle MPQ$.

Solution

Since $JKLM$ is a square, $JK = KL = LM = MJ$.
Since $\triangle JMP$ is equilateral, $MJ = JP = MP$.
Since $\triangle MLQ$ is equilateral, $LM = LQ = QM$.

It follows that $JK = KL = LM = MJ = JP = MP = LQ = QM$. These equal side lengths are shown on the diagram.

Each angle in a square is $90^\circ$. Therefore, $\angle JML = 90^\circ$.

Each angle in an equilateral triangle is $60^\circ$. Therefore, $\angle JMP = 60^\circ$ and $\angle LMQ = 60^\circ$.

A complete revolution is $360^\circ$. Since $\angle PMQ$, $\angle JMP$, $\angle JML$ and $\angle LMQ$ form a complete revolution, then

$$\angle PMQ = 360^\circ - \angle JMP - \angle JML - \angle LMQ$$
$$= 360^\circ - 60^\circ - 90^\circ - 60^\circ$$
$$= 150^\circ$$

In $\triangle MPQ$, $MP = QM$ and the triangle is isosceles. It follows that $\angle MPQ = \angle MQP$.

In a triangle, the sum of the three angles is $180^\circ$. Since $\angle PMQ = 150^\circ$, then the sum of the two remaining equal angles must be $30^\circ$. Therefore, each of the remaining two angles must equal $15^\circ$ and it follows that $\angle MPQ = 15^\circ$. 
Problem of the Week
Problem C
How Far Around is It?

In the diagram, \( \triangle ABC \) is a right triangle with \( \angle ABC = 90^\circ \), \( BD = 6 \text{ m} \), \( AB = 8 \text{ m} \), and the area of \( \triangle ADC \) is 50\% more than the area of \( \triangle ABD \).

Determine the perimeter of \( \triangle ADC \).

The Pythagorean Theorem states, “In a right triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides”

In the following right triangle, \( p^2 = r^2 + q^2 \).

\textbf{STRANDS} \hspace{10pt} \textbf{Number Sense and Numeration, Measurement, Geometry and Spatial Sense}
Problem of the Week
Problem C and Solution
How Far Around is It?

Problem
In the diagram, \( \triangle ABC \) is a right triangle with \( \angle ABC = 90^\circ \), \( BD = 6 \) m, \( AB = 8 \) m, and the area of \( \triangle ADC \) is 50% more than the area of \( \triangle ABD \). Determine the perimeter of \( \triangle ADC \).

Solution
Let \( a \) be the length of side \( DC \), \( b \) be the length of side \( AC \), and \( c \) be the length of side \( AD \). Draw a line through \( A \) parallel to \( BC \). The distance between this line and \( BC \) is 8 m. Note that this distance is also the height of \( \triangle ABD \) and \( \triangle ADC \).

To find the area of a triangle, multiply the length of the base by the height and divide by 2. Therefore,

\[
\text{area of } \triangle ABD = AB \times BD \div 2 = 8 \times 6 \div 2 = 24 \text{ m}^2.
\]

The area of \( \triangle ADC \) is 50% more than the area of \( \triangle ABD \). Therefore,

\[
\text{area of } \triangle ADC = \left( \text{area of } \triangle ABD \right) + \frac{1}{2}(\text{area of } \triangle ABD) = 24 + 12 = 36 \text{ m}^2.
\]

But the area of \( \triangle ADC = (AB)(DC) \div 2 = 8(9) \div 2 = 4(AD) \). Therefore, \( 4(AD) = 36 \) and \( AD = 9 \) m. Then \( BC = BD + DC = 6 + 9 = 15 \) m.

Since \( \triangle ABD \) has a right angle, \( AD^2 = AB^2 + BD^2 = 8^2 + 6^2 = 100 \). Then \( AD = \sqrt{100} = 10 \), since \( AD > 0 \).

Also, \( \triangle ABC \) has a right angle, so \( AC^2 = AB^2 + BC^2 = 8^2 + 15^2 = 64 + 225 = 289 \). Then \( AC = \sqrt{289} = 17 \), since \( AC > 0 \).

\[
\therefore \text{The perimeter of } \triangle ADC = a + b + c
= DC + AC + AD
= 9 + 17 + 10
= 36 \text{ m}
\]

The perimeter of \( \triangle ADC \) is 36 m.
Problem of the Week
Problem C
Cubes, Cubes and More Cubes

A cube measures $10\,\text{cm} \times 10\,\text{cm} \times 10\,\text{cm}$. Three cuts are made parallel to the faces of the cube, as shown, creating 8 identical smaller cubes. What is the difference between the surface of the 8 smaller cubes and the surface area of the original cube?
Problem of the Week
Problem C and Solution
Cubes, Cubes and More Cubes

Problem
A cube measures 10 cm × 10 cm × 10 cm. Three cuts are made parallel to the faces of the cube, as shown, creating 8 identical smaller cubes. What is the difference between the surface of the 8 smaller cubes and the surface area of the original cube?

Solution
Solution 1
Each of the smaller cubes are 5 cm × 5 cm × 5 cm. So each cube’s surface area is \(6 \times 5 \times 5 = 150\) cm\(^2\).
There are 8 small cubes so the new surface area is \(8 \times 150 = 1200\) cm\(^2\).
The surface area of the larger cube is \(6 \times 10 \times 10 = 600\) cm\(^2\).
So the increase in surface area is \(1200\) cm\(^2\) - \(600\) cm\(^2\) = \(600\) cm\(^2\).

Solution 2
Each cut increases the surface area by two 10 cm × 10 cm squares or \(2 \times 10 \times 10 = 200\) cm\(^2\).
There are three cuts. So the increase in area is \(3 \times 200\) cm\(^2\) = \(600\) cm\(^2\).
Locations on the main floor of a house are described using a coordinate system and coordinate geometry.

Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom’s base is located at \((0, 0)\). Vroom starts at its base and moves to the following points, in order, before returning to its base: \((1, 1)\), \((-1, 3)\), \((-3, 3)\), \((-3, 1)\), \((-2, -2)\).

What is the area of the figure that Vroom traced out?
Problem

Locations on the main floor of a house are described using a coordinate system and coordinate geometry. Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom’s base is located at (0, 0). Vroom starts at its base and moves to the following points, in order, before returning to its base: (1, 1), (−1, 3), (−3, 3), (−3, 1), (−2, −2). What is the area of the figure that Vroom traced out?

Solution

Our first task is to draw a diagram to represent Vroom’s path. Label the points $A(0, 0)$, $B(1, 1)$, $C(−1, 3)$, $D(−3, 3)$, $E(−3, 1)$, and $F(−2, −2)$.

We will make a note here about points $F$, $A$ and $B$. If we move 1 unit right and 1 unit up from $F(−2, −2)$, we get to (−1, −1). If we move 1 unit right and 1 unit up from (−1, −1), we get to $A(0, 0)$. If we move 1 unit right and 1 unit up from $A(0, 0)$, we get to $B(1, 1)$. This illustrates that the points $F$, $A$ and $B$ lie on the same line. A detailed discussion of this is provided in high school math courses.

From here, we will present two possible solutions.
Solution 1: Divide the region into triangles and a square.

There are several ways to break the diagram into regions. On the diagram, we have square $CDEG$, triangle $BCG$, and triangle $BEF$. We will determine the area of each figure and then add them together to find the total area of the region that Vroom traced out.

The total area of the region traced out by Vroom is 12 units$^2$.

If we had not used the information that $F$, $A$ and $B$ are three points on the same line we would have to further break up the region $BEF$ into triangles and trapezoids to calculate the area.
Solution 2: Enclose the region with a rectangle.

On the diagram, we have drawn rectangle $DHIG$, the smallest rectangle that encloses the region $ABCDEF$. One of the horizontal sides passes through $C$ and the other passes through $F$. One of the vertical sides passes through $E$ and the other passes through $B$. The rectangle contains four shapes, the figure $ABCDEF$ traced out by Vroom, triangle $CGB$, triangle $BIF$ and triangle $EFH$. To calculate the area of $ABCDEF$, we will subtract the areas of the three triangles from the area of the rectangle.

\[
\text{area of rectangle } DHIG = DH \times DG = 5 \times 4 = 20 \text{ units}^2
\]
\[
\text{area of triangle } CGB = \frac{1}{2} \times CG \times GB = \frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2
\]
\[
\text{area of triangle } BIF = \frac{1}{2} \times FI \times IB = \frac{1}{2} \times 3 \times 3 = 4.5 \text{ units}^2
\]
\[
\text{area of triangle } EFH = \frac{1}{2} \times HF \times HE = \frac{1}{2} \times 1 \times 3 = 1.5 \text{ units}^2
\]
\[
\text{Total area } ABCDEF = 20 - 2 - 4.5 - 1.5 = 12 \text{ units}^2
\]

The total area of the region traced out by Vroom is $12 \text{ units}^2$.

If we had not used the information that $F$, $A$ and $B$ are three points on the same line we would have to break triangle $BIF$ into a triangle and a trapezoid or two triangles and a rectangle to calculate the area.

Extension:
What is the total distance travelled by Vroom? (The full solution is on the next page.)
Solution to the Extension

We will use the diagram from the second solution to help in determining the lengths of the sides of $ABCDEF$. To find the lengths of the sides of the right-angle triangles we will use the Pythagorean Theorem.

In triangle $CGB$, $BC^2 = CG^2 + GB^2 = 2^2 + 2^2 = 4 + 4 = 8$ and $BC = \sqrt{8}$.
In triangle $BIF$, $FB^2 = BF^2 + IF^2 = 3^2 + 3^2 = 9 + 9 = 18$ and $FB = \sqrt{18}$.
In triangle $EFH$, $EF^2 = EH^2 + HF^2 = 3^2 + 1^2 = 9 + 1 = 10$ and $EF = \sqrt{10}$.

The length of $CD = 2$ and the length of $DE = 2$.

The total distance travelled by Vroom is the sum of all of the side lengths of $ABCDEF$.

$BC + CD + DE + EF + FB = \sqrt{8} + 2 + 2 + \sqrt{10} + \sqrt{18} = 4 + \sqrt{8} + \sqrt{10} + \sqrt{18}$

The total distance travelled by Vroom is $(4 + \sqrt{8} + \sqrt{10} + \sqrt{18})$ units. Using a calculator, this distance is approximately 14.2 units.
Measurement
An altitude is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side.

In \( \triangle ABC \), \( CD \) is an altitude. \( AB = 18 \text{ cm} \), \( AC = 20 \text{ cm} \) and \( CD = 16 \text{ cm} \).

An altitude is drawn from \( B \) to \( AC \) intersecting at \( E \). Determine the length of \( BE \).
Problem of the Week
Problem C
New Heights (Original Problem)

An altitude is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side.

In $\triangle ABC$, $CD$ is an altitude. $AB = 16$ cm, $AC = 12$ cm and $CD = 6$ cm.

An altitude is drawn from $B$ to $AC$ extended intersecting at $E$. Determine the length of $BE$.

**STRANDS**  
Measurement, Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
New Heights (Revised)

Problem

An altitude is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side. In $\triangle ABC$, $CD$ is an altitude. $AB = 18$ cm, $AC = 20$ cm and $CD = 16$ cm. An altitude is drawn from $B$ to $AC$ intersecting at $E$. Determine the length of $BE$.

Solution

The area of a triangle is determined using the formula $\text{base} \times \text{height} \div 2$. The height of the triangle is the length of an altitude and the base of the triangle is the length of the side to which a particular altitude is drawn.

\[
\text{Area } \triangle ABC = \frac{(CD) \times (AB)}{2}
\]

\[
= \frac{16 \times 18}{2}
\]

\[
= 144 \text{ cm}^2
\]

But, $\text{Area } \triangle ABC = \frac{(BE) \times (AC)}{2}$

\[
144 = \frac{(BE) \times 20}{2}
\]

\[
144 = 10 \times BE
\]

\[
14.4 \text{ cm} = BE
\]

Therefore, the length of altitude $BE$ is 14.4 cm.
Problem of the Week
Problem C and Solution
New Heights (Original Problem)

Problem

An *altitude* is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side. In $\triangle ABC$, $CD$ is an altitude. $AB = 16$ cm, $AC = 12$ cm and $CD = 6$ cm. An altitude is drawn from $B$ to $AC$ extended intersecting at $E$. Determine the length of $BE$.

Solution

The area of a triangle is determined using the formula $\text{base} \times \text{height} \div 2$. The height of the triangle is the length of an altitude and the base of the triangle is the length of the side to which a particular altitude is drawn.

\[
\text{Area } \triangle ABC = \frac{(CD) \times (AB)}{2}
\]
\[
= \frac{6 \times 16}{2}
\]
\[
= 48 \text{ cm}^2
\]

But, \[
\text{Area } \triangle ABC = \frac{(BE) \times (AC)}{2}
\]
\[
48 = \frac{(BE) \times 12}{2}
\]
\[
48 = 6 \times BE
\]
\[
8 \text{ cm} = BE
\]

Therefore, the length of altitude $BE$ is 8 cm.
Two paths are built from $A$ to $F$ as shown.

The distance from $A$ to $F$ in a straight line is 100 m. Points $B$, $C$, $D$, and $E$ lie along $AF$ such that $AB = BC = CD = DE = EF$.

The upper path, shown with a dashed line, is a semi-circle with diameter $AF$. The lower path, shown with a solid line, consists of five semi-circles with diameters $AB$, $BC$, $CD$, $DE$, and $EF$.

Starting at the same time, Bev and Mike ride their tricycles along these paths from $A$ to $F$. Bev rides along the upper path from $A$ to $F$ while Mike rides along the lower path from $A$ to $F$. If they ride at the same speed, who will get to $F$ first?
Problem of the Week
Problem C and Solution
Curvy Contest

Problem
Two paths are built from $A$ to $F$ as shown on the diagram above. The distance from $A$ to $F$ in a straight line is 100 m. Points $B$, $C$, $D$, and $E$ lie along $AF$ such that $AB = BC = CD = DE = EF$. The upper path, shown with a dashed line, is a semi-circle with diameter $AF$. The lower path, shown with a solid line, consists of five semi-circles with diameters $AB$, $BC$, $CD$, $DE$, and $EF$. Starting at the same time, Bev and Mike ride their tricycles along these paths from $A$ to $F$. Bev rides along the upper path from $A$ to $F$ while Mike rides along the lower path from $A$ to $F$. If they ride at the same speed, who will get to $F$ first?

Solution
The circumference of a circle is found by multiplying its diameter by $\pi$. To find the circumference of a semi-circle, divide its circumference by 2.

The length of the upper path is equal to half the circumference of a circle with diameter 100 m. The length of the upper path equals $\pi \times 100 \div 2 = 50\pi$ m. (This is approximately 157.1 m.)

Each of the semi-circles along the lower path have the same diameter. The diameter of each of these semi-circles is $100 \div 5 = 20$ m. The length of the lower path is equal to half the circumference of five circles, each with diameter 20 m. The distance along the lower path equals $5 \times (\pi \times 20 \div 2) = 5 \times (10\pi) = 50\pi$ m.

Since both Bev and Mike ride at the same speed and both travel the same distance, they will arrive at point $F$ at the same time. Neither wins the race since both arrive at the same time. The answer to the problem may surprise you. Most people, at first glance, would think that the upper path is longer.

If you were to extend the problem so that Bev travels the same route but Mike travels along a lower path made up of 100 semi-circles of equal diameter from $A$ to $F$, they would still both travel exactly the same distance, $50\pi$ m. Check it out!
Problem of the Week
Problem C
No Longer a Rectangle

In the following slightly irregular shape,

- $AB = 50$ cm, $CD = 15$ cm, $EF = 30$ cm;
- the area of the shaded triangle, $\triangle DEF$, is $210$ cm$^2$; and
- the area of the entire figure, $ABCDE$, is $1000$ cm$^2$.

Determine the length of $AE$. 

**Strands**  Measurement, Patterning and Algebra
Problem of the Week
Problem C and Solution
No Longer a Rectangle

Problem
In the following slightly irregular shape, \( AB = 50 \text{ cm}, \ CD = 15 \text{ cm}, \ EF = 30 \text{ cm}; \) the area of the shaded triangle, \( \triangle DEF, \) is \( 210 \text{ cm}^2; \) and the area of the entire figure, \( ABCDE, \) is \( 1000 \text{ cm}^2. \) Determine the length of \( AE. \)

Solution
The first task is to mark the given information on the diagram. This has been completed on the diagram to the right. \( EG \) has been extended to meet \( BC \) at \( H. \)

To find the area of a triangle, multiply the length of the base by the height and divide by 2. In \( \triangle DEF, \) the base, \( EF, \) has length 30 cm. The height of \( \triangle DEF \) is the perpendicular distance from \( EF \) (extended) to vertex \( D, \) namely \( GD. \) The area is given. So

\[
\text{Area } \triangle DEF = \frac{30 \times GD}{2} \\
210 = 15 \times GD \\
14 = GD
\]

We know that \( EH = AB = 50, \ GH = DC = 15, \) and \( EH = EF + FG + GH. \) It follows that \( 50 = 30 + FG + 15 \) and \( FG = 5 \text{ cm}. \)

Now we can relate the total area to the areas contained inside.

\[
\text{Area } ABCDE = \text{Area } ABHE + \text{Area } CDGH + \text{Area } \triangle DFG + \text{Area } \triangle DEF \\
1000 = AB \times AE + GD \times DC + \frac{FG \times GD}{2} + 210 \\
1000 = 50 \times AE + 14 \times 15 + \frac{5 \times 14}{2} + 210 \\
1000 = 50 \times AE + 210 + 35 + 210 \\
1000 = 50 \times AE + 455 \\
1000 - 455 = 50 \times AE \\
545 = 50 \times AE \\
\frac{545}{50} = AE
\]

\( \therefore \ AE = 10.9 \text{ cm}. \)
Problem of the Week
Problem C
Around We Go

A circle with centre $O$ has a point $A$ on the circumference. Radius $OA$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OB$. Point $A$ is then connected to point $B$. Radius $OB$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OC$. Point $B$ is then connected to point $C$.

The process of clockwise rotations continues until some radius rotates back onto $OA$. Every point on the circumference is connected to the points immediately adjacent to it as a result of the process. A polygon is created.

![Construction of the polygon](image1)

![Resulting Polygon](image2)

a) Determine the number of sides of the polygon.

b) Determine the sum of the angles in the polygon. That is, determine the sum of the angles at each of the vertices of the polygon.

**Strands**  Geometry and Spatial Sense, Measurement
Problem of the Week
Problem C and Solution
Around We Go

Problem
A circle with centre $O$ has a point $A$ on the circumference. Radius $OA$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OB$. Point $A$ is then connected to point $B$. Radius $OB$ is rotated $20^\circ$ clockwise about the centre, resulting in the image $OC$. Point $B$ is then connected to point $C$.

The process of clockwise rotations continues until some radius rotates back onto $OA$. Every point on the circumference is connected to the points immediately adjacent to it as a result of the process. A polygon is created.

a) Determine the number of sides of the polygon.

b) Determine the sum of the angles in the polygon. That is, determine the sum of the angles at each of the vertices of the polygon.

Solution
Each time the process is repeated, another congruent triangle is created. Each of these triangles has a $20^\circ$ angle at $O$, the centre of the circle. But a complete rotation at the centre is $360^\circ$. Since each angle in the triangles at the centre of the circle is $20^\circ$ and the total measure at the centre is $360^\circ$, then there are $360 \div 20 = 18$ triangles formed. This means that there are 18 distinct points on the circumference of the circle and the polygon has 18 sides. An 18-sided polygon is called an octadecagon, from octa meaning 8 and deca meaning 10.

The other two angles in the each of the congruent triangles are equal. (Two sides of the triangle are radii of the circle. The triangles are therefore isosceles.) The angles in a triangle sum to $180^\circ$ so after the $20^\circ$ angle is removed, there is $160^\circ$ remaining for the other two angles. It follows that each of the other two angles in each triangle measures $160^\circ \div 2 = 80^\circ$. The following diagram illustrates this information for the two adjacent triangles $AOB$ and $BOC$.

Each angle in the polygon is formed by an $80^\circ$ angle from one triangle and the adjacent $80^\circ$ angle from the next triangle. For example, $\angle ABC = \angle ABO + \angle OBC = 80^\circ + 80^\circ = 160^\circ$. There are 18 vertices in the octadecagon and the angle at each vertex is $160^\circ$. Therefore the sum of the angles in the octadecagon is $18 \times 160^\circ = 2880^\circ$.

Diagrams are provided on the next page to further support the solution.
Angle at the Centre of a Circle

Completing the Construction

The Octadecagon - 18 sided Polygon

Notice the vertices of the octadecagon are labelled $A$ to $S$, but the letter $O$ is missing since it was used in the original construction as the centre of the circle.
Problem of the Week

Problem C

The Inner Square

ABCD is a square with area $64 \text{ m}^2$. E, F, G, and H are points on sides $AB$, $BC$, $CD$, and $DA$, respectively, such that $AE = BF = CG = DH = 2 \text{ m}$. E, F, G, and H are connected to form square $EFGH$.

Determine the area of $EFGH$. 

Strands  Geometry and Spatial Sense, Measurement
Problem of the Week
Problem C and Solution
The Inner Square

Problem

$ABCD$ is a square with area $64 \text{ m}^2$. $E, F, G,$ and $H$ are points on sides $AB, BC, CD,$ and $DA$, respectively, such that $AE = BF = CG = DH = 2 \text{ m}$. $E, F, G,$ and $H$ are connected to form square $EFGH$. Determine the area of $EFGH$.

Solution

The area of square $ABCD$ is $64 \text{ m}^2$. Therefore the side lengths are $8 \text{ m}$ since $8 \times 8 = 64$ and the area is calculated by multiplying the length and the width.

Each of the smaller parts of the sides of square $ABCD$ are $2 \text{ m}$ so the longer parts of the sides are $8 - 2 = 6 \text{ m}$.

Approach #1 to finding the area of square $EFGH$

In right $\triangle HAE$, $AE = 2$ and $AH = 6$. We can use one side as the base and the other as the height in the calculation of the area of the triangle since they are perpendicular to each other. Therefore the area of $\triangle HAE = \frac{AE \times AH}{2} = \frac{2 \times 6}{2} = 6 \text{ m}^2$. Since each of the triangles has the same base length and height, their areas are equal and the total area of the four triangles is $4 \times 6 = 24 \text{ m}^2$.

The area of square $EFGH$ can be determined by subtracting the area of the four triangles from the area of square $ABCD$. Therefore the area of square $EFGH = 64 - 24 = 40 \text{ m}^2$.

Approach #2 to finding the area of square $EFGH$

Some students may be familiar with the Pythagorean Theorem. This theorem states that in a right triangle, the square of the length of the hypotenuse (the longest side) is equal to the sum of the squares of the other two sides. The longest side is located opposite the right angle.

In right $\triangle HAE$, $AE = 2$, $AH = 6$ and $HE$ is the hypotenuse. Therefore,

$$HE^2 = AE^2 + AH^2$$

$$= 2^2 + 6^2$$

$$= 4 + 36$$

$$\therefore HE^2 = 40 \quad \text{(See the note below.)}$$

Taking the square root,

$$HE = \sqrt{40} \text{ m}$$

But $EFGH$ is a square so all of its side lengths equal $\sqrt{40}$. The area is calculated by multiplying the length and the width. The area of $EFGH = \sqrt{40} \times \sqrt{40} = 40 \text{ m}^2$.

Therefore, the area of square $EFGH$ is $40 \text{ m}^2$.

Note: The area of the square is $HE^2 = 40 \text{ m}^2$. We found this when we used the Pythagorean Theorem above.
Problem of the Week
Problem C
How Far Around is It?

In the diagram, \( \triangle ABC \) is a right triangle with \( \angle ABC = 90^\circ \), \( BD = 6 \text{ m} \), \( AB = 8 \text{ m} \), and the area of \( \triangle ADC \) is 50\% more than the area of \( \triangle ABD \).

Determine the perimeter of \( \triangle ADC \).

The Pythagorean Theorem states, “In a right triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides”

In the following right triangle, \( p^2 = r^2 + q^2 \).

**Strands**  
Number Sense and Numeration, Measurement, Geometry and Spatial Sense
Problem of the Week
Problem C and Solution
How Far Around is It?

Problem
In the diagram, \( \triangle ABC \) is a right triangle with \( \angle ABC = 90^\circ \), \( BD = 6 \text{ m} \), \( AB = 8 \text{ m} \), and the area of \( \triangle ADC \) is 50% more than the area of \( \triangle ABD \). Determine the perimeter of \( \triangle ADC \).

Solution
Let \( a \) be the length of side \( DC \), \( b \) be the length of side \( AC \), and \( c \) be the length of side \( AD \). Draw a line through \( A \) parallel to \( BC \). The distance between this line and \( BC \) is 8 m. Note that this distance is also the height of \( \triangle ABD \) and \( \triangle ADC \).

To find the area of a triangle, multiply the length of the base by the height and divide by 2. Therefore,

\[
\text{area of } \triangle ABD = AB \times BD \div 2 = 8 \times 6 \div 2 = 24 \text{ m}^2.
\]

The area of \( \triangle ADC \) is 50% more than the area of \( \triangle ABD \). Therefore,

\[
\text{area of } \triangle ADC = (\text{area of } \triangle ABD) + \frac{1}{2}(\text{area of } \triangle ABD) = 24 + 12 = 36 \text{ m}^2.
\]

But the area of \( \triangle ADC \) = \( (AB)(DC) \div 2 \) = 8\( (DC) \div 2 \) = 4\( (DC) \). Therefore, 4\( (DC) \) = 36 and \( DC = 9 \text{ m} \). Then \( BC = BD + DC = 6 + 9 = 15 \text{ m} \).

Since \( \triangle ABD \) has a right angle, \( AD^2 = AB^2 + BD^2 = 8^2 + 6^2 = 100 \). Then \( AD = \sqrt{100} = 10 \), since \( AD > 0 \).

Also, \( \triangle ABC \) has a right angle, so

\[
AC^2 = AB^2 + BC^2 = 8^2 + 15^2 = 64 + 225 = 289.
\]

Then \( AC = \sqrt{289} = 17 \), since \( AC > 0 \).

Therefore,

\[
\therefore \text{The perimeter of } \triangle ADC = a + b + c = DC + AC + AD = 9 + 17 + 10 = 36 \text{ m}
\]

The perimeter of \( \triangle ADC \) is 36 m.
Problem of the Week
Problem C
Cubes, Cubes and More Cubes

A cube measures 10 cm × 10 cm × 10 cm. Three cuts are made parallel to the faces of the cube, as shown, creating 8 identical smaller cubes. What is the difference between the surface of the 8 smaller cubes and the surface area of the original cube?
Problem of the Week
Problem C and Solution
Cubes, Cubes and More Cubes

Problem
A cube measures 10 cm × 10 cm × 10 cm. Three cuts are made parallel to the faces of the cube, as shown, creating 8 identical smaller cubes. What is the difference between the surface of the 8 smaller cubes and the surface area of the original cube?

Solution
Solution 1
Each of the smaller cubes are 5 cm × 5 cm × 5 cm. So each cube’s surface area is 6 × 5 × 5 = 150 cm²
There are 8 small cubes so the new surface area is 8 × 150 = 1200 cm²
The surface area of the larger cube is 6 × 10 × 10 = 600 cm²
So the increase in surface area is 1200 cm² - 600 cm² = 600 cm²

Solution 2
Each cut increases the surface area by two 10 cm × 10 cm squares or 2 × 10 × 10 = 200 cm².
There are three cuts. So the increase in area is 3 × 200 cm² = 600 cm²
Problem of the Week
Problem C
Clean Up

Locations on the main floor of a house are described using a coordinate system and coordinate geometry.

Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom’s base is located at \((0, 0)\). Vroom starts at its base and moves to the following points, in order, before returning to its base: \((1, 1)\), \((-1, 3)\), \((-3, 3)\), \((-3, 1)\), \((-2, -2)\).

What is the area of the figure that Vroom traced out?
Problem of the Week
Problem C and Solution

Clean Up

Problem
Locations on the main floor of a house are described using a coordinate system and coordinate geometry. Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom’s base is located at \((0, 0)\). Vroom starts at its base and moves to the following points, in order, before returning to its base: \((1, 1)\), \((-1, 3)\), \((-3, 3)\), \((-3, 1)\), \((-2, -2)\). What is the area of the figure that Vroom traced out?

Solution
Our first task is to draw a diagram to represent Vroom’s path. Label the points \(A(0, 0)\), \(B(1, 1)\), \(C(-1, 3)\), \(D(-3, 3)\), \(E(-3, 1)\), and \(F(-2, -2)\).

![Diagram of the figure traced out by Vroom](image)

We will make a note here about points \(F\), \(A\) and \(B\). If we move 1 unit right and 1 unit up from \(F(-2, -2)\), we get to \((-1, -1)\). If we move 1 unit right and 1 unit up from \((-1, -1)\), we get to \(A(0, 0)\). If we move 1 unit right and 1 unit up from \(A(0, 0)\), we get to \(B(1, 1)\). This illustrates that the points \(F\), \(A\) and \(B\) lie on the same line. A detailed discussion of this is provided in high school math courses.

From here, we will present two possible solutions.
Solution 1: Divide the region into triangles and a square.

There are several ways to break the diagram into regions. On the diagram, we have square $CDEG$, triangle $BCG$, and triangle $BEF$. We will determine the area of each figure and then add them together to find the total area of the region that Vroom traced out.

area of square $CDEG = CD \times DE = 2 \times 2 = 4 \text{ units}^2$

area of triangle $BCG = \frac{1}{2} \times GB \times CG = \frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2$

area of triangle $BEF = \frac{1}{2} \times BE \times HF = \frac{1}{2} \times 4 \times 3 = 6 \text{ units}^2$

Total area $ABCDEF = 4 + 2 + 6 = 12 \text{ units}^2$

The total area of the region traced out by Vroom is 12 units$^2$.

If we had not used the information that $F$, $A$ and $B$ are three points on the same line we would have to further break up the region $BEF$ into triangles and trapezoids to calculate the area.
Solution 2: Enclose the region with a rectangle.

On the diagram, we have drawn rectangle $DHIG$, the smallest rectangle that encloses the region $ABCDEF$. One of the horizontal sides passes through $C$ and the other passes through $F$. One of the vertical sides passes through $E$ and the other passes through $B$. The rectangle contains four shapes, the figure $ABCDEF$ traced out by Vroom, triangle $CGB$, triangle $BIF$ and triangle $EFH$. To calculate the area of $ABCDEF$, we will subtract the areas of the three triangles from the area of the rectangle.

\[
\text{area of rectangle } DHIG = DH \times DG = 5 \times 4 = 20 \text{ units}^2
\]

\[
\text{area of triangle } CGB = \frac{1}{2} \times CG \times GB = \frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2
\]

\[
\text{area of triangle } BIF = \frac{1}{2} \times FI \times IB = \frac{1}{2} \times 3 \times 3 = 4.5 \text{ units}^2
\]

\[
\text{area of triangle } EFH = \frac{1}{2} \times HF \times HE = \frac{1}{2} \times 1 \times 3 = 1.5 \text{ units}^2
\]

\[
\text{Total area } ABCDEF = 20 - 2 - 4.5 - 1.5 = 12 \text{ units}^2
\]

The total area of the region traced out by Vroom is 12 units$^2$.

If we had not used the information that $F$, $A$ and $B$ are three points on the same line we would have to break triangle $BIF$ into a triangle and a trapezoid or two triangles and a rectangle to calculate the area.

**Extension:**

What is the total distance travelled by Vroom? (The full solution is on the next page.)
Solution to the Extension

We will use the diagram from the second solution to help in determining the lengths of the sides of $ABCDEF$. To find the lengths of the sides of the right-angle triangles we will use the Pythagorean Theorem.

In triangle $CGB$, $BC^2 = CG^2 + GB^2 = 2^2 + 2^2 = 4 + 4 = 8$ and $BC = \sqrt{8}$.

In triangle $BIF$, $FB^2 = BI^2 + IF^2 = 3^2 + 3^2 = 9 + 9 = 18$ and $FB = \sqrt{18}$.

In triangle $EFH$, $EF^2 = EH^2 + HF^2 = 3^2 + 1^2 = 9 + 1 = 10$ and $EF = \sqrt{10}$.

The length of $CD = 2$ and the length of $DE = 2$.

The total distance travelled by Vroom is the sum of all of the side lengths of $ABCDEF$.

$BC + CD + DE + EF + FB = \sqrt{8} + 2 + 2 + \sqrt{10} + \sqrt{18} = 4 + \sqrt{8} + \sqrt{10} + \sqrt{18}$

The total distance travelled by Vroom is $(4 + \sqrt{8} + \sqrt{10} + \sqrt{18})$ units. Using a calculator, this distance is approximately 14.2 units.
An altitude is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side.

In $\triangle ABC$, $CD$ is an altitude. $AB = 18$ cm, $AC = 20$ cm and $CD = 16$ cm.

An altitude is drawn from $B$ to $AC$ intersecting at $E$. Determine the length of $BE$. 
Problem of the Week
Problem C
New Heights (Original Problem)

An *altitude* is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side.

In $\triangle ABC$, $CD$ is an altitude. $AB = 16$ cm, $AC = 12$ cm and $CD = 6$ cm.

An altitude is drawn from $B$ to $AC$ extended intersecting at $E$. Determine the length of $BE$. 

**STRANDS**  Measurement, Number Sense and Numeration, Patterning and Algebra
Problem of the Week  
Problem C and Solution  
New Heights (Revised)

Problem

An\textit{ altitude} is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side. In $\triangle ABC$, $CD$ is an altitude. $AB = 18$ cm, $AC = 20$ cm and $CD = 16$ cm. An altitude is drawn from $B$ to $AC$ intersecting at $E$. Determine the length of $BE$.

Solution

The area of a triangle is determined using the formula $base \times height \div 2$. The height of the triangle is the length of an altitude and the base of the triangle is the length of the side to which a particular altitude is drawn.

\[
\text{Area } \triangle ABC = \frac{(CD) \times (AB)}{2} = \frac{16 \times 18}{2} = 144 \text{ cm}^2
\]

But, \[
\text{Area } \triangle ABC = \frac{(BE) \times (AC)}{2} = \frac{(BE) \times 20}{2} = 144 = 10 \times BE
\]

Therefore, the length of altitude $BE$ is 14.4 cm.
Problem of the Week
Problem C and Solution
New Heights (Original Problem)

Problem

An *altitude* is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side. In \(\triangle ABC\), \(CD\) is an altitude. \(AB = 16\) cm, \(AC = 12\) cm and \(CD = 6\) cm. An altitude is drawn from \(B\) to \(AC\) extended intersecting at \(E\). Determine the length of \(BE\).

Solution

The area of a triangle is determined using the formula \(\text{base} \times \text{height} \div 2\). The height of the triangle is the length of an altitude and the base of the triangle is the length of the side to which a particular altitude is drawn.

\[
\text{Area } \triangle ABC = \frac{(CD) \times (AB)}{2} = \frac{6 \times 16}{2} = 48 \text{ cm}^2
\]

But, \(\text{Area } \triangle ABC = \frac{(BE) \times (AC)}{2}\)

\[
48 = \frac{(BE) \times 12}{2}
\]

\[
48 = 6 \times BE
48 = 8 \text{ cm} = BE
\]

Therefore, the length of altitude \(BE\) is 8 cm.
“Old MacDonald had a farm, E-I-E-I-O”, says the old children’s song. But Old MacDonald did have a farm! And on that farm he had some horses, cows, pigs and 69 water troughs for the animals to drink from. Only horses drank from the horse troughs, exactly two horses for each trough. Only cows drank from cow troughs, exactly three cows per trough. And only pigs drank from pig troughs, exactly eight pigs per trough. Old MacDonald’s farm has the same number of cows, horses and pigs.

How many animals does Old MacDonald have on his farm?
Problem of the Week
Problem C and Solution
And On This Farm

Problem
“Old MacDonald had a farm, E-I-E-I-O”, says the old children’s song. But Old MacDonald did have a farm! And on that farm he had some horses, cows, pigs and 69 water troughs for the animals to drink from. Only horses drank from the horse troughs, exactly two horses for each trough. Only cows drank from cow troughs, exactly three cows per trough. And only pigs drank from pig troughs, exactly eight pigs per trough. Old MacDonald’s farm has the same number of cows, horses and pigs.

How many animals does Old MacDonald have on his farm?

Solution
Solution 1
Let $n$ represent the number of each type of animal.

Since there are two horses for every horse trough, then $n$ must be divisible by 2.
Since there are three cows for every cow trough, then $n$ must be divisible by 3.
Since there are eight pigs for every pig trough, then $n$ must be divisible by 8.
Therefore, $n$ must be divisible by 2, 3 and 8. The smallest number divisible by 2, 3, and 8 is 24. (This number is called the lowest common multiple or \textit{LCM}, for short.)

If there are 24 of each kind of animal, there would be $24 \div 2 = 12$ troughs for horses, $24 \div 3 = 8$ troughs for cows and $24 \div 8 = 3$ troughs for pigs. This would require a total of $12 + 8 + 3 = 23$ troughs. Since there are 69 troughs and $69 \div 23 = 3$, we require 3 times more of each type of animal. That is, there would be $24 \times 3 = 72$ of each type of animal. The total number of animals is $72 + 72 + 72$ or 216.

We can check the correctness of this solution. Since there are two horses for every horse trough, there are $72 \div 2 = 36$ horse troughs. Since there are three cows for every cow trough, there are $72 \div 3 = 24$ cow troughs. Since there are eight pigs for every pig trough, there are $72 \div 8 = 9$ pig troughs. The total number of troughs is $36 + 24 + 9 = 69$, as expected.
Solution 2

In this solution, algebra and equation solving will be used to solve the problem.

Let \( n \) represent the number of each type of animal.

Since there are \( n \) horses and there are two horses for every horse trough, then there would be \( \frac{n}{2} \) troughs for horses. Since there are \( n \) cows and there are three cows for every cow trough, then there would be \( \frac{n}{3} \) troughs for cows. Since there are \( n \) pigs and there are eight pigs for every pig trough, then there would be \( \frac{n}{8} \) troughs for pigs. Since there are 69 troughs in total,

\[
\frac{n}{2} + \frac{n}{3} + \frac{n}{8} = 69
\]

Common denominator 24

\[
\frac{12n}{24} + \frac{8n}{24} + \frac{3n}{24} = 69 \quad \text{common denominator 24}
\]

Simplify the fractions

\[
\frac{23n}{24} = 69
\]

Multiply both sides by 24

\[
23n = 24 \times 69
\]

Simplify

\[
23n = 1656
\]

Divide both sides by 23

\[
n = \frac{1656}{23}
\]

\[
n = 72
\]

There are 72 of each type of animal, a total of 216 animals.

Solution 3

In this solution, ratios will be used to solve the problem.

Let \( n \) represent the number of each type of animal. The ratio of the number of troughs required for the pigs to the number of troughs required for the cows is

\[
\frac{n}{8} : \frac{n}{3} = 3 \quad \text{ratio of troughs for pigs to troughs for cows}
\]

\[
\frac{3n}{24} : \frac{8n}{24} = 3n : 8n = 3 : 8
\]

Similarly, the ratio of the number of troughs required for the cows to the number of troughs required for the horses is \( 2 : 3 = 8 : 12 \). So the ratio of the number of troughs required for the pigs to the number required for the cows to the number required for the horses is \( 3 : 8 : 12 \).

Let the number of troughs required for the pigs be \( 3k \), for the cows be \( 8k \) and for the horses be \( 12k \), for some positive integer value of \( k \).

Since the total number of troughs required is 69, then

\[
3k + 8k + 12k = 69
\]

\[
23k = 69
\]

\[
k = 3
\]

The number of troughs required for the pigs is \( 3k = 9 \). There are 8 pigs at each trough. There are a total of \( 9 \times 8 = 72 \) pigs. Since there are the same number of each animal, there are also 72 cows and 72 horses. There are a total of \( 72 + 72 + 72 = 216 \) animals on Old MacDonald’s farm.
Problem of the Week
Problem C
Peculiar Perennials

A perennial is a plant that blooms over the spring and summer. The plant dies off over the autumn and winter but returns again the following spring.

There are two known species of the POTW perennial plant, the pink ProbleminusA plant and the red ProbleminusB plant. The ProbleminusA is a pink flowering plant in its first year. The following spring it turns into a red ProbleminusB plant. That is, each ProbleminusA plant blooms as a ProbleminusB plant the next year.

![Diagram of pink ProbleminusA plant turning into red ProbleminusB plant]

The ProbleminusB is a red plant. This plant blooms the following spring and also produces a pink ProbleminusA plant. That is, each ProbleminusB plant returns the following spring along with a new ProbleminusA plant.

![Diagram of red ProbleminusB plant returning as pink ProbleminusA plant]

Every year this cycle reoccurs.

Today in our garden we planted three ProbleminusA plants and two ProbleminusB plants. Assuming that the plants behave exactly as described, how many plants will be in the garden after 10 reproduction cycles?

**Strands** Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
Peculiar Perennials

Problem
A perennial is a plant that blooms over the spring and summer. The plant dies off over the autumn and winter but returns again the following spring. There are two known species of the POTW perennial plant, the pink ProbleminusA plant and the red ProbleminusB plant. The ProbleminusA is a pink flowering plant in its first year. The following spring it turns into a red ProbleminusB plant. That is, each ProbleminusA plant blooms as a ProbleminusB plant the next year. The ProbleminusB is a red plant. This plant blooms the following spring and also produces a pink ProbleminusA plant. That is, each ProbleminusB plant returns the following spring along with a new ProbleminusA plant. Every year this cycle reoccurs. Today in our garden we planted three ProbleminusA plants and two ProbleminusB plants. Assuming that the plants behave exactly as described, how many plants will be in the garden after 10 reproduction cycles?

Solution
Today we have 3 ProbleminusA plants and 2 ProbleminusB plants. In one year the 3 ProbleminusA plants become 3 ProbleminusB plants. In the same year the 2 ProbleminusB plants remain and produce 2 ProbleminusA plants. So after one year there are 2 ProbleminusA plants and $3 + 2 = 5$ ProbleminusB plants.

Proceeding from year 1 to year 2, the 2 ProbleminusA plants are now 2 ProbleminusB plants. The 5 ProbleminusB plants remain and produce 5 ProbleminusA plants. After two years, there are 5 ProbleminusA plants and $2 + 5 = 7$ ProbleminusB plants.

At this point we can make an observation. The number of ProbleminusA plants in a given year equals the number of ProbleminusB plants from the previous year. The number of ProbleminusB plants is the sum of the number of ProbleminusA and the number of ProbleminusB plants from the previous year.

We will use this to produce a chart.

<table>
<thead>
<tr>
<th>Year Number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td># of ProbleminusA</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>19</td>
<td>31</td>
<td>50</td>
<td>81</td>
<td>131</td>
<td>212</td>
</tr>
<tr>
<td># of ProbleminusB</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>19</td>
<td>31</td>
<td>50</td>
<td>81</td>
<td>131</td>
<td>212</td>
<td>343</td>
</tr>
</tbody>
</table>

After 10 reproduction cycles, there are 212 ProbleminusA plants and 343 ProbleminusB plants for a total of 555 plants in the garden.
The number of a particular flower type in a specific year is dependent on the number of flowers of each of the types from the previous year. This is an example of a recursion.

In our problem,

\[
\text{# of ProbleminusA this year} = \text{# of ProbleminusB the previous year}
\]

and

\[
\text{# of ProbleminusB this year} = \text{# of ProbleminusA the previous year} + \text{# of ProbleminusB the previous year}
\]

A famous example of a recursion is known as the Fibonacci Sequence. The first two numbers in the sequence of numbers are defined. The first number, also called a term, is 1 and the second number is 1. Each remaining term in the sequence is equal to the sum of the two previous terms.

So, the third term is equal to the sum of the first and second terms, and is therefore \(1 + 1 = 2\).

We can now determine the fourth term in the sequence. It will be the sum of the second and third terms, and is therefore \(1 + 2 = 3\).

The fifth term in the sequence is the sum of the third and fourth terms, and is therefore \(2 + 3 = 5\).

We can continue generating more terms in the sequence by applying the rule.

The first 20 Fibonacci numbers are

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765
\]

In our problem, the number of ProbleminusB plants is the sum of the number of ProbleminusA and ProbleminusB plants in the previous year. If we had started with only 1 ProbleminusA plant and 0 ProbleminusB plants, the number of ProbleminusB plants in years 1 to 10 would be the first 10 terms of the Fibonacci sequence.

Do an internet search to discover more about the Fibonacci sequence and recursions in general.
Problem of the Week
Problem C
Have YOU Got a Minute?

My clock is a perfectly good clock. It keeps exact time. But it only has an hour hand. Today, in the afternoon, I looked at my clock and discovered that the hour hand was \( \frac{7}{8} \) of the distance between the “4” and the “5”.

Determine the exact time (hours, minutes and seconds).
Problem of the Week
Problem C and Solution
Have YOU Got a Minute?

Problem
My clock is a perfectly good clock. It keeps exact time. But it only has an hour hand. Today, in the afternoon, I looked at my clock and discovered that the hour hand was $\frac{7}{8}$ of the distance between the “4” and the “5”. Determine the exact time (hours, minutes and seconds).

Solution
To solve this problem we note that in one hour, the hour hand travels $\frac{1}{12}$ of a complete revolution while the minute hand travels a complete revolution or 60 minutes.

Since the hour hand is $\frac{7}{8}$ of the distance between the “4” and the “5”, the minute hand will travel $\frac{7}{8}$ of a complete revolution or $\frac{7}{8}$ of 60 minutes which is $\frac{7}{8} \times 60$ or $52\frac{1}{2}$ minutes.

Since we want the time in hours, minutes and seconds, we need to convert $\frac{1}{2}$ minute to seconds.

The number of seconds may be obvious but the calculation, $\frac{1}{2} \text{ minute} \times \frac{60 \text{ seconds}}{1 \text{ minute}} = 30 \text{ seconds}$, is provided for completeness.

Therefore, the precise time is 30 seconds after 4:52 p.m. This can be written 4:52:30 p.m. or 16:52:30 using the twenty-four hour clock.
Problem of the Week
Problem C
Places for Pigeons

One hundred pigeons are to be housed in identical-sized cages under the following conditions:

- each cage must contain at least one pigeon;
- no two cages can contain the same number of pigeons; and
- no cages can go inside any other cage.

Determine the maximum number of cages required to house the pigeons.
Problem of the Week
Problem C and Solution
Places for Pigeons

Problem

One hundred pigeons are to be housed in identical-sized cages under the following conditions:

- each cage must contain at least one pigeon;
- no two cages can contain the same number of pigeons; and
- no cages can go inside any other cage.

Determine the maximum number of cages required to house the pigeons.

Solution

In order to maximize the number of cages, each cage must contain the smallest number of birds possible. However, no two cages can contain the same number of pigeons. The simplest way to determine this number is to put one pigeon in the first cage and then let the number of pigeons in each cage after that be one more than the number of pigeons in the cage before it, until all 100 pigeons are housed.

Put 1 pigeon in the first cage, 2 pigeons in the second cage, 3 pigeons in the third cage, and so on. After filling 12 cages in this manner, we have

\[1 + 2 + 3 + \cdots + 11 + 12 = 78\]

pigeons housed. After putting 13 pigeons in the thirteenth cage,

\[78 + 13 = 91\]

pigeons are housed. There are 9 pigeons left to house. But we already have a cage containing 9 pigeons. The remaining 9 pigeons must be distributed among the existing cages while maintaining the condition that no two cages contain the same number of pigeons.

The most obvious way to do this is to put the 9 pigeons in the last cage which already contains 13 pigeons. This would mean that the final cage would contain

\[13 + 9 = 22\]

birds! A better solution might be to increase the number of birds in each of the final nine cages by one bird each. This solution is summarized below:

<table>
<thead>
<tr>
<th>Cage #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td># of birds</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
</tbody>
</table>

The maximum number of cages required is 13. If you had 14 cages, with the first cage holding 1 pigeon and each cage after that holding one more pigeon than the cage before, you could house 105 pigeons, five more than the number of pigeons that we have.
Problem of the Week

Problem C

Sum Blocks

Ten blocks are arranged as illustrated in the following diagram. Each letter shown on the front of a block represents a number. The sum of the numbers on any three consecutive blocks is 19. Determine the value of $S$. 

4 P Q R S T U V 8 W
Problem of the Week
Problem C and Solution
Sum Blocks

Problem
Ten blocks are arranged as illustrated in the following diagram. Each letter shown on the front of a block represents a number. The sum of the numbers on any three consecutive blocks is 19. Determine the value of $S$.

Solution
Since the sum of the numbers on any three consecutive blocks is the same,

\[
4 + P + Q = P + Q + R \\
4 + P + Q = P + Q + R \quad \text{since } P + Q \text{ is common to both sides}
\]

\[\therefore R = 4\]

Again, since the sum of the numbers on any three consecutive blocks is the same,

\[
T + U + V = U + V + 8 \\
T + U + V = U + V + 8 \quad \text{since } U + V \text{ is common to both sides}
\]

\[\therefore T = 8\]

Since the sum of any three consecutive numbers is 19,

\[
R + S + T = 19 \\
4 + S + 8 = 19 \quad \text{substituting } R = 4 \text{ and } T = 8 \\
S + 12 = 19 \\
\therefore S = 7\]

The value of $S$ is 7.

Extension: Can you determine the values of all remaining letters?
Did you know that 1000 can be written as the sum of 16 consecutive integers? That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70.$$ 

The diagram below illustrates a mathematical short form used for writing the above sum. The notation is called *Sigma Notation*.

Using at least two numbers, what is the minimum number of consecutive integers needed to sum to exactly 1000?
Problem of the Week
Problem C and Solution
Add ’em Up

Problem
The number 1000 can be written as the sum of 16 consecutive integers? That is,

\[ 1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70. \]

The diagram above illustrates a mathematical short form used for writing the above sum. The notation is called Sigma Notation. Using at least two integers, what is the minimum number of consecutive integers needed to sum to exactly 1000?

Solution
In this solution we will examine possible cases until we discover the first one that works.

1. Can 1000 be written using 2 consecutive integers?
   Let \( n \), \( n + 1 \) represent the two integers. Then,
   \[ n + (n + 1) = 1000 \]
   \[ 2n + 1 = 1000 \]
   \[ 2n = 999 \]
   \[ n = 499.5 \]
   Since \( n \) is not an integer, it is not possible to write 1000 using two consecutive integers.

2. Can 1000 be written using 3 consecutive integers?
   Let \( n \), \( n + 1 \), \( n + 2 \) represent the three integers. Then,
   \[ n + (n + 1) + (n + 2) = 1000 \]
   \[ 3n + 3 = 1000 \]
   \[ 3n = 997 \]
   \[ n \doteq 332.3 \]
   Since \( n \) is not integer, it is not possible to write 1000 using three consecutive integers.
   (Refer to the note following the solution for an alternate way to define three consecutive integers.)

3. Can 1000 be written using 4 consecutive integers?
   Let \( n \), \( n + 1 \), \( n + 2 \), \( n + 3 \) represent the four integers. Then,
   \[ n + (n + 1) + (n + 2) + (n + 3) = 1000 \]
   \[ 4n + 6 = 1000 \]
   \[ 4n = 994 \]
   \[ n = 248.5 \]
   Since \( n \) is not an integer, it is not possible to write 1000 using four consecutive integers.
4. Can 1000 be written using 5 consecutive integers?

Let \( n, n + 1, n + 2, n + 3, n + 4 \) represent the five integers. Then,
\[
\begin{align*}
    n + (n+1) + (n+2) + (n+3) + (n+4) &= 1000 \\
    5n + 10 &= 1000 \\
    5n &= 990 \\
    n &= 198
\end{align*}
\]
Since \( n \) is an integer, it is possible to write 1000 using five consecutive integers. That is,
\[
1000 = 198 + 199 + 200 + 201 + 202.
\]

Since we have checked all possible numbers of consecutive integers below five and none of them worked, the minimum number of consecutive integers required to produce a sum of 1000 is five.

Note:
In the second case, when we checked to see if 1000 could be written as the sum of three consecutive integers, we could have proceeded as follows:

Let \( a - 1, a, a + 1 \) represent the three consecutive integers. Then,
\[
\begin{align*}
    (a - 1) + a + (a + 1) &= 1000 \\
    3a &= 1000 \\
    a &= 333.3
\end{align*}
\]
Since \( a \) is not an integer, it is not possible to write 1000 using three consecutive integers.

This “idea” is useful when we are finding the sum of an odd number of consecutive integers. If we applied the same idea to the fourth case by using \( a - 2, a - 1, a, a + 1, a + 2 \) to represent the five consecutive integers,
\[
\begin{align*}
    (a - 2) + (a - 1) + a + (a + 1) + (a + 2) &= 1000 \\
    5a &= 1000 \\
    a &= 200 \\
    a - 2 &= 198 \\
    a - 1 &= 199 \\
    a + 1 &= 201 \\
    a + 2 &= 202
\end{align*}
\]
It is possible to write 1000 using five consecutive integers; namely, 198, 199, 200, 201, and 202.

For Further Thought:
What is the largest odd number of consecutive positive integers that can be used to sum to 1000? What is the smallest number in the list?
How would your answer change if the word positive was removed from the above sentence?
Problem of the Week
Problem C
Sixteen is Just Perfect

A wall is covered with balloons. Each balloon has the number 5, 3 or 2 printed on it. You are given 7 darts to throw at the wall. If your dart breaks a balloon, then you earn the number of points printed on the balloon. If your dart does not break a balloon, then you are awarded 0 points for that shot. You win a prize if your total score is exactly 16 points on seven shots. If your total is over 16 or under 16, then you lose.

Determine the number of different point combinations that can be used to win the game.

Strands  Number Sense and Numeration, Patterning and Algebra
Problem of the Week  
Problem C and Solution  
Sixteen is Just Perfect

Problem  
A wall is covered with balloons. Each balloon has the number 5, 3 or 2 printed on it. You are given 7 darts to throw at the wall. If your dart breaks a balloon, then you earn the number of points printed on the balloon. If your dart does not break a balloon, then you are awarded 0 points for that shot. You win a prize if your total score is exactly 16 points on seven shots. If your total is over 16 or under 16, then you lose. Determine the number of different point combinations that can be used to win the game.

Solution  
Solution 1  
Let us consider cases.

1. You break three balloons with a 5 printed on them. You have a total of $3 \times 5 = 15$ points. There is no possible way to get 16 points since the other balloon values are 2 or 3. There is no way to win by breaking three (or more) balloons with a 5 printed on them.

2. You break two balloons with a 5 printed on them. You have a total of $2 \times 5 = 10$ points. You need to get $16 - 10 = 6$ points by breaking balloons with 2 or 3 printed on them. There are two ways to do this. Break three balloons with a 2 printed on them and miss on two shots or break two balloons with 3 printed on them and miss on three shots. There are 2 ways to win if you break two balloons with a 5 printed on them.

3. You break one balloon with a 5 printed on it. You have a total of 5 points. You need to get $16 - 5 = 11$ points by breaking balloons with 2 or 3 printed on them. You cannot get 11 points breaking only balloons with a 2 printed on them and you cannot get 11 points breaking only balloons with a 3 printed on them. However, you can get 11 points by breaking one 3 and four 2’s or by breaking three 3’s and one 2. There are 2 ways to win if you break one balloon with a 5 printed on it.

4. You break no balloons with a 5 printed on it. You need to make 16 points by breaking only balloons with a 2 or 3 printed on them. You cannot get 16 points by breaking only balloons with a 3 printed on them. You cannot get 16 points by breaking only balloons with a 2 printed on them because you only have seven darts giving a maximum of 14 points. It is possible to get 16 points using combinations of 3 point and 2 point balloons. If you break two 3 point balloons and five 2 point balloons, then you win in seven shots. If you break four 3 point balloons and two 2 point balloons, then you have 16 points in six shots and would have to miss on one of your shots. There are 2 ways to win if you do not break any 5 point balloons.

There are $0 + 2 + 2 + 2 = 6$ combinations that allow you to win by getting 16 points in seven shots.
Solution 2

In this solution we will complete a chart to determine the valid possibilities.

Let the number of 5 point balloons be $a$, the number of 3 point balloons be $b$, and the number of 2 point balloons be $c$. We want $5a + 3b + 2c = 16$ and $a + b + c \leq 7$.

<table>
<thead>
<tr>
<th>Number of 5 Point Balloons</th>
<th>Number of 3 Point Balloons</th>
<th>Number of 2 Point Balloons</th>
<th>Total Points Scored $5a + 3b + 2c$</th>
<th>Number of Missed Shots Needed $7 - a - b - c$</th>
<th>WIN or LOSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>18</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>16</td>
<td>3</td>
<td>WIN</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3</td>
<td>16</td>
<td>2</td>
<td>WIN</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>14</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>16</td>
<td>2</td>
<td>WIN</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>16</td>
<td>1</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>0</td>
<td>18</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>0</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>1</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>16</td>
<td>1</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>4</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>0</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>7</td>
<td>14</td>
<td></td>
<td>LOSE</td>
</tr>
</tbody>
</table>

There are only 6 combinations that allow you to win by getting 16 points in seven shots. (In following a method like the above method, one must be careful to systematically examine all possible cases.)
Problem of the Week

Problem C

Elf Pics

Five of Santa’s elves: Alpha, Beta, Delta, Epsilon and Gamma, are lined up in alphabetical order from left to right. They can each choose one of five festive hats to wear for a photo. The hats are identical except for colour. Three of the hats are red and two of the hats are green.

How many different photos can be taken?
Problem of the Week
Problem C and Solution
Elf Pics

Problem

Five of Santa’s elves: Alpha, Beta, Delta, Epsilon and Gamma, are lined up in alphabetical order from left to right. They can each choose one of five festive hats to wear for a photo. The hats are identical except for colour. Three of the hats are red and two of the hats are green. How many different photos can be taken?

Solution

Since the elves are already organized in alphabetical order, we are looking for the number of different ways that we can distribute the hats among the elves. We will consider cases:

1. If the first elf gets a green hat, there are four ways to give out the second green hat. Once the green hats are distributed, the remaining three elves must each get a red hat. Therefore, there are 4 ways to distribute the hats so that the first elf receives a green hat.
2. If the first elf gets a red hat and the second elf gets a green hat, there are three ways to give out the second green hat. Once the green hats are distributed, the remaining two elves must each get a red hat. Therefore, there are 3 ways to distribute the hats so that the first elf receives a red hat and the second elf receives a green hat.
3. If the first two elves each get a red hat and the third elf gets a green hat, there are two ways to give out the second green hat. Once the green hats are distributed, the remaining elf must get a red hat. Therefore, there are 2 ways to distribute the hats so that the first two elves each receive a red hat and the third elf receives a green hat.
4. If the first three elves each get a red hat and the fourth elf gets a green hat, the fifth elf must get the second green hat. Therefore, there is only 1 way to distribute the hats so that the first three elves each receive a red hat and the fourth elf receives a green hat.

There are no other cases to consider. The total number of ways to distribute the hats is the sum of the number of ways from each of the cases. Therefore, there are \(4 + 3 + 2 + 1 = 10\) ways to distribute the hats. There are 10 different photos that can be taken.
Problem of the Week
Problem C
The Best Laid Plans

In preparing to write an examination, Stu Deus made the following observations:

- The exam had 20 questions.
- He estimated that he would spend 6 minutes per question.
- The exam would take him 2 hours to complete.

However, during the actual examination, Stu discovered some difficult questions which each required 15 minutes to complete. He also discovered some questions which were much easier than he expected and only took him 2 minutes per question to complete. Seven of the questions, however, still required 6 minutes each to complete. Surprisingly, Stu was still able to complete the exam in 2 hours.

How many of the 20 examination questions did Stu find difficult?
Problem of the Week
Problem C and Solution
The Best Laid Plans

Problem

In preparing to write an examination, Stu Deus made the following observations: the exam had 20 questions, he estimated that he would spend 6 minutes per question, and the exam would take him 2 hours to complete. However, during the actual examination, Stu discovered some difficult questions which each required 15 minutes to complete. He also discovered some questions which were much easier than he expected and only took him 2 minutes per question to complete. Seven of the questions, however, still required 6 minutes each to complete. Surprisingly, Stu was still able to complete the exam in 2 hours. How many of the 20 examination questions did Stu find difficult?

Solution

Solution 1: Systematic Trial

Since 7 of the questions required 6 minutes each to complete, it took Stu \(7 \times 6 = 42\) minutes to complete these questions. The total exam took 2 hours or 120 minutes. He had \(120 - 42 = 78\) minutes to complete \(20 - 7 = 13\) questions.

Let \(d\) represent the number of difficult questions and \(e\) represent the number of easier questions. We know that \(d + e = 13\).

Since each difficult question took 15 minutes, it took \(15d\) minutes to complete all of the difficult questions. Since each easier question took 2 minutes, it took \(2e\) minutes to complete all of the easier questions. Since Stu’s total remaining time was 78 minutes, \(15d + 2e = 78\) minutes.

At this point we can pick values for \(d\) and \(e\) that add to 13 and substitute into the equation \(15d + 2e = 78\) to find the combination that works. (We can observe that \(d < 6\) since \(15 \times 6 = 90 > 78\). If this were the case, then \(e\) would have to be a negative number.)

If \(d = 3\) then \(e = 13 - 3 = 10\). The time to complete these would be 
\[15 \times 3 + 2 \times 10 = 45 + 20 = 65\] minutes and he would complete the exam in less than 2 hours.

If \(d = 4\) then \(e = 13 - 4 = 9\). The time to complete these would be 
\[15 \times 4 + 2 \times 9 = 60 + 18 = 78\] minutes and he would complete the exam in exactly 2 hours.

Therefore, Stu found 4 of the questions to be more difficult and time-consuming than he expected.

Solution 2 involves algebra and equation solving.
Solution 2: Using Algebra and Equations

Since 7 of the questions required 6 minutes each to complete, it took Stu $7 \times 6 = 42$ minutes to complete these questions. The total exam took 2 hours or 120 minutes. He had $120 - 42 = 78$ minutes to complete $20 - 7 = 13$ questions.

Let $d$ represent the number of difficult questions and $(13 - d)$ represent the number of easier questions.

Since each difficult question took 15 minutes, it took $15d$ minutes to complete all of the difficult questions. Since each easier question took 2 minutes, it took $2(13 - d)$ minutes to complete all of the easier questions.

Since Stu’s total remaining time was 78 minutes,

\[
15d + 2(13 - d) = 78
\]
\[
15d + 26 - 2d = 78
\]
\[
13d + 26 = 78
\]

Subtracting 26 from both sides:
\[
13d = 52
\]

Dividing both sides by 13:
\[
d = 4
\]

Therefore, Stu found 4 of the questions to be more difficult and time-consuming than he expected.
Four distinct integers are to be chosen from the integers 1, 2, 3, 4, 5, 6, and 7. How many different selections are possible so the sum of the four integers is even?

\[1 + 2 + 3 + 4\] Even Sum

\[1 + 2 + 3 + 5\] Odd Sum

**Strands**  Data Management and Probability, Number Sense and Numeration
Problem of the Week
Problem C and Solution
Probably Even

Problem
Four distinct integers are to be chosen from the integers 1, 2, 3, 4, 5, 6, and 7. How many different selections are possible so the sum of the four integers is even?

Solution
We could look at every possible selection of four distinct numbers from the list, determine the sum of each selection and then count the number of selections for which the sum is even. There are 35 different selections to examine. A justification of this number is provided on the second page of this solution. This would not be an efficient approach!

We will make two simple observations. First, when even numbers are added together the sum is always even. And second, in order to produce an even sum using odd numbers, an even number of odd numbers is required in the sum. We will use these observations to break the problem into cases in which the sum is even. There are three cases to consider.

1. No Odd Numbers are Selected
   Since there are only three even numbers, namely 2, 4, and 6, it is not possible to select only even numbers. Therefore, there are no selections in which there are no odd numbers.

2. Exactly Two Odd Numbers are Selected
   There are four choices for the first odd number. For each of these four choices, there are three choices for the second number producing $4 \times 3 = 12$ choices for two odd numbers. However, each choice is counted twice. For example, 1 could be selected first and 3 could be chosen second or 3 could be selected first and 1 could be chosen second. Therefore, there are only $12 \div 2 = 6$ selections of two odd numbers. They are
   \{1,3\}, \{1,5\}, \{1,7\}, \{3,5\}, \{3,7\}, and \{5,7\}. For each of the 6 possible selections of two distinct odd numbers, we need to select two even numbers from the three even numbers in the list. We could use a similar argument to the selection of the two odd numbers or simply list the (three) possibilities: \{2,4\}, \{2,6\}, and \{4,6\}. Therefore, there are $6 \times 3 = 18$ selections of four distinct numbers in which exactly two of the numbers are odd.

3. Exactly Four Odd Numbers are Selected
   Since there are only four odd numbers in the list to choose from, there is only one way to select four distinct odd numbers from the list.

We have considered every possible case in which the selection produces an even sum. Therefore, there are $0 + 18 + 1 = 19$ selections of four distinct numbers from the list such that the sum is even.
Why are there 35 ways to select four different numbers from the list?

In the solution on the previous page we counted the selections in which the sum was even. There were 19 possibilities. The remaining selections must produce an odd sum. There are two possibilities: either there is 1 odd number and 3 even numbers, or there are 3 odd numbers and 1 even number.

If there is 1 odd number and 3 even numbers, there are only four possible selections, namely, \{1, 2, 4, 6\}, \{3, 2, 4, 6\}, \{5, 2, 4, 6\}, and \{7, 2, 4, 6\}. Once the odd number is selected, the 3 even numbers, \{2, 4, 6\}, must be selected.

If there are 3 odd numbers and 1 even number, there are twelve possible selections. The 3 odd numbers can be selected in four ways, namely, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 5, 7\}, and \{3, 5, 7\}. For each of these 4 selections of three odd numbers, the even number can be selected in 3 ways producing \(4 \times 3 = 12\) possible selections of four distinct numbers in which three of the numbers are odd and the other is even.

We have considered all possible ways in which four distinct numbers can be selected from the list. The total number of selections is \(19 + 4 + 12 = 35\).

We can arrive at this number in a different way.

There are 7 choices for the first number. For each of these choices for the first number, there are 6 choices for the second number, or \(7 \times 6 = 42\) choices for the first two numbers. For each of these 42 choices for the first two numbers, there are 5 choices for the third number, or \(42 \times 5 = 210\) choices for the first three numbers. For each of these 210 choices of the first three numbers, there are 4 choices for the final number, or \(210 \times 4 = 840\) selections of the four numbers. This is considerably higher than the 35 choices shown above!

Our 840 selections assume that the order of selection is important. Each selection has been counted 24 times. To justify this, we will look at the number of ways a specific four number selection can be arranged. Without loss of generality, we will consider the selection \{1, 2, 3, 4\}. The 1 could be placed in four spots. For each of these four placements of the 1, the 2 could be placed in three spots producing \(4 \times 3 = 12\) ways of placing the 1 and 2. For each of these twelve placements of the 1 and 2, the 3 could be placed in two spots producing \(12 \times 2 = 24\) ways of placing the 1, 2 and 3. Once the numbers 1, 2, and 3 are placed, the 4 must be placed in the remaining spot. There are 24 ways of arranging the four numbers. We have to divide 840 by 24 since we have counted each selection 24 times.

Therefore, there are \(840 \div 24 = 35\) ways to select four different numbers from the list of seven numbers.
Problem of the Week
Problem C
About Average

The average of four different positive integers is 100. If the difference between the largest and smallest of these integers is as large as possible, determine the average of the other two integers.

The illustration below is not related to the problem. The illustration is called a picture puzzle. Can you interpret it?

% % % % % % % % % % %

AVERAGE
Problem

The average of four different positive integers is 100. If the difference between the largest and smallest of these integers is as large as possible, determine the average of the other two integers.

Solution

Let $S$, $a$, $b$, and $L$ represent four distinct positive integers such that $S < a < b < L$.

Since the average of the four positive integers is 100, the total sum can be determined by multiplying the average by 4. The sum of the numbers is therefore $4 \times 100 = 400$. That is, $S + a + b + L = 400$.

For the difference between the largest, $L$, and smallest, $S$, to be as large as possible, we want the smallest integer $S$ to be as small as possible. The smallest possible positive integer is 1 so $S = 1$.

Since the sum of the four positive integers is 400 and the smallest integer, $S$, is 1, the sum of the remaining three integers is $a + b + L = 400 - 1 = 399$.

For $L$ to be as large as possible, $a$ and $b$ must be as small as possible. The two positive integers, $a$ and $b$, must be different and cannot equal 1 since the smallest of the four positive integers, $S$, is 1. Therefore, $a = 2$ and $b = 3$, the smallest two distinct remaining positive integers. It follows that $L$, the largest of the four positive integers, is $399 - 2 - 3 = 394$. (This was not required but has been provided for completeness.)

The average of the middle two positive integers, $a$ and $b$, is $\frac{2+3}{2} = \frac{5}{2} = 2.5$.

For Further Thought:

How would your answer change if it was also required that the average of the four distinct positive integers was an integer greater or equal to 3?

Word Picture Solution: Ten percent above average.
Problem of the Week
Problem C
This Problem has Value

A nickel is worth 5 cents (5¢), a dime is worth 10 cents (10¢), a quarter is worth 25 cents (25¢), and a dollar is worth 100 cents (100¢).

A piggy bank contains some quarters, dimes and nickels. There are no other coins in the piggy bank. The ratio of the number of quarters to the number of dimes to the number of nickels in the piggy bank is 9 : 3 : 1. The total value of all of the coins in the piggy bank is $18.20.

Determine the number of coins in the piggy bank.
Problem of the Week
Problem C and Solution
This Problem has Value

Problem
A nickel is worth 5 cents (5¢), a dime is worth 10 cents (10¢), a quarter is worth 25 cents (25¢), and a dollar is worth 100 cents (100¢). A piggy bank contains some quarters, dimes and nickels. There are no other coins in the piggy bank. The ratio of the number of quarters to the number of dimes to the number of nickels in the piggy bank is 9 : 3 : 1. The total value of all of the coins in the piggy bank is $18.20. Determine the number of coins in the piggy bank.

Solution
Solution 1
Suppose the piggy bank contained one nickel. Then, using the ratio 9 : 3 : 1, the bank would contain 9 quarters, 3 dimes and 1 nickel, 13 coins in total. The value of the 13 coins would be $25 \times 9 + 10 \times 3 + 5 \times 1 = 260$ cents or $2.60.

Since the coins in the bank are in the ratio 9 : 3 : 1, then we can group the coins into sets of 9 quarters, 3 dimes and 1 nickel, with each set containing 13 coins and having a value of $2.60.

Since the total value of the coins in the bank is $18.20 and \( \frac{\$18.20}{\$2.60} = 7 \), then there are 7 of these sets of coins. Since each set has 13 coins, then there are \( 7 \times 13 = 91 \) coins in total in the piggy bank.

Solution 2
This solution uses algebra which may be beyond the solver at this point in the school year.

Suppose there are \( n \) nickels in the bank. Then, using the ratio 9 : 3 : 1, the bank would contain 9n quarters, 3n dimes and n nickels, 13n coins in total.

The value of the coins would be

\[
25 \times 9n + 10 \times 3n + 5 \times n = 225n + 30n + 5n = 260n \text{ cents.}
\]

The total value of the coins in the bank is $18.20 or 1820¢. Therefore, \( 260n = 1820 \). Dividing both sides of the equation by 260, \( n = 7 \).

Since there are 13n coins in the bank and \( n = 7 \), there are \( 13 \times 7 = 91 \) coins in the bank.
Solution 3

This solution presents the information in a table.

<table>
<thead>
<tr>
<th># of nickels</th>
<th># of quarters</th>
<th># of dimes</th>
<th># of coins</th>
<th>Total value of coins</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9 × 1 = 9</td>
<td>3 × 1 = 3</td>
<td>1 + 9 + 3 = 13</td>
<td>1 × 5 + 9 × 25 + 3 × 10 = 260 ¢</td>
</tr>
<tr>
<td>2</td>
<td>9 × 2 = 18</td>
<td>3 × 2 = 6</td>
<td>2 + 18 + 6 = 26</td>
<td>2 × 5 + 18 × 25 + 6 × 10 = 520 ¢</td>
</tr>
<tr>
<td>3</td>
<td>9 × 3 = 27</td>
<td>3 × 3 = 9</td>
<td>3 + 27 + 9 = 39</td>
<td>3 × 5 + 27 × 25 + 9 × 10 = 780 ¢</td>
</tr>
<tr>
<td>4</td>
<td>9 × 4 = 36</td>
<td>3 × 4 = 12</td>
<td>4 + 36 + 12 = 52</td>
<td>4 × 5 + 36 × 25 + 12 × 10 = 1040 ¢</td>
</tr>
<tr>
<td>5</td>
<td>9 × 5 = 45</td>
<td>3 × 5 = 15</td>
<td>5 + 45 + 15 = 65</td>
<td>5 × 5 + 45 × 25 + 15 × 10 = 1300 ¢</td>
</tr>
<tr>
<td>6</td>
<td>9 × 6 = 54</td>
<td>3 × 6 = 18</td>
<td>6 + 54 + 18 = 78</td>
<td>6 × 5 + 54 × 25 + 18 × 10 = 1560 ¢</td>
</tr>
<tr>
<td>7</td>
<td>9 × 7 = 63</td>
<td>3 × 7 = 21</td>
<td>7 + 63 + 21 = 91</td>
<td>7 × 5 + 63 × 25 + 21 × 10 = 1820 ¢</td>
</tr>
</tbody>
</table>

From the table, we see that there are a total of 91 coins in the bank.

It is hoped that the solver would recognize a pattern and work with the pattern to find the solution. If the total value had been significantly greater, then this method would provide insight but would be too tedious to completely follow through.

It would also be possible to do some sort of guess and check solution to zero in on the precise solution to the problem.
Problem of the Week
Problem C
Moving Right Along

Six coloured squares are placed beside each other as shown below.

The leftmost square has the number 504 on it and the rightmost square has the number 2017 on it. A number is to be written on each of the four blank squares so that each number after the second number equals the sum of the previous two numbers.

Determine the remaining four numbers that should be written on the front of the boxes as you move from left to right.

**STRANDS** Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
Moving Right Along

Problem
Six coloured squares are placed beside each other as shown below.

```
504   a   ...   ...   2017
```

The leftmost square has the number 504 on it and the rightmost square has the number 2017 on it. A number is to be written on each of the four blank squares so that each number after the second number equals the sum of the previous two numbers.

Determine the remaining four numbers that should be written on the front of the boxes as you move from left to right.

Solution
Let $a$ represent the second number.

Since the third number is the sum of the previous two numbers, the third number is $504 + a$.

Since the fourth number is the sum of the previous two numbers, the fourth number is $(a) + (504 + a) = 504 + 2a$.

Since the fifth number is the sum of the previous two numbers, the fifth number is $(504 + a) + (504 + 2a) = 1008 + 3a$.

Since the sixth number is the sum of the previous two numbers, the sixth number is $(504 + 2a) + (1008 + 3a) = 1512 + 5a$. But the sixth number in the sequence is 2017.

\[
\therefore 1512 + 5a = 2017
\]

\[
1512 + 5a - 1512 = 2017 - 1512
\]

\[
5a = 505
\]

\[
\frac{5a}{5} = \frac{505}{5}
\]

\[
a = 101
\]

We now know that the second number is 101 so we can determine the remaining numbers in the sequence by substituting into the expressions above or by simply using the rule to generate the remaining numbers. Using the rule, the third number is $504 + 101 = 605$, the fourth number is $101 + 605 = 706$, and the fifth number is $605 + 706 = 1311$. As a check, we can use the rule to determine the sixth number obtaining $706 + 1311 = 2017$, as required.

The four missing numbers that would be written on the squares are 101, 605, 706, 1311.
Problem of the Week  
Problem C  
Mystery Materials and Morsels

Xavier, Yvonne and Zak arrived at school at three different times. They each brought one of their favourite snacks to share with the other two (one brought pretzels; one brought cookies; one brought licorice), and their favourite sports apparatus (one brought a baseball; one brought a basketball; one brought a football).

We know a few other facts:

1. The first to arrive did not bring cookies.
2. Xavier arrived second and brought a football.
3. Yvonne arrived before Zak.
4. The person who brought cookies also brought a baseball.
5. The person who brought pretzels did not bring a basketball.

Determine the order they arrived in, what they each brought for a snack, and which sports apparatus they each brought.

**Strand**  Number Sense and Numeration
Problem of the Week
Problem C and Solution
Mystery Materials and Morsels

Problem
Xavier, Yvonne and Zak arrived at school at three different times. They each brought one of their favourite snacks to share with the other two (one brought pretzels; one brought cookies; one brought licorice), and their favourite sports apparatus (one brought a baseball; one brought a basketball; one brought a football).

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4. The person who brought cookies also brought a baseball.
5. The person who brought pretzels did not bring a basketball.

Determine the order they arrived in, what they each brought for a snack, and which sports apparatus they each brought.

Solution
When solving logic problems, setting up a chart to fill in is generally a good way to start.

<table>
<thead>
<tr>
<th>Order of Arrival</th>
<th>Snack</th>
<th>Sports Apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xavier</td>
<td></td>
<td>2nd football</td>
</tr>
<tr>
<td>Yvonne</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zak</td>
<td></td>
<td>3rd</td>
</tr>
</tbody>
</table>

Some of the given information is often more helpful than other information. For example, in the second statement we learn that Xavier arrived second and brought a football. Now the third statement gets us the fact that Yvonne arrived first and Zak arrived third. (This is true since Yvonne arrived before Zak and she could not arrive second leaving only the first and third spots left.) We can add this information to the chart.

<table>
<thead>
<tr>
<th>Order of Arrival</th>
<th>Snack</th>
<th>Sports Apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xavier</td>
<td>2nd</td>
<td>football</td>
</tr>
<tr>
<td>Yvonne</td>
<td>1st</td>
<td></td>
</tr>
<tr>
<td>Zak</td>
<td>3rd</td>
<td></td>
</tr>
</tbody>
</table>
We can combine the first statement and the fourth statement. Yvonne did not bring cookies. The person who brought cookies also brought a baseball. This cannot be Xavier since he brought a football. Therefore, Zak brought cookies and a baseball. Since there is only one piece of sports apparatus unaccounted for, Yvonne must have brought the basketball. We will add this new information to our chart.

<table>
<thead>
<tr>
<th></th>
<th>Order of Arrival</th>
<th>Snack</th>
<th>Sports Apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xavier</td>
<td>2(^{nd})</td>
<td></td>
<td>football</td>
</tr>
<tr>
<td>Yvonne</td>
<td>1(^{st})</td>
<td></td>
<td>basketball</td>
</tr>
<tr>
<td>Zak</td>
<td>3(^{rd})</td>
<td>cookies</td>
<td>baseball</td>
</tr>
</tbody>
</table>

We can now use the fifth statement to conclude that Xavier brought pretzels since the person bringing pretzels did not bring a basketball and Xavier is the only one without a snack accounted for other than Yvonne (who brought the basketball).

<table>
<thead>
<tr>
<th></th>
<th>Order of Arrival</th>
<th>Snack</th>
<th>Sports Apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xavier</td>
<td>2(^{nd})</td>
<td>pretzels</td>
<td>football</td>
</tr>
<tr>
<td>Yvonne</td>
<td>1(^{st})</td>
<td></td>
<td>basketball</td>
</tr>
<tr>
<td>Zak</td>
<td>3(^{rd})</td>
<td>cookies</td>
<td>baseball</td>
</tr>
</tbody>
</table>

The only snack unaccounted for is the licorice and Yvonne is the only one whose snack is unknown. Therefore, Yvonne brought licorice and our chart can be completed.

<table>
<thead>
<tr>
<th></th>
<th>Order of Arrival</th>
<th>Snack</th>
<th>Sports Apparatus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xavier</td>
<td>2(^{nd})</td>
<td>pretzels</td>
<td>football</td>
</tr>
<tr>
<td>Yvonne</td>
<td>1(^{st})</td>
<td>licorice</td>
<td>basketball</td>
</tr>
<tr>
<td>Zak</td>
<td>3(^{rd})</td>
<td>cookies</td>
<td>baseball</td>
</tr>
</tbody>
</table>

The information is summarized in the chart but will be stated below for completeness.

- Yvonne arrived first bringing licorice and a basketball.
- Xavier arrived second bringing pretzels and a football.
- Zak arrived third bringing cookies and a baseball.
A motorcycle and a delivery truck left a roadside diner at the same time. After travelling in the same direction for one and one-quarter hours, the motorcycle had travelled 25 km farther than the delivery truck. If the average speed of the motorcycle was 60 km/h, find the average speed of the delivery truck.
Problem of the Week
Problem C and Solution
Keep on Truckin’

Problem
A motorcycle and a delivery truck left a roadside diner at the same time. After travelling in the same direction for one and one-quarter hours, the motorcycle had travelled 25 km farther than the delivery truck. If the average speed of the motorcycle was 60 km/h, find the average speed of the delivery truck.

Solution
We can calculate distance by multiplying the average speed by the time.

In one and one-quarter hours at 60 km/h, the motorcycle would travel
\[60 \times 1\frac{1}{4} = 60 \times \frac{5}{4} = 75\text{ km}.\]

In the same time, the delivery truck travels 25 km less. The delivery truck has travelled \(75 - 25 = 50\) km. Since the distance travelled equals the average speed multiplied by the time, then the average speed will equal the distance travelled divided by the time travelled. Thus, the average speed of the delivery truck equals \(50 \div 1\frac{1}{4} = 50 \div \frac{5}{4} = 50 \times \frac{4}{5} = 40\text{ km/h}.\)

Therefore the average speed of the delivery truck is 40 km/h.

The calculations in this problem could be done using decimals by converting one and one-quarter hours to 1.25 hours.
Problem of the Week  
Problem C  
How Far Around is It?

In the diagram, \( \triangle ABC \) is a right triangle with \( \angle ABC = 90^\circ \), \( BD = 6 \text{ m} \), \( AB = 8 \text{ m} \), and the area of \( \triangle ADC \) is 50\% more than the area of \( \triangle ABD \).

Determine the perimeter of \( \triangle ADC \).

The Pythagorean Theorem states, “In a right triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides”

In the following right triangle, \( p^2 = r^2 + q^2 \).

**Strands**  
Number Sense and Numeration, Measurement, Geometry and Spatial Sense
Problem of the Week
Problem C and Solution
How Far Around is It?

Problem

In the diagram, \( \triangle ABC \) is a right triangle with \( \angle ABC = 90^\circ \), \( BD = 6 \) m, \( AB = 8 \) m, and the area of \( \triangle ADC \) is 50% more than the area of \( \triangle ABD \). Determine the perimeter of \( \triangle ADC \).

Solution

Let \( a \) be the length of side \( DC \), \( b \) be the length of side \( AC \), and \( c \) be the length of side \( AD \). Draw a line through \( A \) parallel to \( BC \). The distance between this line and \( BC \) is 8 m. Note that this distance is also the height of \( \triangle ABD \) and \( \triangle ADC \).

To find the area of a triangle, multiply the length of the base by the height and divide by 2. Therefore,

\[
\text{area of } \triangle ABD = AB \times BD \div 2 = 8 \times 6 \div 2 = 24 \text{ m}^2.
\]

The area of \( \triangle ADC \) is 50% more than the area of \( \triangle ABD \). Therefore,

\[
\text{area of } \triangle ADC = (\text{area of } \triangle ABD) + \frac{1}{2}(\text{area of } \triangle ABD) = 24 + 12 = 36 \text{ m}^2.
\]

But the area of \( \triangle ADC = (AB)(DC) \div 2 = 8(DC) \div 2 = 4(DC) \). Therefore, \( 4(DC) = 36 \) and \( DC = 9 \) m. Then \( BC = BD + DC = 6 + 9 = 15 \) m.

Since \( \triangle ABD \) has a right angle, \( AD^2 = AB^2 + BD^2 = 8^2 + 6^2 = 100 \). Then \( AD = \sqrt{100} = 10 \), since \( AD > 0 \).

Also, \( \triangle ABC \) has a right angle, so
\[
AC^2 = AB^2 + BC^2 = 8^2 + 15^2 = 64 + 225 = 289.
\]
Then \( AC = \sqrt{289} = 17 \), since \( AC > 0 \).

\[
\therefore \text{The perimeter of } \triangle ADC = a + b + c = DC + AC + AD = 9 + 17 + 10 = 36 \text{ m}
\]

The perimeter of \( \triangle ADC \) is 36 m.
Problem of the Week  
Problem C  
A One in Six Chance

A regular die has faces numbered 1, 2, 3, 4, 5, and 6. The numbers on the faces are arranged so that opposite faces total seven. For example, the face containing 2 is opposite the face containing 5.

The four dice shown have been placed so that the two numbers on the faces touching each other always total nine. The face labelled $P$ is the front of one die as shown. What is the number on the face labelled $P$? If you randomly guessed, you have a one in six chance of getting this right!
Problem of the Week
Problem C and Solution
A One in Six Chance

Problem
A regular die has faces numbered 1, 2, 3, 4, 5, and 6. The numbers on the faces are arranged so that opposite faces total seven. For example, the face containing 2 is opposite the face containing 5. The four dice shown have been placed so that the two numbers on the faces touching each other always total nine. The face labelled $P$ is the front of one die as shown. What is the number on the face labelled $P$? If you randomly guessed, you have a one in six chance of getting this right!

Solution

On Die 1, since 5 is on the front, there is a 2 on the back. Since 4 is on the top, there is a 3 on the bottom. That leaves a 6 and 1 for the sides. Since the sides facing each other add to 9, the right side of die 1 must be a 6. If it were a 1, the left face of die 2 would have to be 8 and that is not possible. Therefore, the right side of die 1 must be a 6.

That means that the left side of die 2 must be a 3 since the sides facing each other total 9. If the left side of die 2 is 3, then the right side of die 2 must be a 4 since opposite sides add to 7.

Then the left side of die 3 must be a 5. If 5 is on the left side, 2 is on the right side. Since 4 is on the top of die 3, there is a 3 on the bottom. That leaves 1 and 6 for the front and back of die 3. The front must be 6 in order for the numbers on the front of die 3 and the back of die 4 to total 9.

Since the front of die 3 is a 6, the back of die 4 must be a 3. If the back of die 4 is a 3, then the front of die 4 must be a 4. But the front of die 4 is $P$. Therefore $P = 4$. 
Problem of the Week
Problem C
Sum Positive Primes

A prime number is a positive integer greater than 1 that has exactly two positive integer factors, 1 and the number itself. A composite number is a positive integer greater than 1 that has more than two positive integer factors. The number 1 is neither prime nor composite.

Four distinct prime numbers have a product of \(d_{10}\): a three-digit number with hundreds digit \(d\). Determine all possible values of the sum of these four prime numbers.
Problem

A prime number is a positive integer greater than 1 that has exactly two positive integer factors, 1 and the number itself. A composite number is a positive integer greater than 1 that has more than two positive integer factors. The number 1 is neither prime nor composite.

Four distinct prime numbers have a product of $d10$: a three-digit number with hundreds digit $d$. Determine all possible values of the sum of these four prime numbers.

Solution

Since the product $d10$ ends in 0, it must be divisible by 10, which is the product of the two primes 2 and 5.

When $d10$ is divided by 10, the quotient is $d10 \div 10 = d1$. This two-digit number must be composite and must be the product of two distinct prime numbers, neither of which is 2 or 5. We can rule out any two-digit prime numbers for $d1$ since these numbers would only have one prime factor. Therefore, we can rule out 11, 31, 41, 61, and 71: the five prime numbers ending with a 1. Then, $d$ cannot be 1, 3, 4, 6, or 7. Since $d10$ is a three-digit number, $d \neq 0$ because $d10 = 010 = 10$ is not a three-digit number. The remaining possibilities for $d$ are 2, 5, 8, and 9.

If $d = 2$, then the two-digit number would be 21, which has prime factors 7 and 3. The four prime factors of $d10 = 210$ are 2, 3, 5, and 7, producing a sum of $2 + 3 + 5 + 7 = 17$.

If $d = 5$, then the two-digit number would be 51, which has prime factors 17 and 3. The four prime factors of $d10 = 510$ are 2, 3, 5, and 17, producing a sum of $2 + 3 + 5 + 17 = 27$.

If $d = 8$, then the two-digit number would be 81, which is the product $9 \times 9$. There are already four factors, two of which are composite so $d10 = 810$ cannot be expressed as the product of four distinct prime numbers. Therefore, 8 can be ruled out as a possible value for $d$. (Note that $810 = 2 \times 3 \times 3 \times 3 \times 3 \times 5$, which is the product of six prime numbers not all of which are distinct.

If $d = 9$, then the two-digit number would be 91, which has prime factors 7 and 13. The four prime factors of $d10 = 910$ are 2, 5, 7, and 13, producing a sum of $2 + 5 + 7 + 13 = 27$.

However, we already have the sum 27.

Since there are no other possible cases to consider, the only possible sums of the four distinct prime factors that multiply to $d10$ are 17 and 27.
Problem of the Week
Problem C
How tall are you?

There are 9 students divided into two groups, the Nanos and the Technos. The heights (in cm) of the 9 students are 151, 153, 157, 161, 153, 157, 156, 159, and 154. The following information is known about the two groups.

• No Nano is taller than any Techno but one of the Nanos is the same height as one of the Technos.

• Two of the Nanos are the same height.

Determine the difference between the mean height of a member of the Technos and the mean height of a member of the Nanos.
Problem of the Week
Problem C and Solution
How tall are you?

Problem
There are 9 students divided into two groups, the Nanos and the Technos. The heights (in cm) of the 9 students are 151, 153, 157, 161, 153, 157, 156, 159, and 154. The following information is known about the two groups. No Nano is taller than any Techno but one of the Nanos is the same height as one of the Technos. Two of the Nanos are the same height. Determine the difference between the mean height of a member of the Technos and the mean height of a member of the Nanos.

Solution
Place the heights in ascending order: 151 153 153 154 156 157 157 159 161.

Since no Nano is taller than any Techno we can introduce a separator “|” that will divide the Nanos and Technos into two distinct groups.

For example, if there are 4 Nanos and 5 Technos, we would place a separator between 154 and 156.

(Nanos in front) 151 153 153 154 | 156 157 157 159 161 (Technos at the end)

Now, the possibilities for the two Nanos with the same height are 157 cm or 153 cm.

If the Nanos with the same height are both 157 cm, then the Nano and Techno with the same height would both be 153 cm. This contradicts the fact that no Nano is taller than any Techno because there would be a Nano whose height is 157 cm and a shorter Techno whose height is 153 cm.

Therefore, the two Nanos with the same height must both be 153 cm, and the Nano and the Techno with the same height must both be 157 cm.

<table>
<thead>
<tr>
<th>Nanos</th>
<th>Technos</th>
</tr>
</thead>
<tbody>
<tr>
<td>151, 153, 153, 154, 156, 157</td>
<td>157, 159, 161</td>
</tr>
</tbody>
</table>

The mean height of a student in the Nano group is \( \frac{151 + 153 + 153 + 154 + 156 + 157}{6} = \frac{924}{6} = 154 \text{ cm} \).

The mean height of a student in the Techno group is \( \frac{157 + 159 + 161}{3} = \frac{477}{3} = 159 \text{ cm} \).

Therefore, the difference in mean heights between the students in the Techno group and the students in the Nano group is 159 – 154 = 5 cm.

NOTE: This question is similar to a question found on the Beaver Computing Challenge (BCC) which is usually written in November. BCC information and past challenges can be found at http://cemc.uwaterloo.ca/contests/bcc.html.
Problem of the Week
Problem C
You Can’t Go Back?

On your 13th birthday you received three different time-travel pedometers. You want to use your pedometers to travel back to your 8th birthday. You may use your pedometers as often as you wish but only one at a time.

- Each time you use Pedometer A, take exactly 7 steps forward. This will result in you going back 4 months in time.
- Each time you use Pedometer B, take exactly 5 steps backward. This will result in you going back 7 months in time.
- Each time you use Pedometer C, take exactly 2 steps forward. This will result in you going back 3 months in time.

In traveling back to your 8th birthday, you made a total of 25 backward steps and had a total of 12 pedometer uses.

How many forward steps did you take while using your pedometers?
Problem
On your 13th birthday you received three different time-travel pedometers. You want to use your pedometers to travel back to your 8th birthday. You may use your pedometers as often as you wish but only one at a time. Each time you use Pedometer A, take exactly 7 steps forward. This will result in you going back 4 months in time. Each time you use Pedometer B, take exactly 5 steps backward. This will result in you going back 7 months in time. Each time you use Pedometer C, take exactly 2 steps forward. This will result in you going back 3 months in time.

In traveling back to your 8th birthday, you made a total of 25 backward steps and had a total of 12 pedometer uses. How many forward steps did you take while using your pedometers?

Solution
From your 13th birthday to your 8th birthday you would travel 5 years back in time. This is equivalent to traveling $5 \times 12 = 60$ months back in time.

Pedometer B is the only pedometer that requires its user to step backward. For every 5 steps backward, you travel 7 months back in time. Therefore, for 25 steps backward, you use Pedometer B five times and travel back in time $5 \times 7 = 35$ months.

You still need to travel $60 - 35 = 25$ more months back in time. You have used a pedometer 5 times and since you only have a total of 12 pedometer uses, you have $12 - 5 = 7$ pedometer uses left. You can now only use Pedometer A and Pedometer C.

If you use Pedometer A and Pedometer C one time each, you travel a total of 7 months back in time. If you use Pedometer A and Pedometer C three times each, this accounts for 6 uses and you travel a total of $7 \times 3 = 21$ months back in time. You have 1 use left and still need to travel 4 more months back in time. This can be accomplished by using Pedometer A once more.

It follows that Pedometer A is used 4 times and Pedometer C is used 3 times. The total number of forward steps is $4 \times 7 + 3 \times 2 = 28 + 6 = 34$.

Note that we could also have looked at each of the possibilities for using Pedometer A. Since there are a total of 7 pedometer uses for Pedometers A and C, the minimum number of uses for Pedometer A would be 0 and the maximum number of uses for Pedometer A would be 7. Once the number of uses for Pedometer A is selected, the number of uses for Pedometer C could be determined by subtracting the number of uses for Pedometer A from 7. For each combination we could determine the number of months traveled back in time. Once the correct combination is determined the total number of forward steps can be calculated. This is summarized in a table on the next page.

An algebraic solution is also provided on the next page.
Using only Pedometer A and Pedometer C a total of 7 times, we want to travel back in time 25 months.

<table>
<thead>
<tr>
<th>Uses of Pedometer A</th>
<th>Uses of Pedometer C</th>
<th>Months Traveled Back in Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>$0 \times 4 + 7 \times 3 = 0 + 21 = 21$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$1 \times 4 + 6 \times 3 = 4 + 18 = 22$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$2 \times 4 + 5 \times 3 = 8 + 15 = 23$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$3 \times 4 + 4 \times 3 = 12 + 12 = 24$</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>$4 \times 4 + 3 \times 3 = 16 + 9 = 25$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>$5 \times 4 + 2 \times 3 = 20 + 6 = 26$</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$6 \times 4 + 1 \times 3 = 24 + 3 = 27$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>$7 \times 4 + 0 \times 3 = 28 + 0 = 28$</td>
</tr>
</tbody>
</table>

Only one combination gives the correct number of Pedometer uses and the correct number of months traveled back in time. Use Pedometer A 4 times and Pedometer C 3 times. The total number of forward steps is $4 \times 7 + 3 \times 2 = 28 + 6 = 34$.

**Algebraic Approach**

This solution is presented for you to get a glimpse of what is coming in future mathematics courses.

Let $a$ be the number of uses of Pedometer A, $b$ be the number of uses of Pedometer B, and $c$ be the number of uses of Pedometer C. Since the total number of uses is 12, then $a + b + c = 12$.

The total number of backward steps is 25 and Pedometer B is the only pedometer requiring backward steps. Since each use of Pedometer B requires 5 backward steps, then we require a total of 5 uses of Pedometer B to go back 25 steps. It follows that $b = 5$ and the equation $a + b + c = 12$ becomes $a + 5 + c = 12$ which simplifies to $a + c = 7$. (1)

In using Pedometer B 5 times, you travel a total of $5 \times 7 = 35$ months back in time. You need to travel 5 years or 60 months back in time altogether. Using Pedometer A and Pedometer C a total of 7 times, you need to travel $60 - 35 = 25$ more months back in time. As an equation this can be written $4a + 3c = 25$. (2)

Rearranging equation (1), we obtain $c = 7 - a$. We can substitute for $c$ in equation (2).

$$4a + 3c = 25$$
$$4a + 3(7 - a) = 25$$
$$4a + 21 - 3a = 25$$
$$a + 21 = 25$$
$$a = 4$$

Since $a = 4$, we can substitute in equation (1) to determine that $c = 3$.

For each use of Pedometer A, 7 forward steps are required. Therefore, you step forward $7a$ steps using Pedometer A. For each use of Pedometer C, 2 forward steps are required. Therefore, you step forward $2c$ steps using Pedometer C. The total number of steps forward is $7a + 2c$. But $a = 4$ and $c = 3$ so the total number of forward steps is $7(4) + 3(2) = 28 + 6 = 34$. 
Patterning & Algebra
Problem of the Week
Problem C
New Heights (Revised)

An altitude is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side.

In \( \triangle ABC \), \( CD \) is an altitude. \( AB = 18 \text{ cm} \), \( AC = 20 \text{ cm} \) and \( CD = 16 \text{ cm} \).

An altitude is drawn from \( B \) to \( AC \) intersecting at \( E \). Determine the length of \( BE \).
Problem of the Week
Problem C
New Heights (Original Problem)

An *altitude* is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side.

In $\triangle ABC$, $CD$ is an altitude. $AB = 16$ cm, $AC = 12$ cm and $CD = 6$ cm.

An altitude is drawn from $B$ to $AC$ extended intersecting at $E$. Determine the length of $BE$.

**STRANDS** MEASUREMENT, NUMBER SENSE AND NUMERATION, PATTERNING AND ALGEBRA
Problem of the Week
Problem C and Solution
New Heights (Revised)

Problem

An *altitude* is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side. In \( \triangle ABC \), \( CD \) is an altitude. \( AB = 18 \text{ cm}, AC = 20 \text{ cm} \) and \( CD = 16 \text{ cm} \). An altitude is drawn from \( B \) to \( AC \) intersecting at \( E \). Determine the length of \( BE \).

Solution

The area of a triangle is determined using the formula \( \text{base} \times \text{height} \div 2 \). The height of the triangle is the length of an altitude and the base of the triangle is the length of the side to which a particular altitude is drawn.

\[
\text{Area } \triangle ABC = \frac{(CD) \times (AB)}{2} = \frac{16 \times 18}{2} = 144 \text{ cm}^2
\]

But, \( \text{Area } \triangle ABC = \frac{(BE) \times (AC)}{2} \)

\[
144 = \frac{(BE) \times 20}{2}
\]

\[
144 = 10 \times BE
\]

\[
14.4 \text{ cm} = BE
\]

Therefore, the length of altitude \( BE \) is 14.4 cm.
Problem of the Week
Problem C and Solution
New Heights (Original Problem)

Problem

An *altitude* is a line segment drawn from a vertex of a triangle to the opposite side or opposite side extended such that the line segment is perpendicular to the opposite side. In \( \triangle ABC \), \( CD \) is an altitude. \( AB = 16 \text{ cm} \), \( AC = 12 \text{ cm} \) and \( CD = 6 \text{ cm} \). An altitude is drawn from \( B \) to \( AC \) extended intersecting at \( E \). Determine the length of \( BE \).

Solution

The area of a triangle is determined using the formula \( \text{base} \times \text{height} \div 2 \). The height of the triangle is the length of an altitude and the base of the triangle is the length of the side to which a particular altitude is drawn.

\[
\text{Area } \triangle ABC = \frac{(CD) \times (AB)}{2} = \frac{6 \times 16}{2} = 48 \text{ cm}^2
\]

But, \( \text{Area } \triangle ABC = \frac{(BE) \times (AC)}{2} \)

\[
48 = \frac{(BE) \times 12}{2}
\]

\[
48 = 6 \times BE
\]

\[
8 \text{ cm} = BE
\]

Therefore, the length of altitude \( BE \) is 8 cm.
“Old MacDonald had a farm, E-I-E-I-O”, says the old children’s song. But Old MacDonald did have a farm! And on that farm he had some horses, cows, pigs and 69 water troughs for the animals to drink from. Only horses drank from the horse troughs, exactly two horses for each trough. Only cows drank from cow troughs, exactly three cows per trough. And only pigs drank from pig troughs, exactly eight pigs per trough. Old MacDonald’s farm has the same number of cows, horses and pigs.

How many animals does Old MacDonald have on his farm?
Problem of the Week
Problem C and Solution
And On This Farm

Problem
“Old MacDonald had a farm, E-I-E-I-O”, says the old children’s song. But Old MacDonald did have a farm! And on that farm he had some horses, cows, pigs and 69 water troughs for the animals to drink from. Only horses drank from the horse troughs, exactly two horses for each trough. Only cows drank from cow troughs, exactly three cows per trough. And only pigs drank from pig troughs, exactly eight pigs per trough. Old MacDonald’s farm has the same number of cows, horses and pigs.

How many animals does Old MacDonald have on his farm?

Solution
Solution 1
Let \( n \) represent the number of each type of animal.

Since there are two horses for every horse trough, then \( n \) must be divisible by 2. Since there are three cows for every cow trough, then \( n \) must be divisible by 3. Since there are eight pigs for every pig trough, then \( n \) must be divisible by 8. Therefore, \( n \) must be divisible by 2, 3 and 8. The smallest number divisible by 2, 3, and 8 is 24. (This number is called the lowest common multiple or LCM, for short.)

If there are 24 of each kind of animal, there would be \( 24 \div 2 = 12 \) troughs for horses, \( 24 \div 3 = 8 \) troughs for cows and \( 24 \div 8 = 3 \) troughs for pigs. This would require a total of \( 12 + 8 + 3 = 23 \) troughs. Since there are 69 troughs and \( 69 \div 23 = 3 \), we require 3 times more of each type of animal. That is, there would be \( 24 \times 3 = 72 \) of each type of animal. The total number of animals is \( 72 + 72 + 72 \) or 216.

We can check the correctness of this solution. Since there are two horses for every horse trough, there are \( 72 \div 2 = 36 \) horse troughs. Since there are three cows for every cow trough, there are \( 72 \div 3 = 24 \) cow troughs. Since there are eight pigs for every pig trough, there are \( 72 \div 8 = 9 \) pig troughs. The total number of troughs is \( 36 + 24 + 9 = 69 \), as expected.
Solution 2

In this solution, algebra and equation solving will be used to solve the problem.

Let $n$ represent the number of each type of animal.

Since there are $n$ horses and there are two horses for every horse trough, then there would be $\frac{n}{2}$ troughs for horses. Since there are $n$ cows and there are three cows for every cow trough, then there would be $\frac{n}{3}$ troughs for cows. Since there are $n$ pigs and there are eight pigs for every pig trough, then there would be $\frac{n}{8}$ troughs for pigs. Since there are 69 troughs in total,

\[ \frac{n}{2} + \frac{n}{3} + \frac{n}{8} = 69 \]

\[ \frac{12n}{24} + \frac{8n}{24} + \frac{3n}{24} = 69 \quad \text{common denominator 24} \]

\[ \frac{23n}{24} = 69 \quad \text{simplify the fractions} \]

\[ 23n = 24 \times 69 \quad \text{multiply both sides by 24} \]

\[ 23n = 1656 \quad \text{simplify} \]

\[ n = \frac{1656}{23} \quad \text{divide both sides by 23} \]

\[ n = 72 \]

There are 72 of each type of animal, a total of 216 animals.

Solution 3

In this solution, ratios will be used to solve the problem.

Let $n$ represent the number of each type of animal. The ratio of the number of troughs required for the pigs to the number of troughs required for the cows is

\[ \frac{n}{8} : \frac{n}{3} = \frac{3n}{24} : \frac{8n}{24} = 3n : 8n = 3 : 8. \]

Similarly, the ratio of the number of troughs required for the cows to the number of troughs required for the horses is $2 : 3 = 8 : 12$. So the ratio of the number of troughs required for the pigs to the number of troughs required for the cows to the number of troughs required for the horses is $3 : 8 : 12$.

Let the number of troughs required for the pigs be $3k$, for the cows be $8k$ and for the horses be $12k$, for some positive integer value of $k$.

Since the total number of troughs required is 69, then

\[ 3k + 8k + 12k = 69 \]

\[ 23k = 69 \]

\[ k = 3 \]

The number of troughs required for the pigs is $3k = 9$. There are 8 pigs at each trough. There are a total of $9 \times 8 = 72$ pigs. Since there are the same number of each animal, there are also 72 cows and 72 horses. There are a total of $72 + 72 + 72 = 216$ animals on Old MacDonald’s farm.
A *perennial* is a plant that blooms over the spring and summer. The plant dies off over the autumn and winter but returns again the following spring.

There are two known species of the POTW perennial plant, the pink ProbleminusA plant and the red ProbleminusB plant. The ProbleminusA is a pink flowering plant in its first year. The following spring it turns into a red ProbleminusB plant. That is, each ProbleminusA plant blooms as a ProbleminusB plant the next year.

The ProbleminusB is a red plant. This plant blooms the following spring and also produces a pink ProbleminusA plant. That is, each ProbleminusB plant returns the following spring along with a new ProbleminusA plant.

Every year this cycle reoccurs.

Today in our garden we planted three ProbleminusA plants and two ProbleminusB plants. Assuming that the plants behave exactly as described, how many plants will be in the garden after 10 reproduction cycles?
Problem

A perennial is a plant that blooms over the spring and summer. The plant dies off over the autumn and winter but returns again the following spring. There are two known species of the POTW perennial plant, the pink ProbleminusA plant and the red ProbleminusB plant. The ProbleminusA is a pink flowering plant in its first year. The following spring it turns into a red ProbleminusB plant. That is, each ProbleminusA plant blooms as a ProbleminusB plant the next year. The ProbleminusB is a red plant. This plant blooms the following spring and also produces a pink ProbleminusA plant. That is, each ProbleminusB plant returns the following spring along with a new ProbleminusA plant. Every year this cycle reoccurs. Today in our garden we planted three ProbleminusA plants and two ProbleminusB plants. Assuming that the plants behave exactly as described, how many plants will be in the garden after 10 reproduction cycles?

Solution

Today we have 3 ProbleminusA plants and 2 ProbleminusB plants. In one year the 3 ProbleminusA plants become 3 ProbleminusB plants. In the same year the 2 ProbleminusB plants remain and produce 2 ProbleminusA plants. So after one year there are 2 ProbleminusA plants and \(3 + 2 = 5\) ProbleminusB plants.

Proceeding from year 1 to year 2, the 2 ProbleminusA plants are now 2 ProbleminusB plants. The 5 ProbleminusB plants remain and produce 5 ProbleminusA plants. After two years, there are 5 ProbleminusA plants and \(2 + 5 = 7\) ProbleminusB plants.

At this point we can make an observation. The number of ProbleminusA plants in a given year equals the number of ProbleminusB plants from the previous year. The number of ProbleminusB plants is the sum of the number of ProbleminusA and the number of ProbleminusB plants from the previous year.

We will use this to produce a chart.

<table>
<thead>
<tr>
<th>Year Number</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td># of ProbleminusA</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>19</td>
<td>31</td>
<td>50</td>
<td>81</td>
<td>131</td>
<td>212</td>
</tr>
<tr>
<td># of ProbleminusB</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>19</td>
<td>31</td>
<td>50</td>
<td>81</td>
<td>131</td>
<td>212</td>
<td>343</td>
</tr>
</tbody>
</table>

After 10 reproduction cycles, there are 212 ProbleminusA plants and 343 ProbleminusB plants for a total of 555 plants in the garden.
The number of a particular flower type in a specific year is dependent on the number of flowers of each of the types from the previous year. This is an example of a recursion.

In our problem,

\[
\text{# of ProbleminusA this year} = \text{# of ProbleminusB the previous year}
\]

and

\[
\text{# of ProbleminusB this year} = \text{# of ProbleminusA the previous year} + \text{# of ProbleminusB the previous year}
\]

A famous example of a recursion is known as the Fibonacci Sequence. The first two numbers in the sequence of numbers are defined. The first number, also called a term, is 1 and the second number is 1. Each remaining term in the sequence is equal to the sum of the two previous terms.

So, the third term is equal to the sum of the first and second terms, and is therefore \(1 + 1 = 2\).

We can now determine the fourth term in the sequence. It will be the sum of the second and third terms, and is therefore \(1 + 2 = 3\).

The fifth term in the sequence is the sum of the third and fourth terms, and is therefore \(2 + 3 = 5\).

We can continue generating more terms in the sequence by applying the rule. The first 20 Fibonacci numbers are

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765\]

In our problem, the number of ProbleminusB plants is the sum of the number of ProbleminusA and ProbleminusB plants in the previous year. If we had started with only 1 ProbleminusA plant and 0 ProbleminusB plants, the number of ProbleminusB plants in years 1 to 10 would be the first 10 terms of the Fibonacci sequence.

Do an internet search to discover more about the Fibonacci sequence and recursions in general.
Problem of the Week
Problem C
Places for Pigeons

One hundred pigeons are to be housed in identical-sized cages under the following conditions:

• each cage must contain at least one pigeon;
• no two cages can contain the same number of pigeons; and
• no cages can go inside any other cage.

Determine the maximum number of cages required to house the pigeons.

Strands  Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
Places for Pigeons

Problem

One hundred pigeons are to be housed in identical-sized cages under the following conditions:

- each cage must contain at least one pigeon;
- no two cages can contain the same number of pigeons; and
- no cages can go inside any other cage.

Determine the maximum number of cages required to house the pigeons.

Solution

In order to maximize the number of cages, each cage must contain the smallest number of birds possible. However, no two cages can contain the same number of pigeons. The simplest way to determine this number is to put one pigeon in the first cage and then let the number of pigeons in each cage after that be one more than the number of pigeons in the cage before it, until all 100 pigeons are housed.

Put 1 pigeon in the first cage, 2 pigeons in the second cage, 3 pigeons in the third cage, and so on. After filling 12 cages in this manner, we have $1 + 2 + 3 + \cdots + 11 + 12 = 78$ pigeons housed. After putting 13 pigeons in the thirteenth cage, $78 + 13 = 91$ pigeons are housed. There are 9 pigeons left to house. But we already have a cage containing 9 pigeons. The remaining 9 pigeons must be distributed among the existing cages while maintaining the condition that no two cages contain the same number of pigeons.

The most obvious way to do this is to put the 9 pigeons in the last cage which already contains 13 pigeons. This would mean that the final cage would contain $13 + 9 = 22$ birds! A better solution might be to increase the number of birds in each of the final nine cages by one bird each. This solution is summarized below:

<table>
<thead>
<tr>
<th>Cage #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td># of birds</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
</tbody>
</table>

The maximum number of cages required is 13. If you had 14 cages, with the first cage holding 1 pigeon and each cage after that holding one more pigeon than the cage before, you could house 105 pigeons, five more than the number of pigeons that we have.
Ten blocks are arranged as illustrated in the following diagram. Each letter shown on the front of a block represents a number. The sum of the numbers on any three consecutive blocks is 19. Determine the value of $S$. 

Strands: Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
Sum Blocks

Problem
Ten blocks are arranged as illustrated in the following diagram. Each letter shown on the front of a block represents a number. The sum of the numbers on any three consecutive blocks is 19. Determine the value of $S$.

Solution
Since the sum of the numbers on any three consecutive blocks is the same,

\[
4 + P + Q = P + Q + R \\
4 + P + Q = P + Q + R \quad \text{since } P + Q \text{ is common to both sides}
\]
\[
\therefore R = 4
\]

Again, since the sum of the numbers on any three consecutive blocks is the same,

\[
T + U + V = U + V + 8 \\
T + U + V = U + V + 8 \quad \text{since } U + V \text{ is common to both sides}
\]
\[
\therefore T = 8
\]

Since the sum of any three consecutive numbers is 19,

\[
R + S + T = 19 \\
4 + S + 8 = 19 \quad \text{substituting } R = 4 \text{ and } T = 8 \\
S + 12 = 19
\]
\[
\therefore S = 7
\]

The value of $S$ is 7.

Extension: Can you determine the values of all remaining letters?
Problem of the Week
Problem C
No Longer a Rectangle

In the following slightly irregular shape,

- $AB = 50\text{ cm}$, $CD = 15\text{ cm}$, $EF = 30\text{ cm}$;
- the area of the shaded triangle, $\triangle DEF$, is $210\text{ cm}^2$; and
- the area of the entire figure, $ABCDE$, is $1000\text{ cm}^2$.

Determine the length of $AE$. 

**STRANDS** Measurement, Patterning and Algebra
Problem of the Week
Problem C and Solution
No Longer a Rectangle

Problem
In the following slightly irregular shape, $AB = 50$ cm, $CD = 15$ cm, $EF = 30$ cm; the area of the shaded triangle, $\triangle DEF$, is $210$ cm$^2$; and the area of the entire figure, $ABCDE$, is $1000$ cm$^2$. Determine the length of $AE$.

Solution
The first task is to mark the given information on the diagram. This has been completed on the diagram to the right. $EG$ has been extended to meet $BC$ at $H$.

To find the area of a triangle, multiply the length of the base by the height and divide by 2. In $\triangle DEF$, the base, $EF$, has length $30$ cm. The height of $\triangle DEF$ is the perpendicular distance from $EF$ (extended) to vertex $D$, namely $GD$. The area is given. So

$$\text{Area } \triangle DEF = \frac{30 \times GD}{2}$$

$$210 = 15 \times GD$$

$$14 = GD$$

We know that $EH = AB = 50$, $GH = DC = 15$, and $EH = EF + FG + GH$. It follows that $50 = 30 + FG + 15$ and $FG = 5$ cm.

Now we can relate the total area to the areas contained inside.

$$\text{Area } ABCDE = \text{Area } ABHE + \text{Area } CDGH + \text{Area } \triangle DFG + \text{Area } \triangle DEF$$

$$1000 = AB \times AE + GD \times DC + \frac{FG \times GD}{2} + 210$$

$$1000 = 50 \times AE + 14 \times 15 + \frac{5 \times 14}{2} + 210$$

$$1000 = 50 \times AE + 210 + 35 + 210$$

$$1000 = 50 \times AE + 455$$

$$1000 - 455 = 50 \times AE$$

$$545 = 50 \times AE$$

$$\frac{545}{50} = AE$$

$. AE = 10.9$ cm.
Problem of the Week
Problem C
Add ’em Up

Did you know that 1000 can be written as the sum of 16 consecutive integers? That is,

$$1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70.$$  

The diagram below illustrates a mathematical short form used for writing the above sum. The notation is called \textit{Sigma Notation}. 

\[
\sum_{i=55}^{70} i = 1000
\]

Using at least two numbers, what is the minimum number of consecutive integers needed to sum to exactly 1000?

\textbf{Strands} Number Sense and Numeration, Patterning and Algebra
Problem

The number 1000 can be written as the sum of 16 consecutive integers? That is,

\[ 1000 = 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70. \]

The diagram above illustrates a mathematical short form used for writing the above sum. The notation is called *Sigma Notation*. Using at least two integers, what is the minimum number of consecutive integers needed to sum to exactly 1000?

Solution

In this solution we will examine possible cases until we discover the first one that works.

1. Can 1000 be written using 2 consecutive integers?

   Let \( n, n+1 \) represent the two integers. Then,
   \[
   n + (n + 1) = 1000 \\
   2n + 1 = 1000 \\
   2n = 999 \\
   n = 499.5
   \]

   Since \( n \) is not an integer, it is not possible to write 1000 using two consecutive integers.

2. Can 1000 be written using 3 consecutive integers?

   Let \( n, n + 1, n + 2 \) represent the three integers. Then,
   \[
   n + (n + 1) + (n + 2) = 1000 \\
   3n + 3 = 1000 \\
   3n = 997 \\
   n \doteq 332.3
   \]

   Since \( n \) is not integer, it is not possible to write 1000 using three consecutive integers.

   (Refer to the note following the solution for an alternate way to define three consecutive integers.)

3. Can 1000 be written using 4 consecutive integers?

   Let \( n, n + 1, n + 2, n + 3 \) represent the four integers. Then,
   \[
   n + (n + 1) + (n + 2) + (n + 3) = 1000 \\
   4n + 6 = 1000 \\
   4n = 994 \\
   n = 248.5
   \]

   Since \( n \) is not an integer, it is not possible to write 1000 using four consecutive integers.
4. Can 1000 be written using 5 consecutive integers?

Let \( n, n + 1, n + 2, n + 3, n + 4 \) represent the five integers. Then,

\[
\begin{align*}
5n + 10 & = 1000 \\
5n & = 990 \\
n & = 198
\end{align*}
\]

Since \( n \) is an integer, it is possible to write 1000 using five consecutive integers. That is,

\[
1000 = 198 + 199 + 200 + 201 + 202
\]

Since we have checked all possible numbers of consecutive integers below five and none of them worked, the minimum number of consecutive integers required to produce a sum of 1000 is five.

**Note:**

In the second case, when we checked to see if 1000 could be written as the sum of three consecutive integers, we could have proceeded as follows:

Let \( a − 1, a, a + 1 \) represent the three consecutive integers. Then,

\[
\begin{align*}
3a & = 1000 \\
a & = 333.3
\end{align*}
\]

Since \( a \) is not an integer, it is not possible to write 1000 using three consecutive integers.

This “idea” is useful when we are finding the sum of an odd number of consecutive integers. If we applied the same idea to the fourth case by using \( a − 2, a − 1, a, a + 1, a + 2 \) to represent the five consecutive integers,

\[
\begin{align*}
5a & = 1000 \\
a & = 200 \\
a − 2 & = 198 \\
a − 1 & = 199 \\
a + 1 & = 201 \\
a + 2 & = 202
\end{align*}
\]

It is possible to write 1000 using five consecutive integers; namely, 198, 199, 200, 201, and 202.

**For Further Thought:**

What is the largest **odd** number of consecutive positive integers that can be used to sum to 1000? What is the smallest number in the list?

How would your answer change if the word positive was removed from the above sentence?
A wall is covered with balloons. Each balloon has the number 5, 3 or 2 printed on it. You are given 7 darts to throw at the wall. If your dart breaks a balloon, then you earn the number of points printed on the balloon. If your dart does not break a balloon, then you are awarded 0 points for that shot. You win a prize if your total score is exactly 16 points on seven shots. If your total is over 16 or under 16, then you lose.

Determine the number of different point combinations that can be used to win the game.
Problem of the Week
Problem C and Solution
Sixteen is Just Perfect

Problem
A wall is covered with balloons. Each balloon has the number 5, 3 or 2 printed on it. You are given 7 darts to throw at the wall. If your dart breaks a balloon, then you earn the number of points printed on the balloon. If your dart does not break a balloon, then you are awarded 0 points for that shot. You win a prize if your total score is exactly 16 points on seven shots. If your total is over 16 or under 16, then you lose. Determine the number of different point combinations that can be used to win the game.

Solution
Solution 1
Let us consider cases.

1. You break three balloons with a 5 printed on them. You have a total of $3 \times 5 = 15$ points. There is no possible way to get 16 points since the other balloon values are 2 or 3. There is no way to win by breaking three (or more) balloons with a 5 printed on them.

2. You break two balloons with a 5 printed on them. You have a total of $2 \times 5 = 10$ points. You need to get $16 - 10 = 6$ points by breaking balloons with 2 or 3 printed on them. There are two ways to do this. Break three balloons with a 2 printed on them and miss on two shots or break two balloons with 3 printed on them and miss on three shots. There are 2 ways to win if you break two balloons with a 5 printed on them.

3. You break one balloon with a 5 printed on it. You have a total of 5 points. You need to get $16 - 5 = 11$ points by breaking balloons with 2 or 3 printed on them. You cannot get 11 points breaking only balloons with a 2 printed on them and you cannot get 11 points breaking only balloons with a 3 printed on them. However, you can get 11 points by breaking one 3 and four 2’s or by breaking three 3’s and one 2. There are 2 ways to win if you break one balloon with a 5 printed on it.

4. You break no balloons with a 5 printed on it. You need to make 16 points by breaking only balloons with a 2 or 3 printed on them. You cannot get 16 points breaking only balloons with a 3 printed on them. You cannot get 16 points breaking only balloons with a 2 printed on them because you only have seven darts giving a maximum of 14 points. It is possible to get 16 points using combinations of 3 point and 2 point balloons. If you break two 3 point balloons and five 2 point balloons, then you win in seven shots. If you break four 3 point balloons and two 2 point balloons, then you have 16 points in six shots and would have to miss on one of your shots. There are 2 ways to win if you do not break any 5 point balloons.

There are $0 + 2 + 2 + 2 = 6$ combinations that allow you to win by getting 16 points in seven shots.
Solution 2

In this solution we will complete a chart to determine the valid possibilities.

Let the number of 5 point balloons be \(a\), the number of 3 point balloons be \(b\), and the number of 2 point balloons be \(c\). We want \(5a + 3b + 2c = 16\) and \(a + b + c \leq 7\).

<table>
<thead>
<tr>
<th>Number of 5 Point Balloons</th>
<th>Number of 3 Point Balloons</th>
<th>Number of 2 Point Balloons</th>
<th>Total Points Scored</th>
<th>Number of Missed Shots Needed</th>
<th>WIN or LOSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(b)</td>
<td>(c)</td>
<td>(5a + 3b + 2c)</td>
<td>(7 - a - b - c)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>18</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>16</td>
<td>3</td>
<td>WIN</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3</td>
<td>16</td>
<td>2</td>
<td>WIN</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>14</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>16</td>
<td>2</td>
<td>WIN</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>16</td>
<td>1</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>0</td>
<td>18</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>0</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>1</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>2</td>
<td>16</td>
<td>1</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>4</td>
<td>17</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>0</td>
<td>WIN</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td></td>
<td>LOSE</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>7</td>
<td>14</td>
<td></td>
<td>LOSE</td>
</tr>
</tbody>
</table>

There are only 6 combinations that allow you to win by getting 16 points in seven shots. (In following a method like the above method, one must be careful to systematically examine all possible cases.)
Problem of the Week
Problem C
The Best Laid Plans

In preparing to write an examination, Stu Deus made the following observations:

• The exam had 20 questions.
• He estimated that he would spend 6 minutes per question.
• The exam would take him 2 hours to complete.

However, during the actual examination, Stu discovered some difficult questions which each required 15 minutes to complete. He also discovered some questions which were much easier than he expected and only took him 2 minutes per question to complete. Seven of the questions, however, still required 6 minutes each to complete. Surprisingly, Stu was still able to complete the exam in 2 hours.

How many of the 20 examination questions did Stu find difficult?
Problem of the Week
Problem C and Solution
The Best Laid Plans

Problem
In preparing to write an examination, Stu Deus made the following observations: the exam had 20 questions, he estimated that he would spend 6 minutes per question, and the exam would take him 2 hours to complete. However, during the actual examination, Stu discovered some difficult questions which each required 15 minutes to complete. He also discovered some questions which were much easier than he expected and only took him 2 minutes per question to complete. Seven of the questions, however, still required 6 minutes each to complete. Surprisingly, Stu was still able to complete the exam in 2 hours. How many of the 20 examination questions did Stu find difficult?

Solution
Solution 1: Systematic Trial
Since 7 of the questions required 6 minutes each to complete, it took Stu $7 \times 6 = 42$ minutes to complete these questions. The total exam took 2 hours or 120 minutes. He had $120 - 42 = 78$ minutes to complete $20 - 7 = 13$ questions.

Let $d$ represent the number of difficult questions and $e$ represent the number of easier questions. We know that $d + e = 13$.

Since each difficult question took 15 minutes, it took $15d$ minutes to complete all of the difficult questions. Since each easier question took 2 minutes, it took $2e$ minutes to complete all of the easier questions. Since Stu’s total remaining time was 78 minutes, $15d + 2e = 78$ minutes.

At this point we can pick values for $d$ and $e$ that add to 13 and substitute into the equation $15d + 2e = 78$ to find the combination that works. (We can observe that $d < 6$ since $15 \times 6 = 90 > 78$. If this were the case, then $e$ would have to be a negative number.)

If $d = 3$ then $e = 13 - 3 = 10$. The time to complete these would be $15 \times 3 + 2 \times 10 = 45 + 20 = 65$ minutes and he would complete the exam in less than 2 hours.

If $d = 4$ then $e = 13 - 4 = 9$. The time to complete these would be $15 \times 4 + 2 \times 9 = 60 + 18 = 78$ minutes and he would complete the exam in exactly 2 hours.

Therefore, Stu found 4 of the questions to be more difficult and time-consuming than he expected.

Solution 2 involves algebra and equation solving.
Solution 2: Using Algebra and Equations

Since 7 of the questions required 6 minutes each to complete, it took Stu \( 7 \times 6 = 42 \) minutes to complete these questions. The total exam took 2 hours or 120 minutes. He had \( 120 - 42 = 78 \) minutes to complete \( 20 - 7 = 13 \) questions.

Let \( d \) represent the number of difficult questions and \( (13 - d) \) represent the number of easier questions.

Since each difficult question took 15 minutes, it took \( 15d \) minutes to complete all of the difficult questions. Since each easier question took 2 minutes, it took \( 2(13 - d) \) minutes to complete all of the easier questions.

Since Stu’s total remaining time was 78 minutes,

\[
15d + 2(13 - d) = 78 \\
15d + 26 - 2d = 78 \\
13d + 26 = 78
\]

Subtracting 26 from both sides:

\[
13d = 52
\]

Dividing both sides by 13:

\[
d = 4
\]

Therefore, Stu found 4 of the questions to be more difficult and time-consuming than he expected.
Problem of the Week
Problem C
This Problem has Value

A nickel is worth 5 cents (5¢), a dime is worth 10 cents (10¢), a quarter is worth 25 cents (25¢), and a dollar is worth 100 cents (100¢).

A piggy bank contains some quarters, dimes and nickels. There are no other coins in the piggy bank. The ratio of the number of quarters to the number of dimes to the number of nickels in the piggy bank is 9 : 3 : 1. The total value of all of the coins in the piggy bank is $18.20.

Determine the number of coins in the piggy bank.
Problem of the Week
Problem C and Solution
This Problem has Value

Problem

A nickel is worth 5 cents (5¢), a dime is worth 10 cents (10¢), a quarter is worth 25 cents (25¢), and a dollar is worth 100 cents (100¢). A piggy bank contains some quarters, dimes and nickels. There are no other coins in the piggy bank. The ratio of the number of quarters to the number of dimes to the number of nickels in the piggy bank is 9 : 3 : 1. The total value of all of the coins in the piggy bank is $18.20. Determine the number of coins in the piggy bank.

Solution

Solution 1

Suppose the piggy bank contained one nickel. Then, using the ratio 9 : 3 : 1, the bank would contain 9 quarters, 3 dimes and 1 nickel, 13 coins in total. The value of the 13 coins would be $25 \times 9 + 10 \times 3 + 5 \times 1 = 260$ cents or $2.60.

Since the coins in the bank are in the ratio 9 : 3 : 1, then we can group the coins into sets of 9 quarters, 3 dimes and 1 nickel, with each set containing 13 coins and having a value of $2.60.

Since the total value of the coins in the bank is $18.20 or 1820 ¢, then there are 7 of these sets of coins. Since each set has 13 coins, then there are $7 \times 13 = 91$ coins in total in the piggy bank.

Solution 2

This solution uses algebra which may be beyond the solver at this point in the school year.

Suppose there are $n$ nickels in the bank. Then, using the ratio 9 : 3 : 1, the bank would contain 9$n$ quarters, 3$n$ dimes and $n$ nickels, 13$n$ coins in total.

The value of the coins would be

$$25 \times 9n + 10 \times 3n + 5 \times n = 225n + 30n + 5n = 260n \text{ cents}.$$ 

The total value of the coins in the bank is $18.20 or 1820 ¢. Therefore, $260n = 1820$. Dividing both sides of the equation by 260, $n = 7$.

Since there are 13$n$ coins in the bank and $n = 7$, there are $13 \times 7 = 91$ coins in the bank.
Solution 3
This solution presents the information in a table.

<table>
<thead>
<tr>
<th># of nickels</th>
<th># of quarters</th>
<th># of dimes</th>
<th># of coins</th>
<th>Total value of coins</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$9 \times 1 = 9$</td>
<td>$3 \times 1 = 3$</td>
<td>$1 + 9 + 3 = 13$</td>
<td>$1 \times 5 + 9 \times 25 + 3 \times 10 = 260 \text{ ¢}$</td>
</tr>
<tr>
<td>2</td>
<td>$9 \times 2 = 18$</td>
<td>$3 \times 2 = 6$</td>
<td>$2 + 18 + 6 = 26$</td>
<td>$2 \times 5 + 18 \times 25 + 6 \times 10 = 520 \text{ ¢}$</td>
</tr>
<tr>
<td>3</td>
<td>$9 \times 3 = 27$</td>
<td>$3 \times 3 = 9$</td>
<td>$3 + 27 + 9 = 39$</td>
<td>$3 \times 5 + 27 \times 25 + 9 \times 10 = 780 \text{ ¢}$</td>
</tr>
<tr>
<td>4</td>
<td>$9 \times 4 = 36$</td>
<td>$3 \times 4 = 12$</td>
<td>$4 + 36 + 12 = 52$</td>
<td>$4 \times 5 + 36 \times 25 + 12 \times 10 = 1040 \text{ ¢}$</td>
</tr>
<tr>
<td>5</td>
<td>$9 \times 5 = 45$</td>
<td>$3 \times 5 = 15$</td>
<td>$5 + 45 + 15 = 65$</td>
<td>$5 \times 5 + 45 \times 25 + 15 \times 10 = 1300 \text{ ¢}$</td>
</tr>
<tr>
<td>6</td>
<td>$9 \times 6 = 54$</td>
<td>$3 \times 6 = 18$</td>
<td>$6 + 54 + 18 = 78$</td>
<td>$6 \times 5 + 54 \times 25 + 18 \times 10 = 1560 \text{ ¢}$</td>
</tr>
<tr>
<td>7</td>
<td>$9 \times 7 = 63$</td>
<td>$3 \times 7 = 21$</td>
<td>$7 + 63 + 21 = 91$</td>
<td>$7 \times 5 + 63 \times 25 + 21 \times 10 = 1820 \text{ ¢}$</td>
</tr>
</tbody>
</table>

From the table, we see that there are a total of 91 coins in the bank.

It is hoped that the solver would recognize a pattern and work with the pattern to find the solution. If the total value had been significantly greater, then this method would provide insight but would be too tedious to completely follow through.

It would also be possible to do some sort of guess and check solution to zero in on the precise solution to the problem.
Six coloured squares are placed beside each other as shown below.

The leftmost square has the number 504 on it and the rightmost square has the number 2017 on it. A number is to be written on each of the four blank squares so that each number after the second number equals the sum of the previous two numbers. Determine the remaining four numbers that should be written on the front of the boxes as you move from left to right.
Problem of the Week  
Problem C and Solution  
Moving Right Along

Problem
Six coloured squares are placed beside each other as shown below.

| 504 | a  |     |     | 2017 |

The leftmost square has the number 504 on it and the rightmost square has the number 2017 on it. A number is to be written on each of the four blank squares so that each number after the second number equals the sum of the previous two numbers.

Determine the remaining four numbers that should be written on the front of the boxes as you move from left to right.

Solution
Let $a$ represent the second number.

Since the third number is the sum of the previous two numbers, the third number is $504 + a$.

Since the fourth number is the sum of the previous two numbers, the fourth number is $(a) + (504 + a) = 504 + 2a$.

Since the fifth number is the sum of the previous two numbers, the fifth number is $(504 + a) + (504 + 2a) = 1008 + 3a$.

Since the sixth number is the sum of the previous two numbers, the sixth number is $(504 + 2a) + (1008 + 3a) = 1512 + 5a$. But the sixth number in the sequence is 2017.

\[
\therefore 1512 + 5a = 2017 \\
1512 + 5a - 1512 = 2017 - 1512 \\
5a = 505 \\
\frac{5a}{5} = \frac{505}{5} \\
a = 101
\]

We now know that the second number is 101 so we can determine the remaining numbers in the sequence by substituting into the expressions above or by simply using the rule to generate the remaining numbers. Using the rule, the third number is $504 + 101 = 605$, the fourth number is $101 + 605 = 706$, and the fifth number is $605 + 706 = 1311$. As a check, we can use the rule to determine the sixth number obtaining $706 + 1311 = 2017$, as required.

The four missing numbers that would be written on the squares are 101, 605, 706, 1311.
Problem of the Week
Problem C
How tall are you?

There are 9 students divided into two groups, the Nanos and the Technos. The heights (in cm) of the 9 students are 151, 153, 157, 161, 153, 157, 156, 159, and 154. The following information is known about the two groups.

• No Nano is taller than any Techno but one of the Nanos is the same height as one of the Technos.

• Two of the Nanos are the same height.

Determine the difference between the mean height of a member of the Technos and the mean height of a member of the Nanos.

Strands  Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
How tall are you?

Problem

There are 9 students divided into two groups, the Nanos and the Technos. The heights (in cm) of the 9 students are 151, 153, 157, 161, 153, 157, 156, 159, and 154. The following information is known about the two groups. No Nano is taller than any Techno but one of the Nanos is the same height as one of the Technos. Two of the Nanos are the same height.

Determine the difference between the mean height of a member of the Technos and the mean height of a member of the Nanos.

Solution

Place the heights in ascending order: 151 153 153 154 156 157 157 159 161.

Since no Nano is taller than any Techno we can introduce a separator “|” that will divide the Nanos and Technos into two distinct groups.

For example, if there are 4 Nanos and 5 Technos, we would place a separator between 154 and 156.

(Nanos in front) 151 153 153 154 | 156 157 157 159 161 (Technos at the end)

Now, the possibilities for the two Nanos with the same height are 157 cm or 153 cm.

If the Nanos with the same height are both 157 cm, then the Nano and Techno with the same height would both be 153 cm. This contradicts the fact that no Nano is taller than any Techno because there would be a Nano whose height is 157 cm and a shorter Techno whose height is 153 cm.

Therefore, the two Nanos with the same height must both be 153 cm, and the Nano and the Techno with the same height must both be 157 cm.

Nanos | Technos
---|---
151 153 153 154 156 157 | 157 159 161

The mean height of a student in the Nano group is \(\frac{151+153+153+154+156+157}{6} = \frac{924}{6} = 154\) cm.

The mean height of a student in the Techno group is \(\frac{157+159+161}{3} = \frac{477}{3} = 159\) cm.

Therefore, the difference in mean heights between the students in the Techno group and the students in the Nano group is 59 – 54 = 5 cm.

NOTE: This question is similar to a question found on the Beaver Computing Challenge (BCC) which is usually written in November. BCC information and past challenges can be found at [http://cemc.uwaterloo.ca/contests/bcc.html](http://cemc.uwaterloo.ca/contests/bcc.html).
Problem of the Week
Problem C
You Can’t Go Back?

On your 13th birthday you received three different time-travel pedometers. You want to use your pedometers to travel back to your 8th birthday. You may use your pedometers as often as you wish but only one at a time.

- Each time you use Pedometer A, take exactly 7 steps forward. This will result in you going back 4 months in time.
- Each time you use Pedometer B, take exactly 5 steps backward. This will result in you going back 7 months in time.
- Each time you use Pedometer C, take exactly 2 steps forward. This will result in you going back 3 months in time.

In traveling back to your 8th birthday, you made a total of 25 backward steps and had a total of 12 pedometer uses.

How many forward steps did you take while using your pedometers?

Strands Number Sense and Numeration, Patterning and Algebra
Problem of the Week
Problem C and Solution
You Can’t Go Back?

Problem
On your 13th birthday you received three different time-travel pedometers. You want to use your pedometers to travel back to your 8th birthday. You may use your pedometers as often as you wish but only one at a time. Each time you use Pedometer A, take exactly 7 steps forward. This will result in you going back 4 months in time. Each time you use Pedometer B, take exactly 5 steps backward. This will result in you going back 7 months in time. Each time you use Pedometer C, take exactly 2 steps forward. This will result in you going back 3 months in time.

In traveling back to your 8th birthday, you made a total of 25 backward steps and had a total of 12 pedometer uses. How many forward steps did you take while using your pedometers?

Solution
From your 13th birthday to your 8th birthday you would travel 5 years back in time. This is equivalent to traveling $5 \times 12 = 60$ months back in time.

Pedometer B is the only pedometer that requires its user to step backward. For every 5 steps backward, you travel 7 months back in time. Therefore, for 25 steps backward, you use Pedometer B five times and travel back in time $5 \times 7 = 35$ months.

You still need to travel $60 - 35 = 25$ more months back in time. You have used a pedometer 5 times and since you only have a total of 12 pedometer uses, you have $12 - 5 = 7$ pedometer uses left. You can now only use Pedometer A and Pedometer C.

If you use Pedometer A and Pedometer C one time each, you travel a total of 7 months back in time. If you use Pedometer A and Pedometer C three times each, this accounts for 6 uses and you travel a total of $7 \times 3 = 21$ months back in time. You have 1 use left and still need to travel 4 more months back in time. This can be accomplished by using Pedometer A once more.

It follows that Pedometer A is used 4 times and Pedometer C is used 3 times. The total number of forward steps is $4 \times 7 + 3 \times 2 = 28 + 6 = 34$.

Note that we could also have looked at each of the possibilities for using Pedometer A. Since there are a total of 7 pedometer uses for Pedometers A and C, the minimum number of uses for Pedometer A would be 0 and the maximum number of uses for Pedometer A would be 7. Once the number of uses for Pedometer A is selected, the number of uses for Pedometer C could be determined by subtracting the number of uses for Pedometer A from 7. For each combination we could determine the number of months traveled back in time. Once the correct combination is determined the total number of forward steps can be calculated. This is summarized in a table on the next page.

An algebraic solution is also provided on the next page.
Using only Pedometer A and Pedometer C a total of 7 times, we want to travel back in time 25 months.

<table>
<thead>
<tr>
<th>Uses of Pedometer A</th>
<th>Uses of Pedometer C</th>
<th>Months Traveled Back in Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>$0 \times 4 + 7 \times 3 = 0 + 21 = 21$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$1 \times 4 + 6 \times 3 = 4 + 18 = 22$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$2 \times 4 + 5 \times 3 = 8 + 15 = 23$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$3 \times 4 + 4 \times 3 = 12 + 12 = 24$</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>$4 \times 4 + 3 \times 3 = 16 + 9 = 25$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>$5 \times 4 + 2 \times 3 = 20 + 6 = 26$</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$6 \times 4 + 1 \times 3 = 24 + 3 = 27$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>$7 \times 4 + 0 \times 3 = 28 + 0 = 28$</td>
</tr>
</tbody>
</table>

Only one combination gives the correct number of Pedometer uses and the correct number of months traveled back in time. Use Pedometer A 4 times and Pedometer C 3 times. The total number of forward steps is $4 \times 7 + 3 \times 2 = 28 + 6 = 34$.

**Algebraic Approach**

This solution is presented for you to get a glimpse of what is coming in future mathematics courses.

Let $a$ be the number of uses of Pedometer A, $b$ be the number of uses of Pedometer B, and $c$ be the number of uses of Pedometer C. Since the total number of uses is 12, then $a + b + c = 12$.

The total number of backward steps is 25 and Pedometer B is the only pedometer requiring backward steps. Since each use of Pedometer B requires 5 backward steps, then we require a total of 5 uses of Pedometer B to go back 25 steps. It follows that $b = 5$ and the equation $a + b + c = 12$ becomes $a + 5 + c = 12$ which simplifies to $a + c = 7$. (1)

In using Pedometer B 5 times, you travel a total of $5 \times 7 = 35$ months back in time. You need to travel 5 years or 60 months back in time altogether. Using Pedometer A and Pedometer C a total of 7 times, you need to travel $60 - 35 = 25$ more months back in time. As an equation this can be written $4a + 3c = 25$. (2)

Rearranging equation (1), we obtain $c = 7 - a$. We can substitute for $c$ in equation (2).

\[
\begin{align*}
4a + 3c & = 25 \\
4a + 3(7 - a) & = 25 \\
4a + 21 - 3a & = 25 \\
4 + a & = 25 \\
a & = 4
\end{align*}
\]

Since $a = 4$, we can substitute in equation (1) to determine that $c = 3$.

For each use of Pedometer A, 7 forward steps are required. Therefore, you step forward $7a$ steps using Pedometer A. For each use of Pedometer C, 2 forward steps are required. Therefore, you step forward $2c$ steps using Pedometer C. The total number of steps forward is $7a + 2c$. But $a = 4$ and $c = 3$ so the total number of forward steps is $7(4) + 3(2) = 28 + 6 = 34$. 