The problems in this booklet are organized into strands. A problem often appears in multiple strands. The problems are suitable for most students in Grade 11 or higher.
Problem of the Week

Problem E

Not That Kind of Median

On highways, medians are used to separate opposing lanes of traffic on divided highways. Our problem is not interested in that kind of median.

In triangles, a median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side, separating the triangle into two triangles of equal area.

In \( \triangle ABC \), \( \angle ABC = 90^\circ \). A median is drawn from \( A \) meeting \( BC \) at \( M \) such that \( AM = 5 \). A second median is drawn from \( C \) meeting \( AB \) at \( N \) such that \( CN = 2\sqrt{10} \).

Determine the length of the longest side of \( \triangle ABC \).
Problem of the Week
Problem E and Solution
Not That Kind of Median

Problem
A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side. In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from $A$ meeting $BC$ at $M$ such that $AM = 5$. A second median is drawn from $C$ meeting $AB$ at $N$ such that $CN = 2\sqrt{10}$. Determine the length of the longest side of $\triangle ABC$.

Solution
Since $AM$ is a median, $M$ is the midpoint of $BC$. Then $BM = MC = y$.
Since $CN$ is a median, $N$ is the midpoint of $AB$. Then $AN = NB = x$.
$\triangle NBC$ is right angled since $\angle B = 90^\circ$. Using the Pythagorean Theorem,

$$NB^2 + BC^2 = CN^2$$
$$x^2 + (2y)^2 = (2\sqrt{10})^2$$
$$x^2 + 4y^2 = 40 \quad (1)$$

$\triangle ABM$ is right angled since $\angle B = 90^\circ$. Using the Pythagorean Theorem,

$$AB^2 + BM^2 = AM^2$$
$$(2x)^2 + y^2 = 5^2$$
$$4x^2 + y^2 = 25 \quad (2)$$

Adding (1) and (2),
$$5x^2 + 5y^2 = 65$$
Dividing by 5,
$$x^2 + y^2 = 13 \quad (3)$$

The longest side of $\triangle ABC$ is the hypotenuse $AC$. Using the Pythagorean Theorem,

$$AC^2 = AB^2 + BC^2$$
$$= (2x)^2 + (2y)^2$$
$$= 4x^2 + 4y^2$$
$$= 4(x^2 + y^2)$$

Substituting from (3) above,
$$AC^2 = 4(13)$$
Taking the square root,
$$AC = 2\sqrt{13}$$

$\therefore$ the length of the longest side is $2\sqrt{13}$ units.

Note: The solver could actually solve a system of equations to find $x = 2$ and $y = 3$ and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate thinking about the solution of this problem.
With the renewed interest in movies based on comic book characters, many clubs for comic book collectors have started. One such club attracts between 15 and 35 members to their monthly meetings.

At their last meeting, they discovered that all of the members in attendance had exactly the same number of comic books, except for one member who had one more comic book than each of the other members. Between them, the members had precisely 1000 comic books.

How many members attended the last meeting?
Problem of the Week
Problem E and Solution
Marvel at This

Problem
With the renewed interest in movies based on comic book characters, many clubs for comic book collectors have started. One such club attracts between 15 and 35 members to their monthly meetings. At their last meeting, they discovered that all of the members in attendance had exactly the same number of comic books, except for one member who had one more comic book than each of the other members. Between them, the members had precisely 1000 comic books. How many members attended the last meeting?

Solution
One could attempt a trial and error solution to this problem. However, a more algebraic solution will be presented here.

Let $n$ represent the number of members present at the last monthly meeting such that $15 < n < 35$ and $n$ is an integer. Let $c$ represent the number of comic books that all but one member had. The one member had $c + 1$ comic books. It follows that $(n - 1)$ members had $c$ comic books each and one member had $c + 1$ comic books producing a total of 1000 comic books.

\[
(n - 1)c + 1(c + 1) = 1000
\]
\[
nc - c + c + 1 = 1000
\]
\[
nc = 999
\]

We are looking for two positive integers with a product of 999 with one of the numbers between 15 and 35. The prime factorization of 999 is $3 \times 3 \times 3 \times 37$. We can combine the factors to produce pairs of positive integers whose product is 999. The possibilities are 1 and 999, 3 and 333, 9 and 111, and 27 and 37. The only possible product which gives one factor between 15 and 35 is $27 \times 37$.

It then follows that there were 27 members present at the last meeting, 26 of the members had 37 comic books each and 1 member had 38 comic books. (This is easily verified: $26 \times 37 + 1 \times 38 = 1000$.)
Problem of the Week
Problem E
Ahead of It’s Time

A stopped watch may be useless but at least it shows the correct time twice a day. A “good” watch which gains or loses time each day, shows the correct time far less often.

When Jeff received a pocket watch from his Grandmother on his 12\textsuperscript{th} birthday it was set at precisely the correct time. However, Jeff soon discovered that his watch gained exactly 10 seconds every day.

Assuming that Jeff never adjusts his watch to correct the time, how many times after his 12\textsuperscript{th} birthday and before his 90\textsuperscript{th} birthday will his watch show the correct time?
Problem of the Week
Problem E and Solution
Ahead of It’s Time

Problem
When Jeff received a pocket watch from his Grandmother on his 12th birthday it was set at precisely the correct time. However, Jeff soon discovered that his watch gained exactly 10 seconds every day. Assuming that Jeff never adjusts his watch to correct the time, how many times after his 12th birthday and before his 90th birthday will his watch show the correct time?

Solution
Solving this problem is not difficult. However, the answer may surprise the solver.

The watch will be correct once it has gained 12 hours.

\[
12 \text{ h} = 12 \times 60 = 720 \text{ minutes} \\
720 \text{ minutes} = 720 \times 60 = 43\,200 \text{ seconds}
\]

Since the watch gains 10 seconds every day, it will take \(43\,200 \div 10 = 4\,320\) days or approximately \(4\,320 \div 365 = 11.8\) years until it is the correct time again.

From Jeff’s 12th birthday to his 90th birthday, 78 years pass. The watch will be accurate \(78 \div (4\,320 \div 365) = 6.6\) times. This means his watch will be accurate only 6 times after his 12th birthday and before his 90th birthday.

The watch will be correct when he is 23.8 years old (between his 23rd and 24th birthday), when he is 35.7 years old (between his 35th and 36th birthday), when he is 47.5 years old (between his 47th and 48th birthday), when he is 59.3 years old (between his 59th and 60th birthday), when he is 71.2 years old (near his 71st birthday), and when he is 83.0 years old (near his 83rd birthday). Jeff may wish to correct his watch periodically or get a more accurate one.

As a concluding note, if the watch gained one second per day, the watch would never be correct again for approximately 120 years. The answer is surprising!
Problem of the Week
Problem E
Putting the Parts Together

Determine the number of solutions to

\[
\frac{P}{Q} - \frac{Q}{P} = \frac{P + Q}{PQ}
\]

where

\[
\begin{align*}
\bullet & \quad P \text{ and } Q \text{ are both integers} \\
\bullet & \quad -9 \leq P \leq 9; \text{ and} \\
\bullet & \quad -9 \leq Q \leq 9
\end{align*}
\]
Problem of the Week
Problem E and Solution
Putting the Parts Together

Problem
Determine the number of solutions to
\[
\frac{P}{Q} - \frac{Q}{P} = \frac{P + Q}{PQ}
\]
where \(P\) and \(Q\) are both integers, \(-9 \leq P \leq 9\) and \(-9 \leq Q \leq 9\).

Solution
\[
\frac{P}{Q} - \frac{Q}{P} = \frac{P + Q}{PQ}
\]
Common Denominator
\[
\frac{P^2 - Q^2}{PQ} = \frac{P + Q}{PQ}
\]
Simplify
\[
\frac{(P - Q)(P + Q)}{PQ} = \frac{(1)(P + Q)}{PQ}
\]
Factor Left Side Numerator

Since the two sides are equal, \(P - Q = 1\) or \(P + Q = 0\). Also, \(P\) and \(Q\) cannot equal zero. Otherwise at least two of the denominators, \(P\), \(Q\), and \(PQ\), would equal zero and division by zero is undefined. We will look at each possibility separately.

1. \(P - Q = 1\), \(P \neq 0\) and \(Q \neq 0\).
   In this case, we see that \(P\) and \(Q\) differ by 1 and \(P > Q\). The largest value of \(P\) is 9. When \(P = 9\), \(Q = 8\). The smallest value of \(Q\) is \(-9\). When \(Q = -9\), \(P = -8\), a value which is 1 more than the value of \(Q\). So \(P\) can take on all of the integer values from \(-8\) to 9 except \(P = 0\). But when \(P = 1\), \(Q = 0\). We would have to remove this value of \(P\) as well. There are 18 integer values for \(P\) from \(-8\) to 9. After removing \(P = 0\) and \(P = 1\), there are 16 values for \(P\) and therefore 16 corresponding values for \(Q\). The equation has 16 solutions such that \(P - Q = 1\), \(P\) and \(Q\) are integers, \(-9 \leq P \leq 9\) and \(-9 \leq Q \leq 9\).

2. \(P + Q = 0\), \(P \neq 0\) and \(Q \neq 0\).
   In this case, \(P + Q = 0\) or \(P = -Q\). When \(P = 9\), the largest integer value it can take on, \(Q = -9\). Similarly, when \(P = -9\), the smallest integer value it can take on, \(Q = 9\). There are 19 integer values that \(P\) can take on but this includes \(P = 0\), \(Q = 0\). So the equation has \(19 - 1 = 18\) solutions such that \(P + Q = 0\), \(P\) and \(Q\) are integers, \(-9 \leq P \leq 9\) and \(-9 \leq Q \leq 9\).

We have considered all possible cases. Therefore, there are \(16 + 18 = 34\) solutions to the equation.
Problem of the Week
Problem E
Keys to Success

The Fine Arts High School Piano Club has 30 student members, some from Grade 11 and the remainder from Grade 12. Each pair of students from the club must play a piano duet together over the course of the year. When two grade 11 students play together, they need 2 hours of practice time. When a grade 11 and a grade 12 student play together, they need 3 hours of practice time. When two grade 12 students play together, they need 4 hours of practice time. In total, the students need 1392 hours of practice time.

How many of the 30 members of the club are Grade 11 students?
Problem of the Week

Problem E and Solution

Keys to Success

Problem

The Fine Arts High School Piano Club has 30 student members, some from Grade 11 and the remainder from Grade 12. Each pair of students from the club must play a piano duet together over the course of the year. When two grade 11 students play together, they need 2 hours of practice time. When a grade 11 and a grade 12 student play together, they need 3 hours of practice time. When two grade 12 students play together, they need 4 hours of practice time. In total, the students need 1392 hours of practice time. How many of the 30 members of the club are Grade 11 students?

Solution

If 3 students \{A, B, C\} are in the same grade, then there will be \(3 \times 2 \div 2 = 3\) duet pairings, namely \{AB\}, \{AC\}, and \{BC\}. (If we look at this using a counting argument, there would be 3 choices for the first student and for each of these choices, there would be 2 choices for the second student, a total of \(3 \times 2 = 6\) pairings, namely \{AB\}, \{AC\}, \{BA\}, \{BC\}, \{CA\}, and \{CB\}. Each pairing appears twice. Since order is not important we must divide by 2, getting us 3 possible pairings.)

If 3 students \{A, B, C\} are in one grade and 2 students \{D, E\} are in the other grade, then there will be \(3 \times 2 = 6\) duet pairings, namely \{AD\}, \{AE\}, \{BD\}, \{BE\}, \{CD\}, and \{CE\}. (There are 3 choices for the grade 11 student in the pairing and for each of these choices, there are 2 possibilities for the choice of the grade 12 student. This gives a total of \(3 \times 2 = 6\) pairings.

This specific argument will now be applied to the general case.

Now, let \(a\) represent the number of Grade 11 students in the club and \((30 - a)\) represent the number of Grade 12 students in the club.
In general, since there are $a$ students in Grade 11 and each must play a duet with every other student in Grade 11, there will be $a \times (a - 1) \div 2$ duets involving only Grade 11 students. Similarly, since there are $(30 - a)$ students in Grade 12 and each must play a duet with every other student in Grade 12, there will be $(30 - a) \times (30 - a - 1) \div 2 = (30 - a) \times (29 - a) \div 2$ duets involving only Grade 12 students. Since every Grade 11 student must play a duet with every Grade 12 student, there will be $a \times (30 - a)$ duets involving one student from each grade.

To determine the total amount of practice time required, take the number of students in each type of pairing and multiply by the number of hours of practice time required for each pairing type.

\[
\text{Total Time} = \text{Time for Grade 11 Pairs} + \text{Time for Grade 12 Pairs} + \text{Time for Grade 11/12 Pairs}
\]

\[
\begin{align*}
1392 &= \left[ \frac{a \times (a - 1)}{2} \right] + 4 \left[ \frac{(30 - a) \times (29 - a)}{2} \right] + 3[a \times (30 - a)] \\
1392 &= a^2 - a + 2(870 - 59a + a^2) + 3(30a - a^2) \\
1392 &= a^2 - a + 1740 - 118a + 2a^2 + 90a - 3a^2 \\
1392 &= -29a + 1740 \\
29a &= 348 \\
a &= 12
\end{align*}
\]

Therefore, 12 of the students in the club are in Grade 11.
Problem of the Week
Problem E
Circle This

In the diagram, $MON$ is a sector of a circle with radius $ON$ which is 6 cm long. If $\angle MON = 60^\circ$, determine the radius of the circle which passes through the points $M$, $N$, and $O$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{circle.png}
\caption{Diagram of circle with $\angle MON = 60^\circ$.}
\end{figure}
Problem of the Week
Problem E and Solution
Circle This

Problem

$MON$ is a sector of a circle with radius $ON$ which is 6 cm long. If $\angle MON = 60^\circ$, determine the radius of the circle which passes through the points $M$, $N$, and $O$.

Solution

Let $C$ be the centre of the circle that passes through $M$, $N$, and $O$. Then $CM$, $CN$, and $CO$ are radii. Therefore, $CM = CN = CO = r$.

In $\triangle CMO$ and $\triangle CNO$, $CM = CN$, $CO$ is common and $OM = ON$. Therefore, $\triangle CMO \cong \triangle CNO$ and it follows that $\angle COM = \angle CON$. But $\angle MON = 60^\circ$. Therefore, $\angle COM = \angle CON = 30^\circ$.

In $\triangle CMO$, $CM = CO = r$ and $\triangle CMO$ is isosceles. Therefore, $\angle CMO = \angle COM = 30^\circ$ and $\angle MCO = 180^\circ - 30^\circ - 30^\circ = 120^\circ$.

**Method 1:** Using the sine law,

\[
\frac{CM}{\sin(\angle COM)} = \frac{OM}{\sin(\angle MCO)}
\]

\[
r = \frac{6}{\sin 120^\circ} \times \sin 30^\circ
\]

\[
r = \frac{6}{\sqrt{3}} \times \frac{1}{2}
\]

\[
r = 6 \times \frac{2}{\sqrt{3}} \times \frac{1}{2}
\]

\[
r = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}
\]

\[
r = 2\sqrt{3} \text{ cm}
\]

The radius of the circle that passes through $M$, $N$, and $O$ is $2\sqrt{3}$ cm.
Method 2: Using the cosine law,

\[ CM^2 = CO^2 + MO^2 - 2 \times CO \times MO \times \cos(\angle COM) \]
\[ r^2 = r^2 + 6^2 - 2(6)(r) \cos 30^\circ \]
\[ 12r \cos 30^\circ = 36 \]
\[ r \cos 30^\circ = 3 \]
\[ r \times \frac{\sqrt{3}}{2} = 3 \]
\[ r \times \sqrt{3} = 6 \]
\[ r \times \sqrt{3} \times \sqrt{3} = 6 \times \sqrt{3} \]
\[ 3r = 6\sqrt{3} \]
\[ r = 2\sqrt{3} \text{ cm} \]

The radius of the circle that passes through \( M, N, \) and \( O \) is \( 2\sqrt{3} \) cm.
Problem of the Week

Problem E

How Low Will It Go?

Suppose \( y = 5x^2 + ax + b, \ a \neq b, \) is a parabola that passes through the points \( A(a, b) \) and \( B(b, a) \).

Determine the minimum value of the parabola.
Problem of the Week
Problem E and Solution
How Low Will It Go?

Problem
Suppose \( y = 5x^2 + ax + b \), \( a \neq b \), is a parabola that passes through the points \( A(a, b) \) and \( B(b, a) \). Determine the minimum value of the parabola.

Solution
Since \( A(a, b) \) is on the parabola, it satisfies the equation of the parabola. We can substitute \( x = a \) and \( y = b \) into the equation \( y = 5x^2 + ax + b \).

\[
\begin{align*}
  b & = 5a^2 + a^2 + b \\
  b & = 6a^2 + b \\
  0 & = 6a^2 \\
  0 & = a^2 \\
  0 & = a
\end{align*}
\]

The equation becomes \( y = 5x^2 + 0x + b \) or simply \( y = 5x^2 + b \).

Since \( B(b, a) \) is on the parabola, it satisfies the equation of the parabola. We can substitute \( x = b \) and \( y = a = 0 \) into the equation \( y = 5x^2 + b \).

\[
\begin{align*}
  0 & = 5b^2 + b \\
  0 & = b(5b + 1) \\
  b = 0 & \text{ or } 5b + 1 = 0 \\
  & \quad \quad b = -\frac{1}{5}
\end{align*}
\]

Since \( a \neq b \) and \( a = 0 \), then \( b = 0 \) is inadmissible. Therefore, \( b = -\frac{1}{5} \) and the equation becomes \( y = 5x^2 - \frac{1}{5} \). The vertex of the parabola is \( \left(0, -\frac{1}{5}\right) \) and so the minimum value is \( -\frac{1}{5} \).
Problem of the Week
Problem E
Positioned Differently

Often we draw parallelograms so that two of the sides are either horizontal or vertical.

The parallelogram, \(ABCD\), is positioned differently. \(A\) lies on the positive \(y\)-axis, \(D\) is on the positive \(x\)-axis, and \(B\) and \(C\) lie in the first quadrant. Three of its vertices, \(A\), \(B\), and \(D\) are located at \((0, 30)\), \((k, 50)\) and \((40, 0)\), respectively. The area of \(ABCD\) is 1340 units\(^2\). If \(k > 0\), determine the coordinates of \(B\) and \(C\).
Problem

Often we draw parallelograms so that two of the sides are either horizontal or vertical. The parallelogram, $ABCD$, is positioned differently. $A$ lies on the positive $y$-axis, $D$ is on the positive $x$-axis, and $B$ and $C$ lie in the first quadrant. Three of its vertices, $A$, $B$, and $D$ are located at $(0,30)$, $(k,50)$ and $(40,0)$, respectively. The area of $ABCD$ is 1340 units². If $k > 0$, determine the coordinates of $B$ and $C$.

Solution

Since $ABCD$ is a parallelogram, $AB = DC$ and $AB \parallel DC$. We can use this to find the coordinates of $C$. To get from $A$ to $B$, we go up 20 units and right $k$ units. Therefore, to get from $D$ to $C$ we do the same. $C$ is located at $(40 + k, 20)$.

In the solution, we will use a method known commonly as “completing the rectangle”.

Enclose $ABCD$ in rectangle $OEFG$ such that $OE$ is on the positive $y$-axis passing through $A$, $EF$ is parallel to the positive $x$-axis passing through $B$, $FG$ is parallel to the positive $y$-axis passing through $C$, and $OG$ lies along the positive $x$-axis passing through $D$.

This information is presented on the following diagram.

The $y$ coordinate of $B$ is the distance from the $x$-axis to $EF$ and also the height, $GF$, of rectangle $OEFG$. It follows that $GF = 50$ units. Similarly, the $x$ coordinate of $C$ is the distance from the $y$-axis to $GF$ and also the width, $OG$, of rectangle $OEFG$. It follows that $OG = (40 + k)$ units. The other dimensions follow. (This information is already marked on the above diagram.)
The diagram from the first page is repeated here.

We can now put the information together using areas to determine the value of $k$.

\[
\text{Area } O\!E\!F\!G = \text{Area } \triangle AEB + \text{Area } \triangle BFC + \text{Area } \triangle CGD + \text{Area } \triangle DOA + \text{Area } ABCD
\]

\[
FG \times OG = \frac{AE \times EB}{2} + \frac{BF \times FC}{2} + \frac{CG \times GD}{2} + \frac{DO \times OA}{2} + 1340
\]

\[
50 \times (40 + k) = \frac{20 \times k}{2} + \frac{40 \times 30}{2} + \frac{20 \times k}{2} + \frac{40 \times 30}{2} + 1340
\]

\[
2000 + 50k = 10k + 600 + 10k + 600 + 1340
\]

\[
2000 + 50k = 20k + 2540
\]

\[
30k = 540
\]

\[
k = 18
\]

Therefore, the value of $k$ is 18 and coordinates of $B$ and $C$ are $B(18, 50)$ and $C(58, 20)$, respectively.

The solver may have approached the problem using linear equations and intersections. This is a very acceptable solution to the problem. However, in this problem, that approach probably would involve considerably more work.
Four numbers are selected such that when each number is added to the average of the other three, the following sums are obtained: 25, 37, 43, and 51.

Determine the average of the four numbers.

The picture puzzle shown above would make an interesting logo for a T-shirt. You may have to research the meanings of the mathematical symbols used in the puzzle. When you determine the meaning, hopefully you will agree that it is worth striving for.
Problem of the Week
Problem E and Solution
Above Average Task

Problem
Four numbers are selected such that when each number is added to the average of the other three, the following sums are obtained: 25, 37, 43, and 51. Determine the average of the four numbers.

Solution
It is possible to precisely determine the four numbers but the problem only asks for their average. Let \(a, b, c, d\) represent each of the four numbers. We are looking for \(\frac{a+b+c+d}{4}\).

When the first number is added to the average of the other three numbers the result is 25.
\[
\therefore a + \frac{b + c + d}{3} = 25 \text{ which simplifies to } 3a + b + c + d = 75 \tag{1}
\]

When the second number is added to the average of the other three numbers the result is 37.
\[
\therefore b + \frac{a + c + d}{3} = 37 \text{ which simplifies to } a + 3b + c + d = 111 \tag{2}
\]

When the third number is added to the average of the other three numbers the result is 43.
\[
\therefore c + \frac{a + b + d}{3} = 43 \text{ which simplifies to } a + b + 3c + d = 129 \tag{3}
\]

When the fourth number is added to the average of the other three numbers the result is 51.
\[
\therefore d + \frac{a + b + c}{3} = 51 \text{ which simplifies to } a + b + c + 3d = 153 \tag{4}
\]

Adding (1), (2), (3), and (4) we obtain \(6a + 6b + 6c + 6d = 468\). Dividing by 6, \(a + b + c + d = 78\). So the sum of the four numbers is 78. Dividing by 4 we determine that the average of the four numbers is 19.5.

Therefore the average of the four numbers is 19.5.

(By solving the system of equations we can actually determine that the numbers are: \(-1.5, 16.5, 25.5, \) and \(37.5\).) The possible T-shirt logo represents “Well Above Average”, something worth striving for.
Problem of the Week  
Problem E  
Every Vote Counts!

Just a small change in how people vote can affect an election result.

In a recent election, the ratio of the number of voters for the Purple Party to the number of voters for the Pink Party was 15:16 and the Pink Party won the election. Had 300 more people voted for the Purple Party and 200 fewer people voted for the Pink Party, the ratio would have been 11:10 and the Purple Party would have won the election.

Determine the total number of votes originally cast.
Problem of the Week
Problem E and Solution
Every Vote Counts!

Problem
In a recent election, the ratio of the number of voters for the Purple Party to the number of
voters for the Pink Party was 15:16 and the Pink Party won the election. Had 300 more people
voted for the Purple Party and 200 fewer people voted for the Pink Party, the ratio would have
been 11:10 and the Purple Party would have won the election. Determine the total number of
votes originally cast.

Solution
Let $a$ represent the number of votes originally cast for the Purple Party.
Let $b$ represent the number of votes originally cast for the Pink Party.
Then the total number of votes originally cast was $a + b$.

The original ratio of votes cast was $a : b = 15 : 16$. This ratio can be written $\frac{a}{b} = \frac{15}{16}$ and
$a = \frac{15}{16}b$ follows. (1)

Had 300 more people voted for the Purple Party, the Purple Party would have received
$(a + 300)$ votes. Had 200 fewer people voted for the Pink Party, the Pink Party would have
received $(b - 200)$ votes. Then

$$\frac{a + 300}{b - 200} = \frac{11}{10}$$

$$10a + 3000 = 11b - 2200$$

$$10a = 11b - 5200$$

$$10 \left( \frac{15b}{16} \right) = 11b - 5200$$

Substituting for $a$ from (1)

$$5 \left( \frac{15b}{8} \right) = 11b - 5200$$

$$75b = 88b - 41600$$

Multiplying by 8

$$-13b = -41600$$

$$b = 3200$$

$$a = \frac{15}{16} \cdot 3200$$

Substituting $b = 3200$ in (1)

$$a = 3000$$

$$a + b = 3000 + 3200$$

$$a + b = 6200$$

There were 6200 total votes originally cast.
Problem of the Week

Problem E

From Altitudes to Angles to Sides

In \( \triangle ABC \), \( \angle BAC \) is the largest angle and \( \angle ACB \) is the smallest angle. \( AQ \), \( BR \), and \( CP \) are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the size of \( \angle ABC \) and the lengths of the sides of \( \triangle ABC \).

The diagram is not necessarily drawn to scale.
Problem of the Week
Problem E and Solution
From Altitudes to Angles to Sides

Problem
In $\triangle ABC$, $\angle BAC$ is the largest angle and $\angle ACB$ is the smallest angle. $AQ$, $BR$, and $CP$ are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the size of $\angle ABC$ and the lengths of the sides of $\triangle ABC$.

Solution
Let $BC = a$, $AC = b$ and $AB = c$.

We can find the area of the triangle by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,

$$\frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}$$

But $AQ = 21$, $BC = a$, $BR = 24$, $AC = b$, $CP = 56$, and $AB = c$. Multiplying through by 2 and substituting we obtain

$$21a = 24b = 56c.$$ 

From $21a = 24b$ we obtain $b = \frac{21}{24}a = \frac{7}{8}a$ and from $21a = 56c$ we obtain $c = \frac{21}{56}a = \frac{3}{8}a$. The ratio of the sides in $\triangle ABC$ is $a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3$. Let $BC = 8x$, $AC = 7x$, and $AB = 3x$, $x > 0$.

Using the cosine law,

$$AC^2 = AB^2 + CB^2 - 2(AB)(CB)\cos(\angle ABC)$$

$$49x^2 = (3x)^2 + (8x)^2 - 2(3x)(8x)\cos(\angle ABC)$$

Dividing by $x^2$ since $x > 0$, $49 = 73 - 48\cos(\angle ABC)$

Rearranging, $48\cos(\angle ABC) = 24$

$$\cos(\angle ABC) = \frac{1}{2}$$

$\therefore \angle ABC = 60^\circ$
In right $\triangle BPC$,

\[
\frac{PC}{BC} = \sin 60^\circ \\
BC = \frac{PC}{\sin 60^\circ} \\
BC = \frac{56}{\frac{\sqrt{3}}{2}} \\
BC = \frac{112\sqrt{3}}{3} \\
\text{But } BC = 8x \\
\therefore 8x = \frac{112\sqrt{3}}{3} \\
x = \frac{14\sqrt{3}}{3} \\
3x = 14\sqrt{3} \\
7x = \frac{98\sqrt{3}}{3}
\]

The side lengths of $\triangle ABC$ are $AB = 3x = 14\sqrt{3}$, $AC = 7x = \frac{98\sqrt{3}}{3}$ and $BC = \frac{112\sqrt{3}}{3}$. 
Problem of the Week
Problem E
Make Gauss Proud

Johann Carl Friedrich Gauss was a mathematician who lived from 1777 to 1855. He made major contributions to number theory and algebra, to name just a few. Some of the earliest stories about Gauss deal with determining the sum of sequences of numbers. Our problem today deals with a long sequence of numbers. So, make Gauss proud as you work with this problem.

A sequence consists of 2018 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2018 terms is 27 243.

Determine the sum of the odd numbered terms. That is, determine the sum of every second term starting with the first term and ending with the second last term.

Johann Carl Friedrich Gauss
April 30, 1777 - February 23, 1855

Some helpful information about sequences is included on the next page. You may or may not wish to refer to it.
A sequence is made up of terms like \( t_1, t_2, t_3, \ldots, t_n \). The subscript indicates the position of the term in the sequence. For example, \( t_8 \) would represent the term in the eighth position in the sequence and \( t_n \) represents the general term in the sequence.

A series is the sum of the terms of a sequence. So \( t_1 + t_2 + t_3 + \cdots + t_n \) would represent the sum of the first \( n \) terms.

The following information may be helpful in the solution of the problem.

An arithmetic sequence is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 is an arithmetic sequence with four terms and constant difference 2.

The general term of an arithmetic sequence is \( t_n = a + (n - 1)d \), where \( a \) is the first term, \( d \) is the constant difference and \( n \) is the number of terms.

The sum, \( S_n \), of the first \( n \) terms of an arithmetic sequence can be found using either \( S_n = \frac{n}{2}[2a + (n - 1)d] \) or \( S_n = n \left( \frac{t_1 + t_n}{2} \right) \), where \( t_1 \) is the first term of the sequence and \( t_n \) is the \( n^{th} \) term of the sequence.

The following example is provided to verify the accuracy of the formulas and to illustrate their use.

For the arithmetic sequence 3, 5, 7, 9, \( a = t_1 = 3 \), \( d = 2 \), \( n = 4 \) and \( t_n = t_4 = 9 \).

\[
S_n = 3 + 5 + 7 + 9 = 24 \\
S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{4}{2}[2(3) + (3)2] = 2[12] = 24 \\
S_n = n \left( \frac{t_1 + t_n}{2} \right) = 4 \left( \frac{3 + 9}{2} \right) = 4(6) = 24
\]

Gauss developed a specific formula for the sum of the first \( n \) positive integers.

\[
1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}
\]

For example, in honour of the current year, the sum of the positive integers from 1 to 2017 is

\[
1 + 2 + 3 + \cdots + 2016 + 2017 = \frac{2017(2018)}{2} = 2035153
\]
Problem of the Week
Problem E and Solution
Make Gauss Proud

Problem
A sequence consists of 2018 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2018 terms is 27243. Determine the sum of the odd numbered terms. That is, determine the sum of every second term starting with the first term and ending with the second last term.

Solution
Solution 1
Let \( S_O \) represent the sum of the terms in the odd numbered positions. Half of the terms of the sequence are in odd numbered positions so there are \( 2018 \div 2 = 1009 \) terms in \( S_O \).

\[
S_O = t_1 + t_3 + t_5 + \cdots + t_{2017}
\]

Let \( S_E \) represent the sum of the terms in the even numbered positions. Half of the terms of the sequence are in even numbered positions so there are 1009 terms in \( S_E \).

\[
S_E = t_2 + t_4 + t_6 + \cdots + t_{2018}
\]

Let \( S \) represent the sum of the 2018 terms. \( S = S_O + S_E = 27243 \quad (1) \)

Since each term after the first term is 1 greater than the term before,

\[
t_2 = t_1 + 1, \ t_4 = t_3 + 1, \ t_6 = t_5 + 1, \cdots, \ t_{2018} = t_{2017} + 1
\]

Now,

\[
S_E = t_2 + t_4 + t_6 + \cdots + t_{2016} + t_{2018}, \quad 1009 \text{ terms}
\]

\[
= (t_1 + 1) + (t_3 + 1) + (t_5 + 1) + \cdots + (t_{2015} + 1) + (t_{2017} + 1)
\]

\[
= (t_1 + t_3 + t_5 + \cdots + t_{2015} + t_{2017}) + 1009(1)
\]

\[
\therefore \ S_E = S_O + 1009 \quad (2)
\]

Substituting \( S_O + 1009 \) for \( S_E \) in (1),

\[
S_O + S_E = 27243
\]

\[
S_O + S_O + 1009 = 27243
\]

\[
2S_O = 27243 - 1009
\]

\[
2S_O = 26234
\]

\[
S_O = 13117
\]

Therefore, the sum of the terms in the odd positions in the sequence is 13117.

Notice that this solution did not require any special formulas.
Solution 2

Let $t_1$ represent the first term in the sequence.

Every term in the sequence can be written in terms of $t_1$. The second term is 1 more than the first term, the third term is 2 more than the first term, the fourth term is 3 more than the first term, and so on.

\[
\begin{align*}
t_1 &+ t_2 + t_3 + t_4 + \cdots + t_{2016} + t_{2017} + t_{2018} = 27243 \\
t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2016) + (t_1 + 2017) &= 27243 \\
2018t_1 + (1 + 2 + 3 + \cdots + 2016 + 2017) &= 27243
\end{align*}
\]

Using the formula for the sum of the first $n$ positive integers with $n = 2017$,

\[
2018t_1 + \frac{2017(2018)}{2} = 27243
\]

Dividing each term by 2018,

\[
\begin{align*}
t_1 + \frac{2017}{2} &= \frac{27243}{2018} \\
t_1 + 1008.5 &= 13.5 \\
t_1 &= -995
\end{align*}
\]

Now that we know that the first term in the sequence is $-995$, we know that the original series is

\[
\begin{align*}
t_1 + t_2 + t_3 + t_4 + \cdots + t_{2016} + t_{2017} + t_{2018} &= t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2016) + (t_1 + 2017) \\
&= -995 - 994 - 993 - 992 - \cdots + 1021 + 1022
\end{align*}
\]

We are interested in the sum

\[-995 - 994 - 993 - 989 - \cdots + 1019 + 1021\]

This is an arithmetic series with 1009 terms, first term $t_1 = -995$, and last term $t_n = 1021$. Then, using $S_n = n \left[\frac{t_1 + t_n}{2}\right]$

\[-995 - 994 - 993 - 992 - \cdots + 1021 + 1022 = 1009 \left[\frac{-995 + 1021}{2}\right] = 1009(13) = 13117\]

Therefore the sum of the odd numbered terms in the sequence is 13117.

Many other approaches could be taken once the first term is found.
Problem of the Week
Problem E
Keep On Chuggin’

Two trains of equal length are on parallel tracks. One train is travelling at 40 km/h and the other at 20 km/h. It takes two minutes longer for the trains to completely pass one another when going in the same direction, than when going in opposite directions.

Determine the length of each train.
Problem of the Week
Problem E and Solution
Keep On Chuggin’

Problem

Two trains of equal length are on parallel tracks. One train is travelling at 40 km/h and the other at 20 km/h. It takes two minutes longer for the trains to completely pass one another when going in the same direction, than when going in opposite directions. Determine the length of each train.

Solution

Solution 1

Let $L$ represent the length, in km, of each train. Let $t_1$ represent the time, in hours, required for the fast train to completely pass the slow train when going in the same direction. Let $t_2$ represent the time, in hours, required for the fast train to completely pass the slow train when going in opposite directions.

In order to completely pass one another when going in the same direction, the faster train must travel two lengths of the train plus whatever distance the slower train travels. Therefore,

\[
40t_1 = 20t_1 + 2L \\
20t_1 = 2L \\
t_1 = \frac{L}{10}
\]

In order to completely pass one another when going in the opposite direction, the total distance travelled by the two trains must be $2L$. Therefore,

\[
40t_2 + 20t_2 = 2L \\
60t_2 = 2L \\
t_2 = \frac{L}{30}
\]

Since it takes two minutes or $\frac{2}{60}$ hours longer for the trains to completely pass one another when going in the same direction than when going in opposite directions,

\[
t_1 - t_2 = \frac{2}{60} \\
\frac{L}{10} - \frac{L}{30} = \frac{1}{30}
\]

Multiplying by 30:

\[
3L - L = 1 \\
2L = 1 \\
L = 0.5
\]

Therefore, the length of each train is 0.5 km.
Solution 2

Let $L$ represent the length, in km, of each train.

When going in the same direction, the faster train is travelling at $40 - 20 = 20$ km/h relative to the slower train. In order to completely pass, the faster train must travel $2L$ km. Therefore, it takes $\frac{2L}{20} = \frac{L}{10}$ hours to completely pass.

When travelling in opposite directions, the faster train is travelling at $40 + 20 = 60$ km/h relative to the slower train. In order to completely pass, the faster train must travel $2L$ km. Therefore, it takes $\frac{2L}{60} = \frac{L}{30}$ hours to completely pass.

Since it takes two minutes or $\frac{2}{60} = \frac{1}{30}$ hours longer for the trains to completely pass one another when going in the same direction than when going in opposite directions,

$\frac{L}{10} - \frac{L}{30} = \frac{1}{30}$

Multiplying by 30:

$3L - L = 1$

$2L = 1$

$L = 0.5$

Therefore, the length of each train is 0.5 km.
Solution 3

Let \( L \) represent the length, in km, of each train.

While the trains are travelling in opposite directions, let \( y \) km be the distance travelled by the slower train from the time the faster train begins to pass until it completely passes. The slower train travels \( y \) km and the faster train travels \((2L - y)\) km. We know that the time travelled will be the same so:

\[
\frac{y}{20} = \frac{2L - y}{40} \\
\frac{2y}{40} = \frac{2L - y}{40} \\
3y = 2L \quad (1)
\]

While the trains are travelling in the same direction, let \( x \) km be the distance travelled by the slower train from the time the faster train begins to pass until it completely passes. The slower train travels \( x \) km and the faster train travels \((x + 2L)\) km. We know that the time travelled will be the same so:

\[
\frac{x}{20} = \frac{x + 2L}{40} \\
\frac{2x}{40} = \frac{x + 2L}{40} \\
x = 2L \quad (2)
\]

We know that it takes two minutes or \( \frac{2}{60} \) hours longer for the trains to completely pass one another when going in the same direction than when going in opposite directions. So,

\[
\frac{x}{20} - \frac{y}{20} = \frac{2}{60} \\
\frac{x}{3y} = \frac{2}{60} \\
\frac{20}{60} = \frac{20}{60}
\]

Substituting \( 2L \) for \( x \) from (2) and \( 2L \) for \( 3y \) from (1),

\[
\frac{2L}{20} - \frac{2L}{60} = \frac{2}{60} \\
\frac{6L}{2L} = \frac{2}{60} \\
\frac{60}{60} = \frac{60}{60} \\
4L = 2 \\
L = 0.5
\]

Therefore, the length of each train is 0.5 km.
Problem of the Week
Problem E
How Far to the Centre

A circle with centre $O$ is drawn with points $P$, $Q$, and $S$ on the circumference such that $PQ = PS = 12$ m. $PO$ is extended to meet $QS$ at $R$ such that $PR \perp QS$ and $OR = 1$ m.

Determine the radius of the circle.
Problem of the Week
Problem E and Solution
How Far to the Centre

Problem
A circle with centre $O$ is drawn with points $P$, $Q$, and $S$ on the circumference such that $PQ = PS = 12$ m. $PO$ is extended to meet $QS$ at $R$ such that $PR \perp QS$ and $OR = 1$ m.

Determine the radius of the circle.

Solution
Since $O$ is the centre of a circle that passes through $P$, $Q$, and $S$, then $OP$, $OQ$, and $OS$ are radii. Then $OP = OQ = OS = x$, $x > 0$. Let $SR = y$.

$\triangle SPR$ is right angled at $R$. Using the Pythagorean Theorem,

$$PR^2 + RS^2 = PS^2$$

$$(PO + OR)^2 + RS^2 = PS^2$$

$$(x + 1)^2 + y^2 = 12^2 \quad (1)$$

$\triangle SOR$ is right angled at $R$. Using the Pythagorean Theorem,

$$OR^2 + RS^2 = OS^2$$

$$1^2 + y^2 = x^2$$

$$y^2 = x^2 - 1$$

Substitute for $y^2$ in (1):

$$(x + 1)^2 + x^2 - 1 = 12^2$$

$$x^2 + 2x + 1 + x^2 - 1 = 144$$

$$2x^2 + 2x - 144 = 0$$

$$x^2 + x - 72 = 0$$

$$(x - 8)(x + 9) = 0$$

$$x = 8 \quad \text{or} \quad x = -9$$

Since $x > 0$, $x = -9$ is inadmissible. Therefore, $x = 8$. But $x$ is the radius of the circle.

$\therefore$ the radius of the circle is 8 m.
Each of the numbers 1, 2, 3, 4, 5, 6 occurs, one to a face, on the faces of a cube. Three people, Bel, Cal and Dan, are seated around a rectangular table. Bel is seated on one side of the table. Cal is seated on the side of the table which is adjacent to Bel and to her right. Dan is seated on the side of the table which is adjacent to Cal and to his right. There is an empty seat along the side which is adjacent to both Bel and Dan.

The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces.

When Bel adds the three numbers that she can see, her total is 9. When Cal adds the three numbers that he can see, his total is 14. When Dan adds the three numbers that he can see, his total is 15.

Determine the number on the bottom face of the cube.
Problem of the Week
Problem E and Solution
A Different Point of View

Problem
Each of the numbers 1, 2, 3, 4, 5, 6 occurs, one to a face, on the faces of a cube. Three people, Bel, Cal and Dan, are seated around a rectangular table. Bel is seated on one side of the table. Cal is seated on the side of the table which is adjacent to Bel and to her right. Dan is seated on the side of the table which is adjacent to Cal and to his right. There is an empty seat along the side which is adjacent to both Bel and Dan. The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces. When Bel adds the three numbers that she can see, her total is 9. When Cal adds the three numbers that he can see, his total is 14. When Dan adds the three numbers that he can see, his total is 15. Determine the number on the bottom face of the cube.

Solution

Let \( a \) represent the number on the top face of the cube.
Let \( a, b \) and \( c \) represent the three numbers that Bel sees.
Let \( a, c \) and \( d \) represent the three numbers that Cal sees.
Let \( a, d \) and \( e \) represent the three numbers that Dan sees.
If there were a person in the fourth seat, that person would see \( a, e \) and \( b \).

From the given information, we are now able to form three equations:

\[
\begin{align*}
a + b + c & = 9 \quad (1) \\
a + c + d & = 14 \quad (2) \\
a + d + e & = 15 \quad (3)
\end{align*}
\]

Comparing equation (1) and equation (2), \( b \) has been replaced by \( d \) and the sum has increased by 5. Therefore, \( b \) and \( d \) differ by 5 and \( b < d \). The only numbers from the set 1, 2, 3, 4, 5, 6 that differ by 5 are 1 and 6. Therefore, \( b = 1 \) and \( d = 6 \).

Comparing equation (2) and equation (3), \( c \) has been replaced by \( e \) and the sum has increased by 1. Therefore, \( c \) and \( e \) differ by 1 and \( c < e \). Since \( b = 1 \) and \( d = 6 \), there are only three possible combinations for \( c \) and \( e \), namely \( c = 2 \) and \( e = 3 \), or \( c = 3 \) and \( e = 4 \), or \( c = 4 \) and \( e = 5 \).

We will check each of these possibilities. First, if \( c = 2, e = 3, b = 1 \) and \( d = 6 \), we can substitute the appropriate values in (1) giving \( a + 1 + 2 = 9 \) or \( a = 6 \). This is not possible since \( d \) would also equal 6. We can rule this case out.
Next, if $c = 3$, $e = 4$, $b = 1$ and $d = 6$, we can substitute the appropriate values in (1) giving $a + 1 + 3 = 9$ or $a = 5$. This is a possible solution. In (2), $a + c + d = 5 + 3 + 6 = 14$ as required. And in (3), $a + d + e = 5 + 6 + 4 = 15$ as required. The only number not used is 2 so the number on the bottom face is 2. But is this the only solution?

Finally, if $c = 4$, $e = 5$, $b = 1$ and $d = 6$, we can substitute the appropriate values in (1) giving $a + 1 + 4 = 9$ or $a = 4$. This is not possible since $c$ would also equal 4. We can rule this case out.

Since we have examined all possible cases, the only possible number on the bottom (unseen) face is 2.

\[
\begin{array}{cccc}
5 & 1 & 3 & 5 \\
3 & 6 & 5 & 4 \\
6 & 4 & 5 & 1 \\
\end{array}
\]

**Notes:**

It is also possible to play with the numbers to solve this problem. The method presented above could be used in a similar way with any list of six different numbers. “Playing” with the numbers might not be as easy.

Instead of “arguing” the difference between equations to obtain the relationship between $b$ and $d$, and $c$ and $e$, we could have used elimination.

\[
\begin{align*}
\ a + b + c &= 9 \\
\ a + c + d &= 14 \\
\ a + d + e &= 15
\end{align*}
\]

For example, equation (1) subtract equation (2) gives $b - d = -5$ which can be written $d - b = 5$. This is the same as saying the difference between $b$ and $d$ is 5.

Similarly, equation (2) subtract equation (3) gives $c - e = -1$ which can be written $e - c = 1$. This is the same as saying that the difference between $c$ and $e$ is 1.
Problem of the Week
Problem E
Volumizing A Triangle

A rectangular prism has dimensions of $2a$, $2b$ and $2c$ as shown in the diagram below on the left.

$H$ is the intersection of the diagonals of the top face of the prism, $J$ is the intersection of the diagonals of the side face of the prism and $K$ is the intersection of the diagonals of the front face of the prism. $\triangle HJK$ is formed by joining $H$, $J$ and $K$. This is shown in the diagram below on the right.

If $HJ = 4$ cm, $HK = 5$ cm, and $JK = 6$ cm, determine the volume of the rectangular prism.
Problem of the Week
Problem E and Solution
Volumizing A Triangle

Problem
A rectangular prism has dimensions of \(2a\), \(2b\) and \(2c\). \(H\) is the intersection of the diagonals of the top face of the prism, \(J\) is the intersection of the diagonals of the side face of the prism and \(K\) is the intersection of the diagonals of the front face of the prism. \(\triangle HJK\) is formed by joining \(H\), \(J\) and \(K\). This is shown in the diagram to the right. If \(HJ = 4\) cm, \(HK = 5\) cm, and \(JK = 6\) cm, determine the volume of the rectangular prism.

Solution
Label the top front edge of the rectangular prism \(AB\) and its midpoint \(M\). Then draw in \(\triangle KMH\).

Since \(H\) and \(K\) are the centres of their respective rectangles, \(MK = a\), \(MH = c\) and \(HK = 5\).

Since \(\angle KMH = 90^\circ\), then

\[ a^2 + c^2 = 25 \quad (1) \]

Similarly, it can be shown that

\[ b^2 + c^2 = 36 \quad (2) \]

and

\[ a^2 + b^2 = 16 \quad (3) \]

Adding (1), (2), and (3),

\[ 2a^2 + 2b^2 + 2c^2 = 77 \]

Then dividing by 2,

\[ a^2 + b^2 + c^2 = \frac{77}{2} \quad (4) \]

Subtracting each of equations (1), (2) and (3) from equation (4) yields

\[ b^2 = \frac{27}{2}, \ a^2 = \frac{5}{2}, \ \text{and} \ c^2 = \frac{45}{2}. \]

Multiplying \(a^2\), \(b^2\) and \(c^2\) gives the product

\[ a^2b^2c^2 = \frac{(5)(27)(45)}{8} = \frac{6075}{8}. \]

Then, taking the positive square root,

\[ abc = \sqrt{\frac{6075}{8}} = \frac{45\sqrt{3}}{2\sqrt{2}} = \frac{45\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{45\sqrt{6}}{4}. \]

To determine the volume of the rectangular prism multiply the side lengths \(2a\), \(2b\) and \(2c\) to obtain

\[ V = (2a)(2b)(2c) = 8abc = 8 \left( \frac{45\sqrt{6}}{4} \right) = (2\sqrt{2})(45\sqrt{3}) = 90\sqrt{6} \ \text{cm}^3. \]

The volume of the rectangular prism is \(90\sqrt{6} \ \text{cm}^3\).
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point \( D \) on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point \( A(9, 8) \), then reaches its peak at \( E(7, 9) \), and finally passes through a hoop located at \( B(b, 5) \) before returning to the ground at point \( C \) and bursting. The ground between \( C \) and \( D \) is flat. Determine the area of quadrilateral \( ABCD \).
Problem of the Week
Problem E and Solution
Balloons Away!

Problem
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point $D$ on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point $A(9,8)$, then reaches its peak at $E(7,9)$, and finally passes through a hoop located at $B(b,5)$ before returning to the ground at point $C$ and bursting. The ground between $C$ and $D$ is flat. Determine the area of quadrilateral $ABCD$.

Solution
We need to find the equation of the parabola. Then, in order to answer the question, we are required to find the $x$-intercepts of the parabola and the $x$-coordinate of point $B$ on the parabola.

We are given the peak $E(7,9)$ so we have the vertex of the parabola. Using the vertex form of the equation of a parabola, $y = a(x - h)^2 + k$, with vertex $(h,k) = (7, 9)$, the equation of the parabola looks like $y = a(x - 7)^2 + 9$.

The point $A(9,8)$ is on the parabola so we can substitute $(x, y) = (9, 8)$ into the equation $y = a(x - 7)^2 + 9$ to find $a$.

$$8 = a(9 - 7)^2 + 9$$
$$8 = a(4) + 9$$
$$-1 = 4a$$
$$-\frac{1}{4} = a$$

The equation of the parabola is $y = -\frac{1}{4}(x - 7)^2 + 9$.

To find the $x$-coordinate of $B(b,5)$, substitute $y = 5$ into the equation of the parabola.

$$5 = -\frac{1}{4}(b - 7)^2 + 9$$
$$-4 = -\frac{1}{4}(b - 7)^2$$
$$16 = (b - 7)^2$$
$$\pm 4 = b - 7$$

It follows that $b - 7 = -4$ or $b - 7 = 4$. Then $b = 3$ or $b = 11$. The point $B$ is to the left of the vertex so $b < 7$. The coordinates of $B$ are $(3, 5)$. 
To find the \( x \)-intercepts of the parabola, substitute \( y = 0 \) into the equation of the parabola. 

\[
0 = -\frac{1}{4}(x - 7)^2 + 9 \\
-9 = -\frac{1}{4}(x - 7)^2 \\
36 = (x - 7)^2 \\
\pm 6 = x - 7
\]

It follows that \( x - 7 = -6 \) or \( x - 7 = 6 \). Then the \( x \)-intercepts of the parabola are 1 and 13. The point \( C \) is to the left of the vertex and the point \( D \) is to the right of the vertex. The coordinates of \( C \) are \((1, 0)\) and \( D \) are \((13, 0)\).

This information has been added to the graph. There are many ways to determine the area of \( ABCD \).

From \( B(3, 5) \) and \( A(9, 8) \) drop perpendiculars to the \( x \)-axis, intersecting at \( F(3, 0) \) and \( G(9, 0) \), respectively. From \( B(3, 5) \) draw a perpendicular to \( AG \), intersecting at \( H(9, 5) \). Draw line segment \( BG \).

We will use the diagram to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

\( BH = 9 - 3 = 6, \ CG = 9 - 1 = 8, \ GD = 13 - 9 = 4, \ BF = 5 - 0 = 5, \) and \( AG = 8 - 0 = 8 \).

To determine the area, we will find the sum of the areas of \( \triangle CGB, \triangle AGD \) and \( \triangle AGB \).

\[
\text{Area } ABCD = \text{Area } \triangle CGB + \text{Area } \triangle AGD + \text{Area } \triangle AGB \\
= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2} \\
= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2} \\
= 20 + 16 + 24 \\
= 60 \text{ units}^2
\]

The area of \( ABCD \) is 60 units \(^2\).
Problem of the Week
Problem E
This is Some Function

For some function $f(x) = ax^3 + bx^2 + cx + d$ where $a$, $b$, $c$ and $d$ are integers, we know the following information:

- the $y$-intercept is 5,
- $f(2) = -3$,
- $40 < f(4) < 50$, and
- $240 < f(6) < 250$.

Determine the value of $f(3)$.

The following notes about solving inequalities may be helpful in solving the above problem. The example, “Solve for $x$ so that $20 < 8x + 5 < 28$”, is used to illustrate steps that can be used in solving inequalities.

- You may add or subtract a constant from each of the parts of the inequality without changing the sense of the inequality. Using the example,

  Subtracting 5 from each part of the inequality
  $20 < 8x + 5 < 28$
  leaves the sense of the inequality unchanged.
  $15 < 8x < 23$

- You may multiply or divide each of the parts of the inequality by a positive number without changing the sense of the inequality. Continuing from where we left off in the example,

  Dividing each part of the inequality by 8
  $15 < 8x < 23$
  leaves the sense of the inequality unchanged.
  $\frac{15}{8} < x < \frac{23}{8}$
Problem of the Week
Problem E and Solution
This is Some Function

Problem
For some function \( f(x) = ax^3 + bx^2 + cx + d \) where \( a, b, c \) and \( d \) are integers, we know the following information: the \( y \)-intercept is 5, \( f(2) = -3 \), \( 40 < f(4) < 50 \), and \( 240 < f(6) < 250 \). Determine the value of \( f(3) \).

Solution
We will process the information in the order that it was provided.

- The \( y \)-intercept is 5 so we know that \( f(0) = 5 \).

\[
\begin{align*}
f(0) &= 5 \\
a(0)^3 + b(0)^2 + c(0) + d &= 5 \\
d &= 5
\end{align*}
\]

The function is now \( f(x) = ax^3 + bx^2 + cx + 5 \).

- We are then given that \( f(2) = -3 \).

\[
\begin{align*}
f(2) &= -3 \\
a(2)^3 + b(2)^2 + c(2) + 5 &= -3 \\
8a + 4b + 2c + 5 &= -3 \\
8a + 4b + 2c &= -8 \\
4a + 2b + c &= -4 \\
c &= -4a - 2b - 4 \quad (1)
\end{align*}
\]

- Next, we know that \( 40 < f(4) < 50 \). We will use the notes provided concerning the solving of inequalities.

\[
\begin{align*}
40 &< f(4) < 50 \\
40 &< a(4)^3 + b(4)^2 + c(4) + 5 < 50 \\
40 &< 64a + 16b + 4c + 5 < 50 \\
\text{Substitute for } c \text{ from (1)} &< 64a + 16b + 4(-4a - 2b - 4) + 5 < 50 \\
40 &< 64a + 16b - 16a - 8b - 16 + 5 < 50 \\
40 &< 48a + 8b - 11 < 50 \\
\text{Adding 11 to each part} &< 48a + 8b < 61 \\
\text{Dividing each part by 8} &< 6a + b < 7.625
\end{align*}
\]

Both \( a \) and \( b \) are integers so \( 6a + b \) will also be an integer. The only integer greater than 6.375 and less than 7.625 is 7. Therefore, it follows that \( 6a + b = 7 \). \quad (2)
The last piece of given information is \(240 < f(6) < 250\). We will use the notes provided concerning the solving of inequalities.

\[
240 < \quad f(6) \quad < 250 \\
240 < \quad a(6)^3 + b(6)^2 + c(6) + 5 \quad < 250 \\
240 < \quad 216a + 36b + 6c + 5 \quad < 250
\]

Substitute for \(c\) from (1)

\[
240 < \quad 216a + 36b + 6(-4a - 2b - 4) + 5 \quad < 250 \\
240 < \quad 216a + 36b - 24a - 12b - 24 + 5 \quad < 250 \\
240 < \quad 192a + 24b - 19 \quad < 250
\]

Adding 19 to each part

\[
259 < \quad 192a + 24b \quad < 269
\]

Dividing each part by 24

\[
10\frac{19}{24} < \quad 8a + b \quad < 11\frac{5}{24}
\]

Both \(a\) and \(b\) are integers so \(8a + b\) will also be an integer. The only integer greater than \(10\frac{19}{24}\) and less than \(11\frac{5}{24}\) is 11. Therefore, it follows that \(8a + b = 11\). (3)

Now we have a system of equations:

\[
6a + b = 7 \quad \text{(2)} \\
8a + b = 11 \quad \text{(3)}
\]

By subtracting (2) from (3), we eliminate \(b\) obtaining \(2a = 4\) and \(a = 2\) follows. Substituting \(a = 2\) in (2), we obtain \(12 + b = 7\) and \(b = -5\) follows.

Substituting \(a = 2\) and \(b = -5\) in (1)

\[
c = -4a - 2b - 4 \\
= -4(2) - 2(-5) - 4 \\
= -8 + 10 - 4 \\
= -2
\]

Since \(f(x) = ax^3 + bx^2 + cx + d\) with \(a = 2\), \(b = -5\), \(c = -2\) and \(d = 5\), then the function becomes \(f(x) = 2x^3 - 5x^2 - 2x + 5\).

To find the value of \(f(3)\), we substitute \(x = 3\) into the function.

\[
f(x) = 2x^3 - 5x^2 - 2x + 5 \\
f(3) = 2(3)^3 - 5(3)^2 - 2(3) + 5 \\
= 2(27) - 5(9) - 6 + 5 \\
= 54 - 45 - 6 + 5 \\
= 8
\]

Therefore, \(f(3) = 8\).
Counting, Probability
&
Data Analysis
Problem of the Week

Problem E

Lucky Lucky

A bag contains ten identical balls, each numbered with a different number from the integers 1 to 10. Lucky Pix draws three balls from the bag and holds them in her hand. She wins if the smallest numbered ball in her hand is odd and the next smallest numbered ball in her hand is even.

Determine the probability that Lucky wins.
Problem of the Week
Problem E and Solution
Lucky Lucky

Problem
A bag contains ten identical balls, each numbered with a different number from the integers 1 to 10. Lucky Pix draws three balls from the bag and holds them in her hand. She wins if the smallest numbered ball in her hand is odd and the next smallest numbered ball in her hand is even. Determine the probability that Lucky wins.

Solution
To determine the probability we need to determine two things: the total number of different three-ball selections and the total number of winning selections.

First, we will determine the total number of different three-ball selections. Since each number is distinct, then there are 10 choices for the first ball, 9 choices for the second ball and 8 choices for the third ball. This produces \(10 \times 9 \times 8 = 720\) ordered selections. But this total includes 6 orderings for each possible selection of three numbers. For example, the three numbers 1, 2, and 3, would be included 6 times: (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1). Basically, each three-ball selection is counted six times. Therefore, there are \(\frac{720}{6} = 120\) different possible three-ball selections.

Next, we will determine the number of winning three-ball selections. That is, the number of selections in which the smallest number is odd and the next smallest number is even. The information is presented in chart form.

<table>
<thead>
<tr>
<th>Smallest Number</th>
<th>Second Smallest Number</th>
<th>Possible Value(s) for the Largest Number</th>
<th>Number of Three-ball Selections</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3,4,5,6,7,8,9,10</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5,6,7,8,9,10</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7,8,9,10</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>9,10</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5,6,7,8,9,10</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7,8,9,10</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>9,10</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7,8,9,10</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>9,10</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9,10</td>
<td>2</td>
</tr>
</tbody>
</table>

The total number of winning selections is
\[(8 + 6 + 4 + 2) + (6 + 4 + 2) + (4 + 2) + (2) = 20 + 12 + 6 + 2 = 40.\]

The probability that Lucky wins is \(\frac{40}{120} = \frac{1}{3}\).
A king loves to travel from his castle to his summer house by coach. He orders his coachman to never go straight when he arrives at an intersection and to never travel along the same section of road twice during a trip. The coachman must turn either right or left when he comes to any intersection.

The map shows all of the roads leading from the palace to the summer house. The thick black lines indicate roads. All roads connecting two adjacent intersections are 1 km long.

The coachman wants to take the shortest route. Determine the length of the shortest route and justify why no shorter route is possible.
Problem

A king loves to travel from his castle to his summer house by coach. He orders his coachman to never go straight when he arrives at an intersection and to never travel along the same section of road twice during a trip. The coachman must turn either right or left when he comes to any intersection. The map shows all of the roads leading from the palace to the summer house. The thick black lines indicate roads. All roads connecting two adjacent intersections are 1 km long. The coachman wants to take the shortest route. Determine the length of the shortest route and justify why no shorter route is possible.

Solution

To get from the castle (bottom left corner) to the summer house (top right corner), the coach must pass through at least one of the four numbered roads on the adjacent diagram.

In the solution, we will examine four cases: routes which take us through each of the four numbered roads. A diagram will be presented for each possibility.

On each diagram, the start point will be labeled $C$, the endpoint will be labeled $H$, the south end of the road used in a particular case will be labeled $S$, and the north end will be labeled $N$.

Some details, for the sake of brevity, will be omitted from the solution and left for the solver to consider further. In the solution, turns will be described in terms of north, south, east and west.

1. **What is the length of the shortest path if the route passes through road 1?**

   If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 15 km long.

   In the part of the route from $C$ to $S$, the shortest route is 4 km and goes north - east - north - west. It would also be possible to extend this route by going north - east - south - east - north - west - north - west, but this route is clearly longer. There is no route starting from $C$ that goes east first which is able to legally pass through road 1. In traveling from $N$ to $H$, the shortest route is shown and is 10 km long. At some points along the route from $N$ to $H$, alternate choices can be made but these choices lead to invalid situations or lengthen the route. The solver may wish to confirm this.
2. What is the length of the shortest path if the route passes through road 2?

If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 13 km long.

In the part of the route from $C$ to $S$, the shortest route is 9 km. In traveling from $N$ to $H$, the shortest route is 3 km long. At some points along the route from $C$ to $S$ or $N$ to $H$, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. The solver may wish to confirm this.

3. What is the length of the shortest path if the route passes through road 3?

If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 17 km long.

In the part of the route from $C$ to $S$, the shortest route is 13 km. In traveling from $N$ to $H$, the shortest route is 3 km long. At some points along the route from $C$ to $S$, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. At two intersections along the route from $N$ to $H$, alternate choices can be made but these choices would both lead to invalid situations. The solver may wish to confirm this. The route from $N$ to $H$ is clearly the shortest one possible.

4. What is the length of the shortest path if the route passes through road 4?

If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 15 km long.

In the part of the route from $C$ to $S$, the shortest route is 12 km. In traveling from $N$ to $H$, the shortest route is 2 km long. At some points along the route from $C$ to $S$, alternate choices can be made but these choices either lead to invalid situations, lengthen the route or have the same length. The solver may wish to confirm this. The route from $N$ to $H$ is clearly the shortest one possible.

After examining the four possible cases, the shortest route is 13 km and passes through road 2.
The Fine Arts High School Piano Club has 30 student members, some from Grade 11 and the remainder from Grade 12. Each pair of students from the club must play a piano duet together over the course of the year. When two grade 11 students play together, they need 2 hours of practice time. When a grade 11 and a grade 12 student play together, they need 3 hours of practice time. When two grade 12 students play together, they need 4 hours of practice time. In total, the students need 1392 hours of practice time.

How many of the 30 members of the club are Grade 11 students?
Problem of the Week
Problem E and Solution
Keys to Success

Problem

The Fine Arts High School Piano Club has 30 student members, some from Grade 11 and the remainder from Grade 12. Each pair of students from the club must play a piano duet together over the course of the year. When two grade 11 students play together, they need 2 hours of practice time. When a grade 11 and a grade 12 student play together, they need 3 hours of practice time. When two grade 12 students play together, they need 4 hours of practice time. In total, the students need 1392 hours of practice time. How many of the 30 members of the club are Grade 11 students?

Solution

If 3 students \( \{A, B, C\} \) are in the same grade, then there will be \( 3 \times 2 \div 2 = 3 \) duet pairings, namely \( \{AB\}, \{AC\}, \) and \( \{BC\} \). (If we look at this using a counting argument, there would be 3 choices for the first student and for each of these choices, there would be 2 choices for the second student, a total of \( 3 \times 2 = 6 \) pairings, namely \( \{AB\}, \{AC\}, \{BA\}, \{BC\}, \{CA\}, \) and \( \{CB\} \). Each pairing appears twice. Since order is not important we must divide by 2, getting us 3 possible pairings.)

If 3 students \( \{A, B, C\} \) are in one grade and 2 students \( \{D, E\} \) are in the other grade, then there will be \( 3 \times 2 = 6 \) duet pairings, namely \( \{AD\}, \{AE\}, \{BD\}, \{BE\}, \{CD\}, \) and \( \{CE\} \). (There are 3 choices for the grade 11 student in the pairing and for each of these choices, there are 2 possibilities for the choice of the grade 12 student. This gives a total of \( 3 \times 2 = 6 \) pairings.

This specific argument will now be applied to the general case.

Now, let \( a \) represent the number of Grade 11 students in the club and \( (30 - a) \) represent the number of Grade 12 students in the club.
In general, since there are $a$ students in Grade 11 and each must play a duet with every other student in Grade 11, there will be $a \times (a - 1) \div 2$ duets involving only Grade 11 students. Similarly, since there are $(30 - a)$ students in Grade 12 and each must play a duet with every other student in Grade 12, there will be $(30 - a) \times (30 - a - 1) \div 2 = (30 - a) \times (29 - a) \div 2$ duets involving only Grade 12 students. Since every Grade 11 student must play a duet with every Grade 12 student, there will be $a \times (30 - a)$ duets involving one student from each grade.

To determine the total amount of practice time required, take the number of students in each type of pairing and multiply by the number of hours of practice time required for each pairing type.

\[
\text{Total Time} = \text{Time for Grade 11 Pairs} + \text{Time for Grade 12 Pairs} + \text{Time for Grade 11/12 Pairs}
\]

\[
1392 = 2 \left[ a \times \frac{(a - 1)}{2} \right] + 4 \left[ \frac{(30 - a) \times (29 - a)}{2} \right] + 3[a \times (30 - a)]
\]

\[
1392 = a^2 - a + 2(870 - 59a + a^2) + 3(30a - a^2)
\]

\[
1392 = a^2 - a + 1740 - 118a + 2a^2 + 90a - 3a^2
\]

\[
1392 = -29a + 1740
\]

\[
29a = 348
\]

\[
a = 12
\]

Therefore, 12 of the students in the club are in Grade 11.
Problem of the Week

Problem E

Faster Adder Required

One day Matt was challenged to find the sum of all the three-digit numbers that could be made by choosing three different digits from the list \(\{1, 2, 3, 4, 5, 6, 7\}\). Unsure of how to proceed, Matt started adding the numbers:

\[
\begin{align*}
1 & \ 2 & \ 3 \\
+ & \ 1 & \ 2 & \ 4 \\
\hline
2 & \ 4 & \ 7 \\
+ & \ 1 & \ 2 & \ 5 \\
\hline
3 & \ 7 & \ 2 \\
+ & \ 1 & \ 2 & \ 6 \\
\hline
4 & \ 9 & \ 8
\end{align*}
\]

After finding the sum of just the first four possible numbers, Matt concluded that there had to be a better way.

Using a “better way”, determine the sum of all the three-digit numbers that can be made by choosing three different numbers from the list \(\{1, 2, 3, 4, 5, 6, 7\}\).
Problem of the Week
Problem E and Solution
Faster Adder Required

Problem
Determine the sum of all the three-digit numbers that can be made by choosing three different numbers from the list \{1, 2, 3, 4, 5, 6, 7\}.

Solution
We need to first determine how many possible three-digit numbers can be formed using three different digits from the list \{1, 2, 3, 4, 5, 6, 7\}. There are 7 choices for the first digit. For each of these choices, there are 6 choices for the second digit giving a total of \(7 \times 6 = 42\) choices for the first two digits. For each of these 42 choices for the first two digits, there are 5 choices for the third digit giving a total of \(42 \times 5 = 210\) different three-digit numbers.

Each of the numbers 1 to 7 has an equal chance of appearing in each of the hundreds, tens and ones positions. Therefore, each digit appears \(210 \div 7 = 30\) times in each place value position.

The sum of the digits in the units position is
\[
30(1) + 30(2) + 30(3) + 30(4) + 30(5) + 30(6) + 30(7) \\
= 30(1 + 2 + 3 + 4 + 5 + 6 + 7) \\
= 30(28) \\
= 840
\]
The units digit of the sum is 0 and 84 is carried to the tens digit column.

The same digits appear in the tens digit column of the sum and again 30 times each. So the sum of the tens digit column is 924 which is the sum of the digits in the column plus 84 carried from the units digit column. The tens digit of the sum is 4 and 92 is carried to the hundreds digit column.

The same digits appear in the hundreds digit column of the sum and again 30 times each. So the sum of the hundreds digit column is 932 which is the sum of the digits in the column plus 92 carried from the tens digit column. The required sum is therefore 93 240.

The same sum would be obtained by adding 111, 222, 333, 444, 555, 666 and 777, and multiplying the sum by 30.
\[
30(111 + 222 + 333 + 444 + 555 + 666 + 777) \\
= 30 \times 111 \times (1 + 2 + 3 + 4 + 5 + 6 + 7) \\
= 3330(28) \\
= 93 240
\]
It is left to the solver to reason this out.
Problem of the Week
Problem E
The Chances of This?

As part of their annual training, Santa’s eight reindeer, Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Donner, Blitzen, participate in reindeer games. Rudolph was not allowed to play in any reindeer games.

In one of the games, the reindeer must arrange themselves in a line. There are a few rules that the reindeer must follow. First, reindeer with the letter \( r \) in their name cannot stand together in adjacent positions in the line. Second, since Donner and Blitzen look so much alike, they can never stand in adjacent positions in the line. The diagram below illustrates one possible valid arrangement of the reindeer.

The reindeer randomly organize themselves in the line. What is the probability that they arrange themselves in a correct order?
Problem of the Week
Problem E and Solution
The Chances of This?

Problem
As part of their annual training, Santa’s eight reindeer, Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Donner, Blitzen, participate in reindeer games. Rudolph was not allowed to play in any reindeer games. In one of the games, the reindeer must arrange themselves in a line. There are a few rules that the reindeer must follow. First, reindeer with the letter r in their name cannot stand together in adjacent positions in the line. Second, since Donner and Blitzen look so much alike, they can never stand in adjacent positions in the line. The reindeer randomly organize themselves in the line. What is the probability that they arrange themselves in a correct order?

Solution
Diagrams will be presented before an explanation of each case. Let Donner’s position in the line be marked with a D. Let the the other reindeer with an r in their name have their positions marked with an R. If a position has a number in it, that will represent the number of ways that position can be filled (by reindeer without r in their name, nor are they Blitzen).

Four of the eight reindeer have an r in their name. The problem is complicated by the fact that Donner and Blitzen cannot stand together. We will break the problem into two cases.

1. Donner is in position 1 or position 8.

\[
\begin{array}{cccccc}
D & 3 & R & _ & R & _ \\
D & 3 & R & _ & R & _ \\
D & 3 & R & _ & R & _ \\
D & 3 & R & _ & R & _ \\
_ & R & _ & R & _ & R \\
R & _ & R & _ & R & _ \\
R & _ & R & _ & R & _ \\
R & _ & R & _ & R & _ \\
\end{array}
\]

There are 8 configurations (shown above) with Donner in the first or last spot. For each of these, there are 3 possible ways for the reindeer to fill in the empty spot immediately adjacent to Donner (these reindeer do not have an r in their name, nor are they Blitzen). For each of these, there are \((3 \times 2 \times 1)\) ways to place the other reindeer with r in their name and \((3 \times 2 \times 1)\) ways to place the remaining reindeer, including Blitzen.

There are \(8 \times 3 \times (3 \times 2 \times 1) \times (3 \times 2 \times 1) = 864\) ways to place the reindeer properly with Donner in the first or last spot.
2. Donner is in positions 2 through 7.

\[
\begin{align*}
3 & \quad D & 2 & \quad R & \_ & \quad R & \_ & \quad R \\
R & \quad 3 & \quad D & 2 & \_ & \quad R & \_ & \quad R \\
R & \quad 3 & \quad D & 2 & R & \_ & \quad R & \_ \\
R & \quad 3 & \quad D & 2 & R & \_ & \_ & \quad R \\
\_ & \quad R & \quad 3 & \quad D & 2 & \_ & \quad R & \_ \\
R & \_ & \_ & \quad R & \_ & \quad 3 & \quad D & 2 & R \\
R & \_ & \_ & \_ & \quad R & \quad 3 & \quad D & 2 & R \\
R & \_ & \_ & \_ & \quad R & \_ & \quad 2 & \quad D & 3
\end{align*}
\]

There are 12 configurations (shown above) with Donner in one of the spots from position 2 to position 7. For each of these, there are 3 possible ways for the reindeer to fill in the first empty spot immediately adjacent to Donner and two ways to fill the other empty spot immediately adjacent to Donner (these reindeer do not have an \(r\) in their name, nor are they Blitzen). For each of these, there are \((3 \times 2 \times 1)\) ways to place the reindeer with \(r\) in their name and \((2 \times 1)\) ways to place the remaining reindeer, including Blitzen.

There are \(12 \times (3 \times 2) \times (3 \times 2 \times 1) \times (2 \times 1) = 864\) ways to place the reindeer properly with Donner in one of the spots 2 through 7.

The cases have no overlapping possibilities and we have considered all of the possible placements of Donner. Therefore, there are \(864 + 864 = 1728\) ways for the reindeer to line up correctly.

If the reindeer could stand in any position in the line, the number of possible ways to line up is

\[8 \times 7 \times 6 \times \cdots \times 3 \times 2 \times 1 = 40\,320.\]

The probability of the reindeer randomly lining up in a correct formation is

\[
\frac{1728}{40\,320} = \frac{3}{70} \approx 0.043. \text{ There is a 4.3\% chance of the reindeer lining up correctly.}
\]

The chances are extremely slight that the reindeer will line up randomly and be ordered correctly.
Problem of the Week

Problem E

E Z Does It Again

E Z Dealer has a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is yellow and the other side of the card is red.

E Z places all the cards, red side up, on the table. He first turns over every card that has a number which is a multiple of 2. He then examines all the cards, and turns over every card that has a number which is a multiple of 3. He again examines all the cards, and turns over every card that has a number which is a multiple of 4. Finally, he examines all the cards and turns over every card that has a number which is a multiple of 5.

After E Z has finished, how many cards have the red side facing up?
Problem of the Week

Problem E and Solution

E Z Does It Again

Problem

E Z Dealer has a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is yellow and the other side of the card is red. E Z places all the cards, red side up, on the table. He first turns over every card that has a number which is a multiple of 2. He then examines all the cards, and turns over every card that has a number which is a multiple of 3. He again examines all the cards, and turns over every card that has a number which is a multiple of 4. Finally, he examines all the cards and turns over every card that has a number which is a multiple of 5. After E Z has finished, how many cards have the red side facing up?

Solution

If a card number is a multiple of 2, 3, 4 and 5, it will be flipped four times. This card will go from red to yellow to red to yellow to red again. So the card will still be red once E Z has finished.

If a card number is a multiple of exactly three of 2, 3, 4 and 5, it will be flipped three times. This card will go from red to yellow to red to yellow. So the card will be yellow once E Z has finished.

If a card number is a multiple of exactly two of 2, 3, 4 and 5, then it will be flipped twice. This card will go from red to yellow to red again. So the card will still be red once E Z has finished.

If a card number is a multiple of exactly one of 2, 3, 4 and 5, it will be flipped once. This card will go from red to yellow. So the card will be yellow once E Z has finished.

If a card number is a multiple of none of 2, 3, 4 and 5, then this card will not be flipped and so the card will still be red once E Z has finished.

To determine how many cards have the red side facing up once E Z has finished, let’s determine how many cards have the yellow side facing up once E Z has finished. To do so, we need to determine how many card numbers are multiples of exactly three of 2, 3, 4 and 5 and how many cards are multiples of exactly one of 2, 3, 4 and 5.

Let’s consider the cases:

- A card number is a multiple of 2, 3 and 4, but not 5
  If a card number is a multiple of 2, 3 and 4, then it must be a multiple of 12, the lowest common multiple of 2, 3 and 4. So we are want card numbers that are multiples of 12 but not 5. If a card number is a multiple of 12 and 5, then it is a multiple of $12 \times 5 = 60$. So we want all multiples of 12 that are not multiples of 60. There are 8 multiples of 12 from 1 to 100, but one is 60. So there are $8 - 1 = 7$ numbers that are multiples of 2, 3 and 4, but not 5.

- A card number is a multiple of 2, 3 and 5, but not 4
  If a card number is a multiple of 2, 3 and 5, then it must be a multiple of 30, the lowest common multiple of 2, 3 and 5. So we want all multiples of 30 that are not multiples of 4. There are 3 multiples of 30 from 1 to 100, but one is 60, which is also a multiple of 4. So there are 2 numbers from 1 to 100 that are multiples of 2, 3 and 5, but not 4.
A card number is a multiple of 2, 4 and 5, but not 3
If a card number is a multiple of 2, 4 and 5, then it must be a multiple of 20, the lowest common multiple of 2, 4 and 5. So we want all multiples of 20 that are not multiples of 3. There are 5 multiples of 20 from 1 to 100, but one is 60, which is a multiple of 3. So there are 4 numbers from 1 to 100 that are multiples of 2, 4 and 5, but not 3.

A card number is a multiple of 3, 4 and 5, but not 2
It is not possible for a card number to be a multiple of 4 but not 2. So there are no card numbers in this case.

A card number is a multiple of 2 but not 3, 4, or 5
There are 50 numbers from 1 to 100 which are multiples of 2 and 25 numbers from 1 to 100 which are multiples of 4 (and thus 2). So there are $50 - 25 = 25$ numbers from 1 to 100 multiples of by 2 but not 4. These are
\[ \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98\}. \]

We still need to remove numbers that are still multiples of 3 or 5. After doing so we are left with
\[ \{2, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 98\}. \]

So there are 14 numbers from 1 to 100 that are multiples of 2 but not 3, 4 or 5.

A card number is a multiple of 3 but not 2, 4, or 5
There are 33 multiples of 3 from 1 to 100,
\[ \{3, 6, 9, 12, 15, \ldots, 87, 90, 93, 96, 99\}. \]

In this group of multiples, there are 17 numbers that are odd. So there are 17 numbers from 1 to 100 that are multiples of 3 but not 2. These numbers are
\[ \{3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99\}. \]

We still need to remove numbers that are multiples of 5. After doing so we are left with
\[ \{3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99\}. \]

So there are 14 numbers from 1 to 100 that are multiples of 3 but not 2, 4 or 5.

A card number is a multiple of 4 but not 2, 3, or 5
It is not possible for a card number to be a multiple of 4 but not 2. So there are no card numbers in this case.

A card number is a multiple of 5 but not 2, 3, or 4
There are 20 multiples of 5 from 1 to 100, but half of those are multiples of 2. The multiples of 5 which are not multiples of 2 are
\[ \{5, 15, 25, 35, 45, 55, 65, 75, 85, 95\}. \]

We still need to remove numbers that are multiples of 3. After doing so we are left with
\[ \{5, 25, 35, 55, 65, 85, 95\}. \]

So there are 7 numbers from 1 to 100 that are multiples of 5 but not 2, 3 or 4.

Therefore, once he has finished, E Z Dealer is left with $100 - (7 + 2 + 4 + 14 + 14 + 7) = 100 - 48 = 52$ cards with the red side facing up.

**Extension:** Suppose E Z Dealer continues flipping cards in this manner. So, after he has flipped all cards whose number is a multiple of 5, he then flips all cards whose card number is a multiple of 6, then 7, then 8, and so on until he flips all cards whose number is a multiple of 100. Once E Z has finished, how many cards will have the red side facing up?
Problem of the Week
Problem E
The Truth of the Matter

Four people, Andy, Barb, Carl and Dana, each said two statements such that:
• one person lied in both statements;
• one person told the truth in both statements; and
• two people told the truth in one statement and a lie in the other statement.

Andy said, “Barb lied once” and “Dana lied twice.”
Barb said, “I never lie” and “Andy never lied.”
Carl said, “Dana lied twice” and “Barb never lied.”
Dana said, “Andy lied twice” and “I never lie.”

Who lied twice? Who never lied? Who lied exactly once?
Problem

Four people, Andy, Barb, Carl and Dana, each said two statements such that:
- one person lied in both statements;
- one person told the truth in both statements; and
- two people told the truth in one statement and a lie in the other statement.

Andy said, “Barb lied once” and “Dana lied twice.” Barb said, “I never lie” and “Andy never lied.” Carl said, “Dana lied twice” and “Barb never lied.” Dana said, “Andy lied twice” and “I never lie.”

Who lied twice? Who never lied? Who lied exactly once?

Solution

1. Who lied twice?
   a) Assume that Andy lied twice. If so, “Barb lied once” is a lie. Therefore, “Barb never lied” or “Barb lied twice.” But “Barb never lied” cannot be true because Barb says, “Andy never lied.” This contradicts our assumption that Andy lied twice. “Barb lied twice” cannot be true since that means Barb and Andy both lied twice and this contradicts the fact that only one person lied twice. Therefore, our assumption that Andy lied twice is false.

   b) Assume that Barb lied twice. If so, “Andy never lied” is a lie. Then, Andy lied twice or Andy lied once. Andy cannot have lied twice since both he and Barb would have lied twice and this contradicts the fact that only one person lied twice. But “Andy lied once” is also false since “Barb lied once” is a lie (we assumed she lied twice) and “Dana lied twice” is a lie because it contradicts the assumption that Barb lied twice (and only one person can lie twice). Therefore, our assumption that Barb lied twice is false.

   c) Assume that Carl lied twice. If so, “Barb never lied” is a lie and “Dana lied twice” is a lie. Since “Barb never lied” is a lie, then she lied twice or she lied once. But if Barb lied twice our assumption that Carl lied twice cannot be true since only one person lied twice.

   If Barb lied once, then “I never lie” must be the lie and “Andy never lied” must be true. But if Andy never lied, then “Dana lied twice” must be true and this contradicts the fact that only one person can lie twice. Therefore, our assumption that Carl lied twice is false.

   We are told that one person lied twice and none of Andy, Barb or Carl lied twice. Therefore, by elimination, Dana is the one who lied twice.
2. Who never lied?

   a) Assume that Barb never lied. Then her statement that “Andy never lied” must be true. There are then two people who never lied. This contradicts the fact that only one person never lied. Therefore, our assumption that Barb never lied is false.

   b) Assume that Carl never lied. Then his statement that “Barb never lied” must be true. There are then two people who never lied. This contradicts the fact that only one person never lied. Therefore, our assumption that Carl never lied is false.

   Dana lied twice. Barb and Carl lied. Therefore, by elimination, Andy is the one who never lied. It then follows that Barb and Carl each make one true statement and tell one lie.

   We can now check our results.

   Andy never lied. Then his statements are both true. Barb lied once is true and Dana lied twice is true.

   Barb lied once. Then one of her statements is true and the other is a lie. Her statement that she never lies is a lie and her statement that Andy never lies is true.

   Carl lied once. Then one of his statements is true and the other is a lie. His statement that Dana lied twice is true and his statement that Barb never lied is a lie.

   Dana lied twice. Then both of her statements are lies. Andy lied twice is a lie. And her statement the she never lies is a lie.

   TRUE OR FALSE
Each of the numbers 1, 2, 3, 4, 5, 6 occurs, one to a face, on the faces of a cube. Three people, Bel, Cal and Dan, are seated around a rectangular table. Bel is seated on one side of the table. Cal is seated on the side of the table which is adjacent to Bel and to her right. Dan is seated on the side of the table which is adjacent to Cal and to his right. There is an empty seat along the side which is adjacent to both Bel and Dan.

The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces.

When Bel adds the three numbers that she can see, her total is 9. When Cal adds the three numbers that he can see, his total is 14. When Dan adds the three numbers that he can see, his total is 15.

Determine the number on the bottom face of the cube.
Problem of the Week
Problem E and Solution
A Different Point of View

Problem
Each of the numbers 1, 2, 3, 4, 5, 6 occurs, one to a face, on the faces of a cube. Three people, Bel, Cal and Dan, are seated around a rectangular table. Bel is seated on one side of the table. Cal is seated on the side of the table which is adjacent to Bel and to her right. Dan is seated on the side of the table which is adjacent to Cal and to his right. There is an empty seat along the side which is adjacent to both Bel and Dan. The cube is placed on the table so that from their different seat locations, each one can see the top face and two adjacent side faces. When Bel adds the three numbers that she can see, her total is 9. When Cal adds the three numbers that he can see, his total is 14. When Dan adds the three numbers that he can see, his total is 15. Determine the number on the bottom face of the cube.

Solution

Let $a$ represent the number on the top face of the cube. Let $a$, $b$ and $c$ represent the three numbers that Bel sees. Let $a$, $c$ and $d$ represent the three numbers that Cal sees. Let $a$, $d$ and $e$ represent the three numbers that Dan sees.

If there were a person in the fourth seat, that person would see $a$, $e$ and $b$.

From the given information, we are now able to form three equations:

\[
\begin{align*}
    a + b + c &= 9 \quad (1) \\
    a + c + d &= 14 \quad (2) \\
    a + d + e &= 15 \quad (3)
\end{align*}
\]

Comparing equation (1) and equation (2), $b$ has been replaced by $d$ and the sum has increased by 5. Therefore, $b$ and $d$ differ by 5 and $b < d$. The only numbers from the set 1, 2, 3, 4, 5, 6 that differ by 5 are 1 and 6. Therefore, $b = 1$ and $d = 6$.

Comparing equation (2) and equation (3), $c$ has been replaced by $e$ and the sum has increased by 1. Therefore, $c$ and $e$ differ by 1 and $c < e$. Since $b = 1$ and $d = 6$, there are only three possible combinations for $c$ and $e$, namely $c = 2$ and $e = 3$, or $c = 3$ and $e = 4$, or $c = 4$ and $e = 5$.

We will check each of these possibilities. First, if $c = 2$, $e = 3$, $b = 1$ and $d = 6$, we can substitute the appropriate values in (1) giving $a + 1 + 2 = 9$ or $a = 6$. This is not possible since $d$ would also equal 6. We can rule this case out.
Next, if $c = 3$, $e = 4$, $b = 1$ and $d = 6$, we can substitute the appropriate values in (1) giving $a + 1 + 3 = 9$ or $a = 5$. This is a possible solution. In (2), $a + c + d = 5 + 3 + 6 = 14$ as required. And in (3), $a + d + e = 5 + 6 + 4 = 15$ as required. The only number not used is 2 so the number on the bottom face is 2. But is this the only solution?

Finally, if $c = 4$, $e = 5$, $b = 1$ and $d = 6$, we can substitute the appropriate values in (1) giving $a + 1 + 4 = 9$ or $a = 4$. This is not possible since $c$ would also equal 4. We can rule this case out.

Since we have examined all possible cases, the only possible number on the bottom (unseen) face is 2.

Notes:

It is also possible to play with the numbers to solve this problem. The method presented above could be used in a similar way with any list of six different numbers. “Playing” with the numbers might not be as easy.

Instead of “arguing” the difference between equations to obtain the relationship between $b$ and $d$, and $c$ and $e$, we could have used elimination.

\[
\begin{align*}
  a + b + c & = 9 \quad (1) \\
  a + c + d & = 14 \quad (2) \\
  a + d + e & = 15 \quad (3)
\end{align*}
\]

For example, equation (1) subtract equation (2) gives $b - d = -5$ which can be written $d - b = 5$. This is the same as saying the difference between $b$ and $d$ is 5.

Similarly, equation (2) subtract equation (3) gives $c - e = -1$ which can be written $e - c = 1$. This is the same as saying that the difference between $c$ and $e$ is 1.
Problem of the Week
Problem E
Search and Swap

Randi has a deck consisting of 10 cards. One side of each card is red and the other side of each card has one of the letters A, B, C, D, E, F, G, H, I, or J on it. Each letter occurs exactly once. The cards are shuffled and placed letter-side down on a table from left to right.

Every time Randi looks for a letter, she turns over cards one by one starting with the leftmost card and moving to the right. If a card does not have the letter she is looking for on it, Randi puts it back letter-side down in the same location and continues with the next card. If a card does have the letter she is looking for on it, Randi swaps the locations of this card and the card on its immediate left placing both cards letter-side down. One exception is when she finds the letter she is looking for on the leftmost card. In this case, Randi puts the card back letter-side down in the same location and no swap occurs. Either way, once Randi finds the letter she is looking for, she does not look at any more cards. Also, Randi never remembers the locations of any cards on the table.

For example, suppose Randi is asked to find the letter E and the cards were on the table as shown below.

```
G   B   F   J   A   I   E   H   C   D
```

Randi would look at each of the first 6 cards and return each of them, letter-side down, to the same place on the table. When she looked at the seventh card and found that it was the E, she would swap the location of the E and the I. So to locate the letter E, Randi looked at 7 cards and the resulting card ordering would be as follows.

```
G   B   F   J   A   E   I   H   C   D
```

To find another card, Randi must begin her search with the leftmost card. For example, if next she wanted to search for the F, she would look at the G and B and not change their locations. She would then look at the third card, see the F, and swap the locations of the third and second card. The resulting card ordering would be as follows.

```
G   F   B   J   A   E   I   H   C   D
```

After searching for the E and the F, Randi has looked at a total of $7 + 3 = 10$ cards.

If the ten cards begin in some unknown order and Randi searches for each of the ten letters exactly once, what is the maximum possible number of cards that Randi looks at?
Problem

Randi has a deck consisting of 10 cards. One side of each card is red and the other side of each card has one of the letters A, B, C, D, E, F, G, H, I, or J on it. Each letter occurs exactly once. The cards are shuffled and placed letter-side down on a table from left to right. Every time Randi looks for a letter, she turns over cards one by one starting with the leftmost card and moving to the right. If a card does not have the letter she is looking for on it, Randi puts it back letter-side down in the same location and continues with the next card. If a card does have the letter she is looking for on it, Randi swaps the locations of this card and the card on its immediate left placing both cards letter-side down. One exception is when she finds the letter she is looking for on the leftmost card. In this case, Randi puts the card back letter-side down in the same location and no swap occurs. Either way, once Randi finds the letter she is looking for, she does not look at any more cards. Also, Randi never remembers the locations of any cards on the table.

If the ten cards begin in some unknown order and Randi searches for each of the ten letters exactly once, what is the maximum possible number of cards that Randi looks at?

Solution

If no swaps were required as a result of finding a card, how many cards would Randi have to look at in total?

At some point she is looking for the card in position 1. She would have to look at 1 card to find it. At some point she is looking for the card in position 2. She would have to look at 2 cards to find it. At some point she is looking for the card in position 3. She would have to look at 3 cards to find it. This continues until at some point she is looking for the card in position 10. She would have to look at 10 cards to find it. To locate all 10 cards, Randi would have to look at $1 + 2 + 3 + \cdots + 10 = 55$ cards.

Since Randi only looks for each letter exactly once, swapping the position of one letter with the position of another letter can only have the effect of increasing the number of cards looked at for the letter on the preceding card by one. The number of cards looked at to find other letters would not be affected. Therefore, swapping can only increase the number of cards looked at (by one) for all but the first search. These means swapping can increase the number of cards looked at by at most 9 in total making the maximum total number of cards looked at equal to $55 + 9 = 64$.

On the next page, an illustration of how this maximum can be achieved is illustrated.
Is 64 an achievable maximum?

Put the cards in order, left to right, from A to J.

\[
\begin{array}{ccccccccccc}
A & B & C & D & E & F & G & H & I & J \\
\end{array}
\]

Now search for each letter in order from B to J and search for A last.

Since B is in the second position, we must look at 2 cards to find it. We then swap A and B.

\[
\begin{array}{ccccccccccc}
B & A & C & D & E & F & G & H & I & J \\
\end{array}
\]

Since C is in the third position, we must look at 3 cards to find it. We then swap A and C.

\[
\begin{array}{ccccccccccc}
B & C & A & D & E & F & G & H & I & J \\
\end{array}
\]

Since D is in the fourth position, we must look at 4 cards to find it. We then swap A and D.

\[
\begin{array}{ccccccccccc}
B & C & D & A & E & F & G & H & I & J \\
\end{array}
\]

Since E is in the fifth position, we must look at 5 cards to find it. We then swap A and E.

\[
\begin{array}{ccccccccccc}
B & C & D & E & A & F & G & H & I & J \\
\end{array}
\]

Since F is in the sixth position, we must look at 6 cards to find it. We then swap A and F.

\[
\begin{array}{ccccccccccc}
B & C & D & E & F & A & G & H & I & J \\
\end{array}
\]

Since G is in the seventh position, we must look at 7 cards to find it. We then swap A and G.

\[
\begin{array}{ccccccccccc}
B & C & D & E & F & G & A & H & I & J \\
\end{array}
\]

Since H is in the eighth position, we must look at 8 cards to find it. We then swap A and H.

\[
\begin{array}{ccccccccccc}
B & C & D & E & F & G & H & A & I & J \\
\end{array}
\]

Since I is in the ninth position, we must look at 9 cards to find it. We then swap A and I.

\[
\begin{array}{ccccccccccc}
B & C & D & E & F & G & H & I & A & J \\
\end{array}
\]

Since J is in the tenth position, we must look at 10 cards to find it. We then swap A and J.

\[
\begin{array}{ccccccccccc}
B & C & D & E & F & G & H & I & J & A \\
\end{array}
\]

Finally, since A is in the tenth position, we must look at 10 cards to find it. We then swap A and J (again).

\[
\begin{array}{ccccccccccc}
B & C & D & E & F & G & H & I & A & J \\
\end{array}
\]

We have looked at a total of \(2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 = 64\) cards to locate each of the cards.

On the next page, we look at a possible extension and a connection to Computer Science.
Extension:
Suppose you have \( n \) cards, each with something different on them. You lay the cards out in a similar manner to how we handled the 10 different cards. You search for each of the different cards, one at a time. What is the maximum number of cards you must look at in order to locate all of the cards using the search described in the problem?

Connection to Computer Science:

One of the fundamental problems in computer science is how to organize data in order to search within it quickly. There are many ways to do this: using binary trees, splay trees, skip lists, sorted arrays, etc. The technique outlined in this problem is the idea of moving found items closer to the “front,” with the assumption that if we search for something once, it is quite likely that the same item will be searched for again. The transpose (swap) heuristic used by Randi in this problem is one technique for doing this. Other heuristics include move-to-front, which moves a found element to the very front of the list. Moreover, this problem highlights the process of performing worst-case analysis for an algorithm. Computer scientists care about “what is the worst possible input for this algorithm, and how long will it take to execute on that input?” In this question, we are asking about the worst-case performance of the transpose heuristic on a list of size 10.
Functions
(includes Trigonometry)
In the diagram, $MON$ is a sector of a circle with radius $ON$ which is 6 cm long. If $\angle MON = 60^\circ$, determine the radius of the circle which passes through the points $M$, $N$, and $O$. 
Problem of the Week
Problem E and Solution
Circle This

Problem

MON is a sector of a circle with radius ON which is 6 cm long. If \( \angle MON = 60^\circ \), determine the radius of the circle which passes through the points \( M, N, \) and \( O \).

Solution

Let \( C \) be the centre of the circle that passes through \( M, N, \) and \( O \). Then \( CM, CN, \) and \( CO \) are radii. Therefore, \( CM = CN = CO = r \).

In \( \triangle CMO \) and \( \triangle CNO \), \( CM = CN, CO \) is common and \( OM = ON \). Therefore, \( \triangle CMO \cong \triangle CNO \) and it follows that \( \angle COM = \angle CON \). But \( \angle MON = 60^\circ \). Therefore, \( \angle COM = \angle CON = 30^\circ \).

In \( \triangle CMO \), \( CM = CO = r \) and \( \triangle CMO \) is isosceles. Therefore, \( \angle CMO = \angle COM = 30^\circ \) and \( \angle MCO = 180^\circ - 30^\circ - 30^\circ = 120^\circ \).

Method 1: Using the sine law,

\[
\frac{CM}{\sin (\angle COM)} = \frac{OM}{\sin (\angle MCO)}
\]

\[
r = \frac{6}{\sin 120^\circ} \times \sin 30^\circ
\]

\[
r = \frac{6}{\frac{\sqrt{3}}{2}} \times \frac{1}{2}
\]

\[
r = 6 \times \frac{2}{\sqrt{3}} \times \frac{1}{2}
\]

\[
r = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}
\]

\[
r = 2\sqrt{3} \text{ cm}
\]

The radius of the circle that passes through \( M, N, \) and \( O \) is \( 2\sqrt{3} \) cm.
Method 2: Using the cosine law,

\[ CM^2 = CO^2 + MO^2 - 2 \times CO \times MO \times \cos(\angle COM) \]

\[ r^2 = r^2 + 6^2 - 2(6)(r) \cos 30^\circ \]

\[ 12r \cos 30^\circ = 36 \]

\[ r \cos 30^\circ = 3 \]

\[ r \times \frac{\sqrt{3}}{2} = 3 \]

\[ r \times \sqrt{3} = 6 \]

\[ r \times \sqrt{3} \times \sqrt{3} = 6 \times \sqrt{3} \]

\[ 3r = 6\sqrt{3} \]

\[ r = 2\sqrt{3} \text{ cm} \]

The radius of the circle that passes through \( M, N, \) and \( O \) is \( 2\sqrt{3} \text{ cm} \).
Problem of the Week
Problem E
How Low Will It Go?

Suppose \( y = 5x^2 + ax + b, \ a \neq b \), is a parabola that passes through the points \( A(a, b) \) and \( B(b, a) \).

Determine the minimum value of the parabola.
Problem of the Week
Problem E and Solution
How Low Will It Go?

Problem
Suppose \( y = 5x^2 + ax + b, \ a \neq b, \) is a parabola that passes through the points \( A(a, b) \) and \( B(b, a) \). Determine the minimum value of the parabola.

Solution
Since \( A(a, b) \) is on the parabola, it satisfies the equation of the parabola. We can substitute \( x = a \) and \( y = b \) into the equation \( y = 5x^2 + ax + b \).

\[
\begin{align*}
  b &= 5a^2 + a^2 + b \\
  b &= 6a^2 + b \\
  0 &= 6a^2 \\
  0 &= a^2 \\
  0 &= a
\end{align*}
\]

The equation becomes \( y = 5x^2 + 0x + b \) or simply \( y = 5x^2 + b \).

Since \( B(b, a) \) is on the parabola, it satisfies the equation of the parabola. We can substitute \( x = b \) and \( y = a = 0 \) into the equation \( y = 5x^2 + b \).

\[
\begin{align*}
  0 &= 5b^2 + b \\
  0 &= b(5b + 1) \\
  b = 0 \text{ or } 5b + 1 &= 0 \\
  b &= -\frac{1}{5}
\end{align*}
\]

Since \( a \neq b \) and \( a = 0 \), then \( b = 0 \) is inadmissible. Therefore, \( b = -\frac{1}{5} \) and the equation becomes \( y = 5x^2 - \frac{1}{5} \). The vertex of the parabola is \( \left(0, -\frac{1}{5}\right) \) and so the minimum value is \(-\frac{1}{5}\).
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point $D$ on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point $A(9, 8)$, then reaches its peak at $E(7, 9)$, and finally passes through a hoop located at $B(b, 5)$ before returning to the ground at point $C$ and bursting. The ground between $C$ and $D$ is flat. Determine the area of quadrilateral $ABCD$. 
Problem of the Week
Problem E and Solution
Balloons Away!

Problem
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point $D$ on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point $A(9, 8)$, then reaches its peak at $E(7, 9)$, and finally passes through a hoop located at $B(b, 5)$ before returning to the ground at point $C$ and bursting. The ground between $C$ and $D$ is flat. Determine the area of quadrilateral $ABCD$.

Solution
We need to find the equation of the parabola. Then, in order to answer the question, we are required to find the $x$-intercepts of the parabola and the $x$-coordinate of point $B$ on the parabola.

We are given the peak $E(7, 9)$ so we have the vertex of the parabola. Using the vertex form of the equation of a parabola, $y = a(x - h)^2 + k$, with vertex $(h, k) = (7, 9)$, the equation of the parabola looks like $y = a(x - 7)^2 + 9$.

The point $A(9, 8)$ is on the parabola so we can substitute $(x, y) = (9, 8)$ into the equation $y = a(x - 7)^2 + 9$ to find $a$.

\[
8 = a(9 - 7)^2 + 9 \\
8 = a(4) + 9 \\
-1 = 4a \\
\frac{-1}{4} = a
\]

The equation of the parabola is $y = -\frac{1}{4}(x - 7)^2 + 9$.

To find the $x$-coordinate of $B(b, 5)$, substitute $y = 5$ into the equation of the parabola.

\[
5 = -\frac{1}{4}(b - 7)^2 + 9 \\
-4 = -\frac{1}{4}(b - 7)^2 \\
16 = (b - 7)^2 \\
\pm 4 = b - 7
\]

It follows that $b - 7 = -4$ or $b - 7 = 4$. Then $b = 3$ or $b = 11$. The point $B$ is to the left of the vertex so $b < 7$. The coordinates of $B$ are $(3, 5)$. 


To find the \(x\)-intercepts of the parabola, substitute \(y = 0\) into the equation of the parabola.

\[
0 = -\frac{1}{4}(x - 7)^2 + 9 \\
-9 = -\frac{1}{4}(x - 7)^2 \\
36 = (x - 7)^2 \\
\pm 6 = x - 7
\]

It follows that \(x - 7 = -6\) or \(x - 7 = 6\). Then the \(x\)-intercepts of the parabola are 1 and 13. The point \(C\) is to the left of the vertex and the point \(D\) is to the right of the vertex. The coordinates of \(C\) are \((1, 0)\) and \(D\) are \((13, 0)\).

This information has been added to the graph. There are many ways to determine the area of \(ABCD\).

From \(B(3, 5)\) and \(A(9, 8)\) drop perpendiculars to the \(x\)-axis, intersecting at \(F(3, 0)\) and \(G(9, 0)\), respectively. From \(B(3, 5)\) draw a perpendicular to \(AG\), intersecting at \(H(9, 5)\). Draw line segment \(BG\).

We will use the diagram to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

\(BH = 9 - 3 = 6\), \(CG = 9 - 1 = 8\), \(GD = 13 - 9 = 4\), \(BF = 5 - 0 = 5\), and \(AG = 8 - 0 = 8\).

To determine the area, we will find the sum of the areas of \(\triangle CGB\), \(\triangle AGD\) and \(\triangle AGB\).

\[
\text{Area } ABCD = \frac{\text{Area } \triangle CGB}{2} + \frac{\text{Area } \triangle AGD}{2} + \frac{\text{Area } \triangle AGB}{2}
\]

\[
= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2}
\]

\[
= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2}
\]

\[
= 20 + 16 + 24
\]

\[
= 60 \text{ units}^2
\]

The area of \(ABCD\) is 60 \text{ units}^2.
Problem of the Week

Problem E

This is Some Function

For some function $f(x) = ax^3 + bx^2 + cx + d$ where $a$, $b$, $c$ and $d$ are integers, we know the following information:

- the $y$-intercept is 5,
- $f(2) = -3$,
- $40 < f(4) < 50$, and
- $240 < f(6) < 250$.

Determine the value of $f(3)$.

The following notes about solving inequalities may be helpful in solving the above problem. The example, “Solve for $x$ so that $20 < 8x + 5 < 28$”, is used to illustrate steps that can be used in solving inequalities.

- You may add or subtract a constant from each of the parts of the inequality without changing the sense of the inequality. Using the example,

  Subtracting 5 from each part of the inequality leaves the sense of the inequality unchanged.

  $20 < 8x + 5 < 28$

  $15 < 8x < 23$

- You may multiply or divide each of the parts of the inequality by a positive number without changing the sense of the inequality. Continuing from where we left off in the example,

  Dividing each part of the inequality by 8 leaves the sense of the inequality unchanged.

  $15 < 8x < 23$

  $\frac{15}{8} < x < \frac{23}{8}$
Problem of the Week
Problem E and Solution
This is Some Function

Problem

For some function \( f(x) = ax^3 + bx^2 + cx + d \) where \( a, b, c \) and \( d \) are integers, we know the following information: the \( y \)-intercept is 5, \( f(2) = -3 \), \( 40 < f(4) < 50 \), and \( 240 < f(6) < 250 \). Determine the value of \( f(3) \).

Solution

We will process the information in the order that it was provided.

- The \( y \)-intercept is 5 so we know that \( f(0) = 5 \).

\[
f(0) = 5 \\
a(0)^3 + b(0)^2 + c(0) + d = 5 \\
d = 5
\]

The function is now \( f(x) = ax^3 + bx^2 + cx + 5 \).

- We are then given that \( f(2) = -3 \).

\[
f(2) = -3 \\
a(2)^3 + b(2)^2 + c(2) + 5 = -3 \\
8a + 4b + 2c + 5 = -3 \\
8a + 4b + 2c = -8 \\
4a + 2b + c = -4 \\
c = -4a - 2b - 4 \quad (1)
\]

- Next, we know that \( 40 < f(4) < 50 \). We will use the notes provided concerning the solving of inequalities.

\[
40 < f(4) < 50 \\
40 < a(4)^3 + b(4)^2 + c(4) + 5 < 50 \\
40 < 64a + 16b + 4c + 5 < 50 \\
\text{Substitute for } c \text{ from (1)} \\
40 < 64a + 16b + 4(-4a - 2b - 4) + 5 < 50 \\
40 < 64a + 16b - 16a - 8b - 16 + 5 < 50 \\
40 < 48a + 8b - 11 < 50 \\
\text{Adding 11 to each part} \\
51 < 48a + 8b < 61 \\
\text{Dividing each part by 8} \\
6.375 < 6a + b < 7.625
\]

Both \( a \) and \( b \) are integers so \( 6a + b \) will also be an integer. The only integer greater than 6.375 and less than 7.625 is 7. Therefore, it follows that \( 6a + b = 7 \). \quad (2)
The last piece of given information is $240 < f(6) < 250$. We will use the notes provided concerning the solving of inequalities.

$$
240 < f(6) < 250 \\
240 < a(6)^3 + b(6)^2 + c(6) + 5 < 250 \\
240 < 216a + 36b + 6c + 5 < 250
$$

Substitute for $c$ from (1)

$$
240 < 216a + 36b + 6(-4a - 2b - 4) + 5 < 250 \\
240 < 216a + 36b - 24a - 12b - 24 + 5 < 250 \\
240 < 192a + 24b - 19 < 250
$$

Adding 19 to each part

$$
259 < 192a + 24b < 269
$$

Dividing each part by 24

$$
10\frac{19}{24} < 8a + b < 11\frac{5}{24}
$$

Both $a$ and $b$ are integers so $8a + b$ will also be an integer. The only integer greater than $10\frac{19}{24}$ and less than $11\frac{5}{24}$ is 11. Therefore, it follows that $8a + b = 11$. (3)

Now we have a system of equations:

$$
6a + b = 7 \quad \text{(2)} \\
8a + b = 11 \quad \text{(3)}
$$

By subtracting (2) from (3), we eliminate $b$ obtaining $2a = 4$ and $a = 2$ follows. Substituting $a = 2$ in (2), we obtain $12 + b = 7$ and $b = -5$ follows.

Substituting $a = 2$ and $b = -5$ in (1)

$$
c = -4a - 2b - 4 \\
= -4(2) - 2(-5) - 4 \\
= -8 + 10 - 4 \\
= -2
$$

Since $f(x) = ax^3 + bx^2 + cx + d$ with $a = 2$, $b = -5$, $c = -2$ and $d = 5$, then the function becomes $f(x) = 2x^3 - 5x^2 - 2x + 5$.

To find the value of $f(3)$, we substitute $x = 3$ into the function.

$$
f(x) = 2x^3 - 5x^2 - 2x + 5 \\
f(3) = 2(3)^3 - 5(3)^2 - 2(3) + 5 \\
= 2(27) - 5(9) - 6 + 5 \\
= 54 - 45 - 6 + 5 \\
= 8
$$

Therefore, $f(3) = 8$. 
Geometry & Measurement
Problem of the Week
Problem E
Not That Kind of Median

On highways, medians are used to separate opposing lanes of traffic on divided highways. Our problem is not interested in that kind of median.

In triangles, a *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side, separating the triangle into two triangles of equal area.

In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from $A$ meeting $BC$ at $M$ such that $AM = 5$. A second median is drawn from $C$ meeting $AB$ at $N$ such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$. 
Problem of the Week
Problem E and Solution
Not That Kind of Median

Problem
A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side. In \( \triangle ABC \), \( \angle ABC = 90^\circ \). A median is drawn from \( A \) meeting \( BC \) at \( M \) such that \( AM = 5 \). A second median is drawn from \( C \) meeting \( AB \) at \( N \) such that \( CN = 2\sqrt{10} \).

Determine the length of the longest side of \( \triangle ABC \).

Solution
Since \( AM \) is a median, \( M \) is the midpoint of \( BC \). Then \( BM = MC = y \).
Since \( CN \) is a median, \( N \) is the midpoint of \( AB \). Then \( AN = NB = x \).
\( \triangle NBC \) is right angled since \( \angle B = 90^\circ \). Using the Pythagorean Theorem,
\[
NB^2 + BC^2 = CN^2
\]
\[
x^2 + (2y)^2 = (2\sqrt{10})^2
\]
\[
x^2 + 4y^2 = 40 \quad (1)
\]
\( \triangle ABM \) is right angled since \( \angle B = 90^\circ \). Using the Pythagorean Theorem,
\[
AB^2 + BM^2 = AM^2
\]
\[
(2x)^2 + y^2 = 5^2
\]
\[
4x^2 + y^2 = 25 \quad (2)
\]
Adding (1) and (2), \( 5x^2 + 5y^2 = 65 \)
Dividing by 5, \( x^2 + y^2 = 13 \quad (3) \)

The longest side of \( \triangle ABC \) is the hypotenuse \( AC \). Using the Pythagorean Theorem,
\[
AC^2 = AB^2 + BC^2
\]
\[
= (2x)^2 + (2y)^2
\]
\[
= 4x^2 + 4y^2
\]
\[
= 4(x^2 + y^2)
\]
Substituting from (3) above, \( AC^2 = 4(13) \)
Taking the square root, \( AC = 2\sqrt{13} \)

\[ \therefore \] the length of the longest side is \( 2\sqrt{13} \) units.

Note: The solver could actually solve a system of equations to find \( x = 2 \) and \( y = 3 \) and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate thinking about the solution of this problem.
Problem of the Week
Problem E
This Picture Looks Like ...

The shape of the head and ears of a famous mouse are contained in a rectangle.

The two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to the sides of the rectangle. The width of the rectangle is 4 m.

Determine the area of the rectangle not covered by the head and ears of the famous mouse.

For this problem, the following known results about circles may be useful:

- If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.

- If two circles are tangent to each other at point $P$, a line segment through the point of tangency can be drawn connecting the two centres, $C_1$ and $C_2$. 
Problem of the Week
Problem E and Solution
This Picture Looks Like ...

Problem
The shape of the head and ears of a famous mouse appears to be contained in a rectangle. The two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to the sides of the rectangle. The width of the rectangle is 4 m.

Solution
Since the larger circle is tangent to two opposite sides of the rectangle, its diameter is 4 m, the width of the rectangle. It follows that the radius of the larger circle is 2 m.

The two smaller circles have equal radii, are tangent to each other and to opposite sides of the rectangle. It follows that the diameter of each of the smaller circles is half the width of the rectangle, namely 2 m. The radius of each of the smaller circles is 1 m.

Let the centre of the large circle and leftmost small circle be $B$ and $E$ respectively. Let the two small circles be tangent at $C$. Let the leftmost small circle and the larger circle be tangent at $F$. Position line segment $AD$ so that it is parallel to the longer side such that $A$ and $D$ are midpoints of the shorter sides of the rectangle. $AD$ will pass through $C$ and $B$.

Let the length of the rectangle be $d$. This is the same as the distance from $A$ to $D$ on the diagram. We know that $AB = 2$ m, the radius of the larger circle, and $CD = 1$ m, the radius of the smaller circle. We need to find the length of $BC$.

$AD$ is tangent to the smaller circles at $C$. Using the first property, we know that $EC \perp AD$ at $C$. Using the second property, $EFB$ is a straight line segment and $EB = EF + FB = 1 + 2 = 3$ m.
Combining the information, $\triangle ECB$ is right angled at $C$. Using the Pythagorean Theorem, $BC^2 = EB^2 - EC^2 = 3^2 - 1^2 = 8$ and $BC = \sqrt{8}$ follows. Then the length of the rectangle is

$$d = AB + BC + CD = 2 + \sqrt{8} + 1 = 3 + \sqrt{8}.$$

To find the area not covered by the head and ears we need to find the shaded area. To do this we find the area of the rectangle and subtract the area of the large circle and the area of the two equal radii smaller circles.

**Shaded Area**

$$= \text{Area of Rectangle} - \text{Area of Large Circle} - \text{Area of two smaller circles}
= 4 \times (3 + \sqrt{8}) - \pi \times 2^2 - 2 \times (\pi \times 1^2)
= 12 + 4\sqrt{8} - 4\pi - 2\pi
= 12 + 4\sqrt{8} - 6\pi$$

Some students have learned to simplify radicals and know that $\sqrt{8} = \sqrt{4\sqrt{2}} = 2\sqrt{2}$. The shaded area can then be written

$$12 + 4 \times 2\sqrt{2} - 6\pi = 12 + 8\sqrt{2} - 6\pi.$$

The shaded area is $(12 + 4\sqrt{8} - 6\pi)$ m$^2$ or $(12 + 8\sqrt{2} - 6\pi)$ m$^2$ (approximately 4.5 m$^2$).
Problem of the Week
Problem E
Paper Folding 101

The problem today involves simple paper folding. In fact, only one fold is required.

A rectangular piece of paper is 30 cm wide and 40 cm long. The paper has a pattern on one side and is plain on the other. The paper is folded so that the two diagonally opposite corners, $A$ and $C$, coincide. (This is illustrated on the diagram to the right.)

Determine the length of the crease, $FE$, created by the fold.
Problem of the Week
Problem E and Solution
Paper Folding 101

Problem
The problem today involves simple paper folding. In fact, only one fold is required.
A rectangular piece of paper is 30 cm wide and 40 cm long. The paper has a pattern on one side and is plain on the other. The paper is folded so that the two diagonally opposite corners, A and C, coincide. (This is illustrated on the diagram to the right.)

Determine the length of the crease, FE, created by the fold.

Solution
After the fold, C coincides with A and D folds to G. The angle at G is the same as the angle at D. Since ABCD is a rectangle, \( \angle ADC = 90^\circ \) and it follows that \( \angle AGF = 90^\circ \).

Let \( a \) represent the length of BE and \( b \) represent the length of FD.
Then \( EC = CB - BE = 40 - a \) and \( AF = AD - FD = 40 - b \).

The distance from the top of the crease at F to D is the same length as the distance from F to G. It follows that \( FG = FD = b \).

The distance from the bottom of the crease at E to C is the same length as the distance from E to A. It follows that \( AE = EC = 40 - a \).

All of the information is recorded on the following diagram.
Since △ABE and △AGF are both right angled, we can use the Pythagorean Theorem to find \(a\) and \(b\).

\[
BE^2 + AB^2 = AE^2 \\
\quad \quad \quad \quad \quad \quad a^2 + 30^2 = (40 - a)^2 \\
\quad \quad \quad \quad \quad \quad a^2 + 900 = 1600 - 80a + a^2 \\
\quad \quad \quad \quad \quad \quad 80a = 700 \\
\quad \quad \quad \quad \quad \quad a = \frac{35}{4}
\]

\[
FG^2 + AG^2 = AF^2 \\
\quad \quad \quad \quad \quad \quad b^2 + 30^2 = (40 - b)^2 \\
\quad \quad \quad \quad \quad \quad b^2 + 900 = 1600 - 80b + b^2 \\
\quad \quad \quad \quad \quad \quad 80b = 700 \\
\quad \quad \quad \quad \quad \quad b = \frac{35}{4}
\]

\(\therefore a = b = \frac{35}{4}\)

We still need to find the length of the crease.

From \(F\) drop a perpendicular to \(BC\) intersecting at \(H\). \(FHCD\) is a rectangle. It follows that \(FH = DC = 30\) and \(HC = FD = b\).

Also, \(EH = BC - BE - HC = 40 - a - b = 40 - \frac{35}{4} - \frac{35}{4} = \frac{90}{4} = \frac{45}{2}\).

Using the Pythagorean Theorem in △EFH,

\[
EF^2 = FH^2 + EH^2 \\
\quad \quad \quad = 30^2 + \left( \frac{45}{2} \right)^2 \\
\quad \quad \quad = 900 + \frac{2025}{4} \\
\quad \quad \quad = \frac{5625}{4} \\
\quad \quad \quad = \frac{75^2}{2}
\]

\(EF = \frac{75}{2} \quad (EF > 0)\)

The length of the crease is \(\frac{75}{2}\) cm (37.5 cm).
Problem of the Week

Problem E

Circle This

In the diagram, $MON$ is a sector of a circle with radius $ON$ which is 6 cm long. If $\angle MON = 60^\circ$, determine the radius of the circle which passes through the points $M$, $N$, and $O$. 
Problem of the Week
Problem E and Solution
Circle This

Problem

MON is a sector of a circle with radius ON which is 6 cm long. If \( \angle MON = 60^\circ \), determine the radius of the circle which passes through the points M, N, and O.

Solution

Let \( C \) be the centre of the circle that passes through \( M, N, \) and \( O \). Then \( CM, CN, \) and \( CO \) are radii. Therefore, \( CM = CN = CO = r \).

In \( \triangle CMO \) and \( \triangle CNO \), \( CM = CN, CO \) is common and \( OM = ON \). Therefore, \( \triangle CMO \cong \triangle CNO \) and it follows that \( \angle COM = \angle CON \). But \( \angle MON = 60^\circ \). Therefore, \( \angle COM = \angle CON = 30^\circ \).

In \( \triangle CMO \), \( CM = CO = r \) and \( \triangle CMO \) is isosceles. Therefore, \( \angle CMO = \angle COM = 30^\circ \) and \( \angle MCO = 180^\circ - 30^\circ - 30^\circ = 120^\circ \).

Method 1: Using the sine law,

\[
\frac{CM}{\sin(\angle COM)} = \frac{OM}{\sin(\angle MCO)}
\]

\[
\frac{r}{\sin 30^\circ} = \frac{6}{\sin 120^\circ}
\]

\[
r = \frac{6}{\sqrt{3}} \times \frac{1}{2}
\]

\[
r = 6 \times \frac{2}{\sqrt{3}} \times \frac{1}{2}
\]

\[
r = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}
\]

\[
r = 2\sqrt{3} \text{ cm}
\]

The radius of the circle that passes through \( M, N, \) and \( O \) is \( 2\sqrt{3} \) cm.
Method 2: Using the cosine law,

\[ CM^2 = CO^2 + MO^2 - 2 \times CO \times MO \times \cos(\angle COM) \]
\[ r^2 = r^2 + 6^2 - 2(6)(r) \cos 30^\circ \]
\[ 12r \cos 30^\circ = 36 \]
\[ r \cos 30^\circ = 3 \]
\[ r \times \frac{\sqrt{3}}{2} = 3 \]
\[ r \times \sqrt{3} = 6 \]
\[ r \times \sqrt{3} \times \sqrt{3} = 6 \times \sqrt{3} \]
\[ 3r = 6\sqrt{3} \]
\[ r = 2\sqrt{3} \text{ cm} \]

The radius of the circle that passes through \( M, N, \) and \( O \) is \( 2\sqrt{3} \text{ cm} \).
Often we draw parallelograms so that two of the sides are either horizontal or vertical.

The parallelogram, $ABCD$, is positioned differently. $A$ lies on the positive $y$-axis, $D$ is on the positive $x$-axis, and $B$ and $C$ lie in the first quadrant. Three of its vertices, $A$, $B$, and $D$ are located at $(0,30)$, $(k,50)$ and $(40,0)$, respectively. The area of $ABCD$ is 1340 units$^2$. If $k > 0$, determine the coordinates of $B$ and $C$. 
Problem of the Week
Problem E and Solution
Positioned Differently

Problem
Often we draw parallelograms so that two of the sides are either horizontal or vertical. The parallelogram, $ABCD$, is positioned differently. $A$ lies on the positive $y$-axis, $D$ is on the positive $x$-axis, and $B$ and $C$ lie in the first quadrant. Three of its vertices, $A$, $B$, and $D$ are located at $(0, 30)$, $(k, 50)$ and $(40, 0)$, respectively. The area of $ABCD$ is $1340$ units$^2$. If $k > 0$, determine the coordinates of $B$ and $C$.

Solution
Since $ABCD$ is a parallelogram, $AB = DC$ and $AB \parallel DC$. We can use this to find the coordinates of $C$. To get from $A$ to $B$, we go up 20 units and right $k$ units. Therefore, to get from $D$ to $C$ we do the same. $C$ is located at $(40 + k, 20)$.

In the solution, we will use a method known commonly as “completing the rectangle”.

Enclose $ABCD$ in rectangle $OEFG$ such that $OE$ is on the positive $y$-axis passing through $A$, $EF$ is parallel to the positive $x$-axis passing through $B$, $FG$ is parallel to the positive $y$-axis passing through $C$, and $OG$ lies along the positive $x$-axis passing through $D$.

This information is presented on the following diagram.

The $y$ coordinate of $B$ is the distance from the $x$-axis to $EF$ and also the height, $GF$, of rectangle $OEFG$. It follows that $GF = 50$ units. Similarly, the $x$ coordinate of $C$ is the distance from the $y$-axis to $GF$ and also the width, $OG$, of rectangle $OEFG$. It follows that $OG = (40 + k)$ units. The other dimensions follow. (This information is already marked on the above diagram.)
The diagram from the first page is repeated here.

We can now put the information together using areas to determine the value of $k$.

\[
\text{Area } OEF = \text{Area } \triangle AEB + \text{Area } \triangle BFC + \text{Area } \triangle CGD + \text{Area } \triangle DOA + \text{Area } ABCD
\]

\[
FG \times OG = \frac{AE \times EB}{2} + \frac{BF \times FC}{2} + \frac{CG \times GD}{2} + \frac{DO \times OA}{2} + 1340
\]

\[
50 \times (40 + k) = \frac{20 \times k}{2} + \frac{40 \times 30}{2} + \frac{20 \times k}{2} + \frac{40 \times 30}{2} + 1340
\]

\[
2000 + 50k = 10k + 600 + 10k + 600 + 1340
\]

Therefore, the value of $k$ is 18 and coordinates of $B$ and $C$ are $B(18, 50)$ and $C(58, 20)$, respectively.

The solver may have approached the problem using linear equations and intersections. This is a very acceptable solution to the problem. However, in this problem, that approach probably would involve considerably more work.
Problem of the Week

Problem E

How Far to the Centre

A circle with centre $O$ is drawn with points $P$, $Q$, and $S$ on the circumference such that $PQ = PS = 12$ m. $PO$ is extended to meet $QS$ at $R$ such that $PR \perp QS$ and $OR = 1$ m.

Determine the radius of the circle.
Problem of the Week
Problem E and Solution
How Far to the Centre

Problem
A circle with centre $O$ is drawn with points $P$, $Q$, and $S$ on the circumference such that $PQ = PS = 12$ m. $PO$ is extended to meet $QS$ at $R$ such that $PR \perp QS$ and $OR = 1$ m.

Determine the radius of the circle.

Solution
Since $O$ is the centre of a circle that passes through $P$, $Q$, and $S$, then $OP$, $OQ$, and $OS$ are radii. Then $OP = OQ = OS = x$, $x > 0$. Let $SR = y$.

$\triangle SPR$ is right angled at $R$. Using the Pythagorean Theorem,

\[
PR^2 + RS^2 = PS^2
\]

\[
(PO + OR)^2 + RS^2 = PS^2
\]

\[
(x + 1)^2 + y^2 = 12^2 \quad (1)
\]

$\triangle SOR$ is right angled at $R$. Using the Pythagorean Theorem,

\[
OR^2 + RS^2 = OS^2
\]

\[
1^2 + y^2 = x^2
\]

\[
y^2 = x^2 - 1
\]

Substitute for $y^2$ in (1):

\[
(x + 1)^2 + x^2 - 1 = 12^2
\]

\[
x^2 + 2x + 1 + x^2 - 1 = 144
\]

\[
2x^2 + 2x - 144 = 0
\]

\[
x^2 + x - 72 = 0
\]

\[
(x - 8)(x + 9) = 0
\]

\[
x = 8 \quad \text{or} \quad x = -9
\]

Since $x > 0$, $x = -9$ is inadmissible. Therefore, $x = 8$. But $x$ is the radius of the circle.

$\therefore$ the radius of the circle is 8 m.
A rectangular prism has dimensions of $2a$, $2b$ and $2c$ as shown in the diagram below on the left.

$H$ is the intersection of the diagonals of the top face of the prism, $J$ is the intersection of the diagonals of the side face of the prism and $K$ is the intersection of the diagonals of the front face of the prism. $\triangle HJK$ is formed by joining $H$, $J$ and $K$. This is shown in the diagram below on the right.

If $HJ = 4$ cm, $HK = 5$ cm, and $JK = 6$ cm, determine the volume of the rectangular prism.
Problem of the Week
Problem E and Solution
Volumizing A Triangle

Problem
A rectangular prism has dimensions of $2a$, $2b$ and $2c$. $H$ is the intersection of the diagonals of the top face of the prism, $J$ is the intersection of the diagonals of the side face of the prism and $K$ is the intersection of the diagonals of the front face of the prism. $\triangle HJK$ is formed by joining $H$, $J$ and $K$. This is shown in the diagram to the right. If $HJ = 4$ cm, $HK = 5$ cm, and $JK = 6$ cm, determine the volume of the rectangular prism.

Solution
Label the top front edge of the rectangular prism $AB$ and its midpoint $M$. Then draw in $\triangle KMH$.

Since $H$ and $K$ are the centres of their respective rectangles, $MK = a, MH = c$ and $HK = 5$.

Since $\angle KMH = 90^\circ$, then
\[
 a^2 + c^2 = 25 \quad (1)
\]
Similarly, it can be shown that
\[
 b^2 + c^2 = 36 \quad (2)
\]
and
\[
 a^2 + b^2 = 16 \quad (3)
\]
Adding (1), (2), and (3),
\[
 2a^2 + 2b^2 + 2c^2 = 77
\]
Then dividing by 2,
\[
 a^2 + b^2 + c^2 = \frac{77}{2} \quad (4)
\]
Subtracting each of equations (1), (2) and (3) from equation (4) yields
\[
 b^2 = \frac{27}{2}, \quad a^2 = \frac{5}{2}, \quad \text{and} \quad c^2 = \frac{45}{2}.
\]
Multiplying $a^2$, $b^2$ and $c^2$ gives the product $a^2b^2c^2 = \frac{(5)(27)(45)}{8} = \frac{6075}{8}$.

Then, taking the positive square root, $abc = \sqrt{\frac{6075}{8}} = \frac{45\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{45\sqrt{6}}{4}$.

To determine the volume of the rectangular prism multiply the side lengths $2a$, $2b$ and $2c$ to obtain $V = (2a)(2b)(2c) = 8abc = 8 \left( \frac{45\sqrt{6}}{4} \right) = (2\sqrt{2})(45\sqrt{3}) = 90\sqrt{6}$ cm$^3$.

The volume of the rectangular prism is $90\sqrt{6}$ cm$^3$. 
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point $D$ on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point $A(9, 8)$, then reaches its peak at $E(7, 9)$, and finally passes through a hoop located at $B(b, 5)$ before returning to the ground at point $C$ and bursting. The ground between $C$ and $D$ is flat. Determine the area of quadrilateral $ABCD$. 
Problem of the Week
Problem E and Solution
Balloons Away!

Problem
John has just built a balloon launcher. He is using coordinates and coordinate geometry to describe the location of the balloon.

John launches a balloon from point $D$ on the ground. The balloon follows a perfect parabolic trajectory so that it passes through the point $A(9, 8)$, then reaches its peak at $E(7, 9)$, and finally passes through a hoop located at $B(b, 5)$ before returning to the ground at point $C$ and bursting. The ground between $C$ and $D$ is flat. Determine the area of quadrilateral $ABCD$.

Solution
We need to find the equation of the parabola. Then, in order to answer the question, we are required to find the $x$-intercepts of the parabola and the $x$-coordinate of point $B$ on the parabola.

We are given the peak $E(7, 9)$ so we have the vertex of the parabola. Using the vertex form of the equation of a parabola, $y = a(x - h)^2 + k$, with vertex $(h, k) = (7, 9)$, the equation of the parabola looks like $y = a(x - 7)^2 + 9$.

The point $A(9, 8)$ is on the parabola so we can substitute $(x, y) = (9, 8)$ into the equation $y = a(x - 7)^2 + 9$ to find $a$.

\[
8 = a(9 - 7)^2 + 9 \\
8 = a(4) + 9 \\
-1 = 4a \\
-\frac{1}{4} = a
\]

The equation of the parabola is $y = -\frac{1}{4}(x - 7)^2 + 9$.

To find the $x$-coordinate of $B(b, 5)$, substitute $y = 5$ into the equation of the parabola.

\[
5 = -\frac{1}{4}(b - 7)^2 + 9 \\
-4 = -\frac{1}{4}(b - 7)^2 \\
16 = (b - 7)^2 \\
\pm 4 = b - 7
\]

It follows that $b - 7 = -4$ or $b - 7 = 4$. Then $b = 3$ or $b = 11$. The point $B$ is to the left of the vertex so $b < 7$. The coordinates of $B$ are $(3, 5)$.
To find the $x$-intercepts of the parabola, substitute $y = 0$ into the equation of the parabola.

\[
0 = -\frac{1}{4}(x - 7)^2 + 9 \\
-9 = -\frac{1}{4}(x - 7)^2 \\
36 = (x - 7)^2 \\
\pm 6 = x - 7
\]

It follows that $x - 7 = -6$ or $x - 7 = 6$. Then the $x$-intercepts of the parabola are 1 and 13. The point $C$ is to the left of the vertex and the point $D$ is to the right of the vertex. The coordinates of $C$ are $(1, 0)$ and $D$ are $(13, 0)$.

This information has been added to the graph. There are many ways to determine the area of $ABCD$.

From $B(3, 5)$ and $A(9, 8)$ drop perpendiculars to the $x$-axis, intersecting at $F(3, 0)$ and $G(9, 0)$, respectively. From $B(3, 5)$ draw a perpendicular to $AG$, intersecting at $H(9, 5)$. Draw line segment $BG$.

We will use the diagram to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

$BH = 9 - 3 = 6$, $CG = 9 - 1 = 8$, $GD = 13 - 9 = 4$, $BF = 5 - 0 = 5$, and $AG = 8 - 0 = 8$.

To determine the area, we will find the sum of the areas of $\triangle CGB$, $\triangle AGD$ and $\triangle AGB$.

\[
\text{Area } ABCD = \text{Area } \triangle CGB + \text{Area } \triangle AGD + \text{Area } \triangle AGB \\
= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2} \\
= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2} \\
= 20 + 16 + 24 \\
= 60 \text{ units}^2
\]

The area of $ABCD$ is 60 units $^2$. 
Number Theory

&

Proportions

TAKE ME TO THE COVER
With the renewed interest in movies based on comic book characters, many clubs for comic book collectors have started. One such club attracts between 15 and 35 members to their monthly meetings.

At their last meeting, they discovered that all of the members in attendance had exactly the same number of comic books, except for one member who had one more comic book than each of the other members. Between them, the members had precisely 1000 comic books.

How many members attended the last meeting?
Problem of the Week
Problem E and Solution
Marvel at This

Problem
With the renewed interest in movies based on comic book characters, many clubs for comic book collectors have started. One such club attracts between 15 and 35 members to their monthly meetings. At their last meeting, they discovered that all of the members in attendance had exactly the same number of comic books, except for one member who had one more comic book than each of the other members. Between them, the members had precisely 1000 comic books. How many members attended the last meeting?

Solution
One could attempt a trial and error solution to this problem. However, a more algebraic solution will be presented here.

Let \( n \) represent the number of members present at the last monthly meeting such that \( 15 < n < 35 \) and \( n \) is an integer. Let \( c \) represent the number of comic books that all but one member had. The one member had \( c + 1 \) comic books. It follows that \((n - 1)\) members had \( c \) comic books each and one member had \( c + 1 \) comic books producing a total of 1000 comic books.

\[
(n - 1)c + 1(c + 1) = 1000
\]
\[
nc - c + c + 1 = 1000
\]
\[
nc = 999
\]

We are looking for two positive integers with a product of 999 with one of the numbers between 15 and 35. The prime factorization of 999 is \( 3 \times 3 \times 3 \times 37 \). We can combine the factors to produce pairs of positive integers whose product is 999. The possibilities are 1 and 999, 3 and 333, 9 and 111, and 27 and 37. The only possible product which gives one factor between 15 and 35 is 27 \( \times \) 37.

It then follows that there were 27 members present at the last meeting, 26 of the members had 37 comic books each and 1 member had 38 comic books. (This is easily verified: \( 26 \times 37 + 1 \times 38 = 1000 \).)
A king loves to travel from his castle to his summer house by coach. He orders his coachman to never go straight when he arrives at an intersection and to never travel along the same section of road twice during a trip. The coachman must turn either right or left when he comes to any intersection.

The map shows all of the roads leading from the palace to the summer house. The thick black lines indicate roads. All roads connecting two adjacent intersections are 1 km long.

The coachman wants to take the shortest route. Determine the length of the shortest route and justify why no shorter route is possible.
Problem
A king loves to travel from his castle to his summer house by coach. He orders his coachman to never go straight when he arrives at an intersection and to never travel along the same section of road twice during a trip. The coachman must turn either right or left when he comes to any intersection. The map shows all of the roads leading from the palace to the summer house. The thick black lines indicate roads. All roads connecting two adjacent intersections are 1 km long. The coachman wants to take the shortest route. Determine the length of the shortest route and justify why no shorter route is possible.

Solution
To get from the castle (bottom left corner) to the summer house (top right corner), the coach must pass through at least one of the four numbered roads on the adjacent diagram.

In the solution, we will examine four cases: routes which take us through each of the four numbered roads. A diagram will be presented for each possibility.

On each diagram, the start point will be labeled \(C\), the endpoint will be labeled \(H\), the south end of the road used in a particular case will be labeled \(S\), and the north end will be labeled \(N\).

Some details, for the sake of brevity, will be omitted from the solution and left for the solver to consider further. In the solution, turns will be described in terms of north, south, east and west.

1. **What is the length of the shortest path if the route passes through road 1?**

   If we travel from \(S\) to \(N\), travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along \(S\) to \(N\) there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 15 km long.

In the part of the route from \(C\) to \(S\), the shortest route is 4 km and goes north - east - north - west. It would also be possible to extend this route by going north - east - south - east - north - west - north - west, but this route is clearly longer. There is no route starting from \(C\) that goes east first which is able to legally pass through road 1. In traveling from \(N\) to \(H\), the shortest route is shown and is 10 km long. At some points along the route from \(N\) to \(H\), alternate choices can be made but these choices lead to invalid situations or lengthen the route. The solver may wish to confirm this.
2. **What is the length of the shortest path if the route passes through road 2?**

   If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 13 km long.

   In the part of the route from $C$ to $S$, the shortest route is 9 km. In traveling from $N$ to $H$, the shortest route is 3 km long. At some points along the route from $C$ to $S$ or $N$ to $H$, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. The solver may wish to confirm this.

3. **What is the length of the shortest path if the route passes through road 3?**

   If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 17 km long.

   In the part of the route from $C$ to $S$, the shortest route is 13 km. In traveling from $N$ to $H$, the shortest route is 3 km long. At some points along the route from $C$ to $S$, alternate choices can be made but these choices either lead to invalid situations or lengthen the route. At two intersections along the route from $N$ to $H$, alternate choices can be made but these choices would both lead to invalid situations. The solver may wish to confirm this.

4. **What is the length of the shortest path if the route passes through road 4?**

   If we travel from $S$ to $N$, travel on the north-south roads immediately above and below would violate the king’s order. On the diagram there is an X on each of these roads. However, to travel along $S$ to $N$ there are roads which must be traveled on. These are marked on the diagram with a ✓. The route shown on the diagram is the shortest route. This route is 15 km long.

   In the part of the route from $C$ to $S$, the shortest route is 12 km. In traveling from $N$ to $H$, the shortest route is 2 km long. At some points along the route from $C$ to $S$, alternate choices can be made but these choices either lead to invalid situations, lengthen the route or have the same length. The solver may wish to confirm this. The route from $N$ to $H$ is clearly the shortest one possible.

   After examining the four possible cases, the shortest route is 13 km and passes through road 2.
Problem of the Week
Problem E
This Picture Looks Like ...

The shape of the head and ears of a famous mouse are contained in a rectangle.

The two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to the sides of the rectangle. The width of the rectangle is 4 m.

Determine the area of the rectangle not covered by the head and ears of the famous mouse.

For this problem, the following known results about circles may be useful:

• If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.

• If two circles are tangent to each other at point \( P \), a line segment through the point of tangency can be drawn connecting the two centres, \( C_1 \) and \( C_2 \).
Problem of the Week
Problem E and Solution
This Picture Looks Like ...

Problem
The shape of the head and ears of a famous mouse appears to be contained in a rectangle. The two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to the sides of the rectangle. The width of the rectangle is 4 m.

Solution
Since the larger circle is tangent to two opposite sides of the rectangle, its diameter is 4 m, the width of the rectangle. It follows that the radius of the larger circle is 2 m.

The two smaller circles have equal radii, are tangent to each other and to opposite sides of the rectangle. It follows that the diameter of each of the smaller circles is half the width of the rectangle, namely 2 m. The radius of each of the smaller circles is 1 m.

Let the centre of the large circle and leftmost small circle be $B$ and $E$ respectively. Let the two small circles be tangent at $C$. Let the leftmost small circle and the larger circle be tangent at $F$. Position line segment $AD$ so that it is parallel to the longer side such that $A$ and $D$ are midpoints of the shorter sides of the rectangle. $AD$ will pass through $C$ and $B$.

Let the length of the rectangle be $d$. This is the same as the distance from $A$ to $D$ on the diagram. We know that $AB = 2$ m, the radius of the larger circle, and $CD = 1$ m, the radius of the smaller circle. We need to find the length of $BC$.

$AD$ is tangent to the smaller circles at $C$. Using the first property, we know that $EC \perp AD$ at $C$. Using the second property, $EFB$ is a straight line segment and $EB = EF + FB = 1 + 2 = 3$ m.
Combining the information, \( \triangle ECB \) is right angled at \( C \). Using the Pythagorean Theorem, \( BC^2 = EB^2 - EC^2 = 3^2 - 1^2 = 8 \) and \( BC = \sqrt{8} \) follows. Then the length of the rectangle is

\[
d = AB + BC + CD = 2 + \sqrt{8} + 1 = 3 + \sqrt{8}.
\]

To find the area not covered by the head and ears we need to find the shaded area. To do this we find the area of the rectangle and subtract the area of the large circle and the area of the two equal radii smaller circles.

Shaded Area

\[
= \text{Area of Rectangle} - \text{Area of Large Circle} - \text{Area of two smaller circles}
\]

\[
= 4 \times (3 + \sqrt{8}) - \pi \times 2^2 - 2 \times (\pi \times 1^2)
\]

\[
= 12 + 4\sqrt{8} - 4\pi - 2\pi
\]

\[
= 12 + 4\sqrt{8} - 6\pi
\]

Some students have learned to simplify radicals and know that \( \sqrt{8} = \sqrt{4\sqrt{2}} = 2\sqrt{2} \). The shaded area can then be written

\[
12 + 4 \times 2\sqrt{2} - 6\pi = 12 + 8\sqrt{2} - 6\pi.
\]

The shaded area is \((12 + 4\sqrt{8} - 6\pi) \text{ m}^2\) or \((12 + 8\sqrt{2} - 6\pi) \text{ m}^2\) (approximately 4.5 m²).
Problem of the Week
Problem E
Ahead of It’s Time

A stopped watch may be useless but at least it shows the correct time twice a day. A “good” watch which gains or loses time each day, shows the correct time far less often.

When Jeff received a pocket watch from his Grandmother on his 12\textsuperscript{th} birthday it was set at precisely the correct time. However, Jeff soon discovered that his watch gained exactly 10 seconds every day.

Assuming that Jeff never adjusts his watch to correct the time, how many times after his 12\textsuperscript{th} birthday and before his 90\textsuperscript{th} birthday will his watch show the correct time?
Problem of the Week
Problem E and Solution
Ahead of It’s Time

Problem
When Jeff received a pocket watch from his Grandmother on his 12th birthday it was set at precisely the correct time. However, Jeff soon discovered that his watch gained exactly 10 seconds every day. Assuming that Jeff never adjusts his watch to correct the time, how many times after his 12th birthday and before his 90th birthday will his watch show the correct time?

Solution
Solving this problem is not difficult. However, the answer may surprise the solver.

The watch will be correct once it has gained 12 hours.

\[
12 \text{ h} = 12 \times 60 = 720 \text{ minutes}
\]
\[
720 \text{ minutes} = 720 \times 60 = 43200 \text{ seconds}
\]

Since the watch gains 10 seconds every day, it will take \(43200 ÷ 10 = 4320\) days or approximately \(4320 ÷ 365 = 11.8\) years until it is the correct time again.

From Jeff’s 12th birthday to his 90th birthday, 78 years pass. The watch will be accurate \(78 ÷ (4320 ÷ 365) ≈ 6.6\) times. This means his watch will be accurate only 6 times after his 12th birthday and before his 90th birthday.

The watch will be correct when he is 23.8 years old (between his 23rd and 24th birthday), when he is 35.7 years old (between his 35th and 36th birthday), when he is 47.5 years old (between his 47th and 48th birthday), when he is 59.3 years old (between his 59th and 60th birthday), when he is 71.2 years old (near his 71st birthday), and when he is 83.0 years old (near his 83rd birthday). Jeff may wish to correct his watch periodically or get a more accurate one.

As a concluding note, if the watch gained one second per day, the watch would never be correct again for approximately 120 years. The answer is surprising!
Problem of the Week

Problem E

Doughnuts!

“Baker’s Dozen Doughnut Shop” doughnuts are sold only in boxes of 7, 13, or 25. To buy 14 doughnuts you must order two boxes of 7, but you cannot buy exactly 15 doughnuts since no combination of boxes contains 15 doughnuts.

What is the maximum number of doughnuts that cannot be ordered using combinations of the three different size boxes from “Baker’s Dozen Doughnut Shop”?
Problem of the Week
Problem E and Solution
Doughnuts!

Problem
“Baker’s Dozen Doughnut Shop” doughnuts are sold only in boxes of 7, 13, or 25. To buy 14 doughnuts you must order two boxes of 7, but you cannot buy exactly 15 doughnuts since no combination of boxes contains 15 doughnuts. What is the maximum number of doughnuts that cannot be ordered using combinations of the three different size boxes from “Baker’s Dozen Doughnut Shop”?

Solution
We can fill any order size which is a multiple of 7. Therefore, we can fill orders for \{7, 14, 21, 28, 35, 42, 49, \ldots\} doughnuts. We can also fill any order size which is a multiple of 13. Therefore, we can fill orders for \{13, 26, 39, 52, \ldots\} doughnuts. And we can fill any order size which is a multiple of 25. Therefore, we can fill orders for \{25, 50, 75, \ldots\} doughnuts.

Using the multiples above and combinations of the three different size boxes, we can fill orders of the following sizes:
7, 13, 14, 20 (7 + 13), 21, 25, 26, 27 (14 + 13), 28, 32 (7 + 25), 33 (7 + 26),
34 (21 + 13), 35, 38 (13 + 25), 39, 40 (14 + 26), 41 (28 + 13), 42,
45 (7 + 13 + 25), 46 (21 + 25), 47 (21 + 26), 48 (35 + 13), 49, 50, and 51 (26 + 25).

The missing numbers from the above list correspond to the order sizes that cannot be filled. The largest order that we are unable to fill in the above list appears to be 44. But we must justify that this is the maximum order size which cannot be filled. To do this, we note that orders of sizes 45, 46, 47, 48, 49, 50 and 51 can all be filled. This corresponds to 7 consecutive order sizes. If we add a 7-pak to each of these order sizes, we can fill the next seven consecutive order sizes. That is, we can fill orders of 52, 53, 54, 55, 56, 57 and 58. If we add a 7-pak to each of these orders, we can fill the next seven consecutive order sizes. In fact, every order size of 45 or more doughnuts can be filled. Since an order of size 44 doughnuts cannot be filled, this is the maximum size order which cannot be filled.

It turns out that there are only 26 order sizes that “Baker’s Dozen Doughnut Shop” cannot fill using the three different size boxes of doughnuts. If the shop were to add a box containing 3 doughnuts, how many orders would be impossible to fill?
Problem of the Week
Problem E
Putting the Parts Together

Determine the number of solutions to

\[
\frac{P}{Q} - \frac{Q}{P} = \frac{P + Q}{PQ}
\]

where \(P\) and \(Q\) are both integers

- \(-9 \leq P \leq 9\); and
- \(-9 \leq Q \leq 9\)
Problem of the Week
Problem E and Solution
Putting the Parts Together

Problem
Determine the number of solutions to
\[ \frac{P}{Q} - \frac{Q}{P} = \frac{P + Q}{PQ} \]
where \( P \) and \( Q \) are both integers, \(-9 \leq P \leq 9\) and \(-9 \leq Q \leq 9\).

Solution

\[ \frac{P}{Q} - \frac{Q}{P} = \frac{P + Q}{PQ} \]

Common Denominator

\[ \frac{P^2 - Q^2}{PQ} = \frac{P + Q}{PQ} \]

Simplify

\[ \frac{(P - Q)(P + Q)}{PQ} = \frac{(1)(P + Q)}{PQ} \]
Factor Left Side Numerator

Since the two sides are equal, \( P - Q = 1 \) or \( P + Q = 0 \). Also, \( P \) and \( Q \) cannot equal zero. Otherwise at least two of the denominators, \( P \), \( Q \), and \( PQ \), would equal zero and division by zero is undefined. We will look at each possibility separately.

1. \( P - Q = 1 \), \( P \neq 0 \) and \( Q \neq 0 \).
   In this case, we see that \( P \) and \( Q \) differ by 1 and \( P > Q \). The largest value of \( P \) is 9. When \( P = 9 \), \( Q = 8 \). The smallest value of \( Q \) is \(-9 \). When \( Q = -9 \), \( P = -8 \), a value which is 1 more than the value of \( Q \). So \( P \) can take on all of the integer values from \(-8 \) to \( 9 \) except \( P = 0 \). But when \( P = 1 \), \( Q = 0 \). We would have to remove this value of \( P \) as well. There are 18 integer values that \( P \) can take on from \(-8 \) to \( 9 \). After removing \( P = 0 \) and \( P = 1 \), there are 16 values for \( P \) and therefore 16 corresponding values for \( Q \). The equation has 16 solutions such that \( P - Q = 1 \), \( P \) and \( Q \) are integers, \(-9 \leq P \leq 9 \) and \(-9 \leq Q \leq 9 \).

2. \( P + Q = 0 \), \( P \neq 0 \) and \( Q \neq 0 \).
   In this case, \( P + Q = 0 \) or \( P = -Q \). When \( P = 9 \), the largest integer value it can take on, \( Q = -9 \). Similarly, when \( P = -9 \), the smallest integer value it can take on, \( Q = 9 \). There are 19 integer values that \( P \) can take on but this includes \( P = 0 \), \( Q = 0 \). So the equation has \( 19 - 1 = 18 \) solutions such that \( P + Q = 0 \), \( P \) and \( Q \) are integers, \(-9 \leq P \leq 9 \) and \(-9 \leq Q \leq 9 \).

We have considered all possible cases. Therefore, there are \( 16 + 18 = 34 \) solutions to the equation.
The problem today involves simple paper folding. In fact, only one fold is required.

A rectangular piece of paper is 30 cm wide and 40 cm long. The paper has a pattern on one side and is plain on the other. The paper is folded so that the two diagonally opposite corners, $A$ and $C$, coincide. (This is illustrated on the diagram to the right.)

Determine the length of the crease, $FE$, created by the fold.
Problem of the Week
Problem E and Solution
Paper Folding 101

Problem
The problem today involves simple paper folding. In fact, only one fold is required.

A rectangular piece of paper is 30 cm wide and 40 cm long. The paper has a pattern on one side and is plain on the other. The paper is folded so that the two diagonally opposite corners, A and C, coincide. (This is illustrated on the diagram to the right.)

Determine the length of the crease, FE, created by the fold.

Solution
After the fold, C coincides with A and D folds to G. The angle at G is the same as the angle at D. Since ABCD is a rectangle, \( \angle ADC = 90^\circ \) and it follows that \( \angle AGF = 90^\circ \).

Let \( a \) represent the length of \( BE \) and \( b \) represent the length of \( FD \). Then \( EC = CB - BE = 40 - a \) and \( AF = AD - FD = 40 - b \).

The distance from the top of the crease at F to D is the same length as the distance from F to G. It follows that \( FG = FD = b \).

The distance from the bottom of the crease at E to C is the same length as the distance from E to A. It follows that \( AE = EC = 40 - a \).

All of the information is recorded on the following diagram.
Since $\triangle ABE$ and $\triangle AGF$ are both right angled, we can use the Pythagorean Theorem to find $a$ and $b$.

\[
BE^2 + AB^2 = AE^2 \quad \text{and} \quad FG^2 + AG^2 = AF^2
\]

\[
a^2 + 30^2 = (40 - a)^2
\]

\[
a^2 + 900 = 1600 - 80a + a^2
\]

\[
80a = 700
\]

\[
a = \frac{35}{4}
\]

\[
\therefore a = b = \frac{35}{4}
\]

We still need to find the length of the crease.

From $F$ drop a perpendicular to $BC$ intersecting at $H$. $FHCD$ is a rectangle. It follows that $FH = DC = 30$ and $HC = FD = b$.

Also, $EH = BC - BE - HC = 40 - a - b = 40 - \frac{35}{4} - \frac{35}{4} = \frac{90}{4} = \frac{45}{2}$.

Using the Pythagorean Theorem in $\triangle EFH$,

\[
EF^2 = FH^2 + EH^2
\]

\[
= 30^2 + \left(\frac{45}{2}\right)^2
\]

\[
= 900 + \frac{2025}{4}
\]

\[
= \frac{5625}{4}
\]

\[
EF = \frac{75}{2} \quad (EF > 0)
\]

The length of the crease is $\frac{75}{2}$ cm (37.5 cm).
One day Matt was challenged to find the sum of all the three-digit numbers that could be made by choosing three different digits from the list \{1, 2, 3, 4, 5, 6, 7\}. Unsure of how to proceed, Matt started adding the numbers:

\[
\begin{array}{c}
1 & 2 & 3 \\
+ & 1 & 2 & 4 \\
\hline
2 & 4 & 7 \\
+ & 1 & 2 & 5 \\
\hline
3 & 7 & 2 \\
+ & 1 & 2 & 6 \\
\hline
4 & 9 & 8 \\
\vdots & & \\
\end{array}
\]

After finding the sum of just the first four possible numbers, Matt concluded that there had to be a better way.

Using a “better way”, determine the sum of all the three-digit numbers that can be made by choosing three different numbers from the list \{1, 2, 3, 4, 5, 6, 7\}.
Problem of the Week
Problem E and Solution
Faster Adder Required

Problem
Determine the sum of all the three-digit numbers that can be made by choosing three different numbers from the list $\{1, 2, 3, 4, 5, 6, 7\}$.

Solution
We need to first determine how many possible three-digit numbers can be formed using three different digits from the list $\{1, 2, 3, 4, 5, 6, 7\}$. There are 7 choices for the first digit. For each of these choices, there are 6 choices for the second digit giving a total of $7 \times 6 = 42$ choices for the first two digits. For each of these 42 choices for the first two digits, there are 5 choices for the third digit giving a total of $42 \times 5 = 210$ different three-digit numbers.

Each of the numbers 1 to 7 has an equal chance of appearing in each of the hundreds, tens and ones positions. Therefore, each digit appears $210 \div 7 = 30$ times in each place value position.

The sum of the digits in the units position is

$$30(1 + 2 + 3 + 4 + 5 + 6 + 7)$$
$$= 30(28)$$
$$= 840$$

The units digit of the sum is 0 and 84 is carried to the tens digit column.

The same digits appear in the tens digit column of the sum and again 30 times each. So the sum of the tens digit column is 924 which is the sum of the digits in the column plus 84 carried from the units digit column. The tens digit of the sum is 4 and 92 is carried to the hundreds digit column.

The same digits appear in the hundreds digit column of the sum and again 30 times each. So the sum of the hundreds digit column is 932 which is the sum of the digits in the column plus 92 carried from the tens digit column. The required sum is therefore 93,240.

The same sum would be obtained by adding 111, 222, 333, 444, 555, 666 and 777, and multiplying the sum by 30.

$$30(111 + 222 + 333 + 444 + 555 + 666 + 777)$$
$$= 30 \times 111 \times (1 + 2 + 3 + 4 + 5 + 6 + 7)$$
$$= 3330(28)$$
$$= 93,240$$

It is left to the solver to reason this out.
Problem of the Week
Problem E
E Z Does It Again

E Z Dealer has a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is yellow and the other side of the card is red.

E Z places all the cards, red side up, on the table. He first turns over every card that has a number which is a multiple of 2. He then examines all the cards, and turns over every card that has a number which is a multiple of 3. He again examines all the cards, and turns over every card that has a number which is a multiple of 4. Finally, he examines all the cards and turns over every card that has a number which is a multiple of 5.

After E Z has finished, how many cards have the red side facing up?
Problem of the Week
Problem E and Solution
E Z Does It Again

Problem
E Z Dealer has a deck consisting of 100 cards numbered from 1 to 100. Each card has the same number printed on both sides. One side of the card is yellow and the other side of the card is red. E Z places all the cards, red side up, on the table. He first turns over every card that has a number which is a multiple of 2. He then examines all the cards, and turns over every card that has a number which is a multiple of 3. He again examines all the cards, and turns over every card that has a number which is a multiple of 4. Finally, he examines all the cards and turns over every card that has a number which is a multiple of 5. After E Z has finished, how many cards have the red side facing up?

Solution
If a card number is a multiple of 2, 3, 4 and 5, it will be flipped four times. This card will go from red to yellow to red to yellow to red again. So the card will still be red once E Z has finished.
If a card number is a multiple of exactly three of 2, 3, 4 and 5, it will be flipped three times. This card will go from red to yellow to red to yellow. So the card will be yellow once E Z has finished.
If a card number is a multiple of exactly two of 2, 3, 4 and 5, then it will be flipped twice. This card will go from red to yellow to red again. So the card will still be red once E Z has finished.
If a card number is a multiple of exactly one of 2, 3, 4 and 5, it will be flipped once. This card will go from red to yellow. So the card will be yellow once E Z has finished.
If a card number is a multiple of none of 2, 3, 4 and 5, then this card will not be flipped and so the card will still be red once E Z has finished.

To determine how many cards have the red side facing up once E Z has finished, let’s determine how many cards have the yellow side facing up once E Z has finished. To do so, we need to determine how many card numbers are multiples of exactly three of 2, 3, 4 and 5 and how many cards are multiples of exactly one of 2, 3, 4 and 5.

Let’s consider the cases:

- A card number is a multiple of 2, 3 and 4, but not 5
  If a card number is a multiple of 2, 3 and 4, then it must be a multiple of 12, the lowest common multiple of 2, 3 and 4. So we want card numbers that are multiples of 12 but not 5. If a card number is a multiple of 12 and 5, then it is a multiple of \(12 \times 5 = 60\). So we want all multiples of 12 that are not multiples of 60.
  There are 8 multiples of 12 from 1 to 100, but one is 60. So there are \(8 - 1 = 7\) numbers that are multiples of 2, 3 and 4, but not 5.

- A card number is a multiple of 2, 3 and 5, but not 4
  If a card number is a multiple of 2, 3 and 5, then it must be a multiple of 30, the lowest common multiple of 2, 3 and 5. So we want all multiples of 30 that are not multiples of 4.
  There are 3 multiples of 30 from 1 to 100, but one is 60, which is also a multiple of 4. So there are \(2\) numbers from 1 to 100 that are multiples of 2, 3 and 5, but not 4.
• A card number is a multiple of 2, 4 and 5, but not 3
  If a card number is a multiple of 2, 4 and 5, then it must be a multiple of 20, the lowest common multiple of 2, 4 and 5. So we want all multiples of 20 that are not multiples of 3. There are 5 multiples of 20 from 1 to 100, but one is 60, which is a multiple of 3. So there are 4 numbers from 1 to 100 that are multiples of 2, 4 and 5, but not 3.

• A card number is a multiple of 3, 4 and 5, but not 2
  It is not possible for a card number to be a multiple of 4 but not 2. So there are no card numbers in this case.

• A card number is a multiple of 2 but not 3, 4, or 5
  There are 50 numbers from 1 to 100 which are multiples of 2 and 25 numbers from 1 to 100 which are multiples of 4 (and thus 2). So there are $50 - 25 = 25$ numbers from 1 to 100 multiples of by 2 but not 4. These are
  \[ \{2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98\}. \]
  We need to remove numbers that are still multiples of 3 or 5. After doing so we are left with
  \[ \{2, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94, 98\}. \]
  So there are 14 numbers from 1 to 100 that are multiples of 2 but not 3, 4 or 5.

• A card number is a multiple of 3 but not 2, 4, or 5
  There are 33 multiples of 3 from 1 to 100,
  \[ \{3, 6, 9, 12, 15, \ldots, 87, 90, 93, 96, 99\}. \]
  In this group of multiples, there are 17 numbers that are odd.
  So there are 17 numbers from 1 to 100 that are multiples of 3 but not 2. These numbers are
  \[ \{3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99\}. \]
  We still need to remove numbers that are multiples of 5. After doing so we are left with
  \[ \{3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99\}. \]
  So there are 14 numbers from 1 to 100 that are multiples of 3 but not 2, 4 or 5.

• A card number is a multiple of 4 but not 2, 3, or 5
  It is not possible for a card number to be a multiple of 4 but not 2. So there are no card numbers in this case.

• A card number is a multiple of 5 but not 2, 3, or 4
  There are 20 multiples of 5 from 1 to 100, but half of those are multiples of 2. The multiples of 5 which are not multiples of 2 are
  \[ \{5, 15, 25, 35, 45, 55, 65, 75, 85, 95\}. \]
  We still need to remove numbers that are multiples of 3. After doing so we are left with
  \[ \{5, 25, 35, 55, 65, 85, 95\}. \]
  So there are 7 numbers from 1 to 100 that are multiples of 5 but not 2, 3 or 4.

Therefore, once he has finished, E Z Dealer is left with $100 - (7 + 2 + 4 + 14 + 14 + 7) = 100 - 48 = 52$ cards with the red side facing up.

**Extension:** Suppose E Z Dealer continues flipping cards in this manner. So, after he has flipped all cards whose number is a multiple of 5, he then flips all cards whose card number is a multiple of 6, then 7, then 8, and so on until he flips all cards whose number is a multiple of 100. Once E Z has finished, how many cards will have the red side facing up?
Problem of the Week

Problem E

Above Average Task

Four numbers are selected such that when each number is added to the average of the other three, the following sums are obtained: 25, 37, 43, and 51.

Determine the average of the four numbers.

The picture puzzle shown above would make an interesting logo for a T-shirt. You may have to research the meanings of the mathematical symbols used in the puzzle. When you determine the meaning, hopefully you will agree that it is worth striving for.
Problem of the Week
Problem E and Solution
 Above Average Task

Problem
Four numbers are selected such that when each number is added to the average of the other three, the following sums are obtained: 25, 37, 43, and 51. Determine the average of the four numbers.

Solution
It is possible to precisely determine the four numbers but the problem only asks for their average. Let $a, b, c, d$ represent each of the four numbers. We are looking for $\frac{a+b+c+d}{4}$.

When the first number is added to the average of the other three numbers the result is 25.

$$\therefore a + \frac{b + c + d}{3} = 25$$

which simplifies to $3a + b + c + d = 75$ \hspace{1cm} (1)

When the second number is added to the average of the other three numbers the result is 37.

$$\therefore b + \frac{a + c + d}{3} = 37$$

which simplifies to $a + 3b + c + d = 111$ \hspace{1cm} (2)

When the third number is added to the average of the other three numbers the result is 43.

$$\therefore c + \frac{a + b + d}{3} = 43$$

which simplifies to $a + b + 3c + d = 129$ \hspace{1cm} (3)

When the fourth number is added to the average of the other three numbers the result is 51.

$$\therefore d + \frac{a + b + c}{3} = 51$$

which simplifies to $a + b + c + 3d = 153$ \hspace{1cm} (4)

Adding (1), (2), (3), and (4) we obtain $6a + 6b + 6c + 6d = 468$. Dividing by 6, $a + b + c + d = 78$. So the sum of the four numbers is 78. Dividing by 4 we determine that the average of the four numbers is 19.5.

Therefore the average of the four numbers is 19.5.

(By solving the system of equations we can actually determine that the numbers are: $-1.5, 16.5, 25.5, \text{ and } 37.5$.) The possible T-shirt logo represents “Well Above Average”, something worth striving for.
Problem of the Week

Problem E

From Altitudes to Angles to Sides

In $\triangle ABC$, $\angle BAC$ is the largest angle and $\angle ACB$ is the smallest angle. $AQ$, $BR$, and $CP$ are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the size of $\angle ABC$ and the lengths of the sides of $\triangle ABC$.

The diagram is not necessarily drawn to scale.
Problem of the Week
Problem E and Solution
From Altitudes to Angles to Sides

Problem
In \( \triangle ABC \), \( \angle BAC \) is the largest angle and \( \angle ACB \) is the smallest angle. \( AQ \), \( BR \), and \( CP \) are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.

Determine the size of \( \angle ABC \) and the lengths of the sides of \( \triangle ABC \).

Solution
Let \( BC = a \), \( AC = b \) and \( AB = c \).

We can find the area of the triangle by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,

\[
\frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}
\]

But \( AQ = 21 \), \( BC = a \), \( BR = 24 \), \( AC = b \), \( CP = 56 \), and \( AB = c \). Multiplying through by 2 and substituting we obtain

\[21a = 24b = 56c.\]

From \( 21a = 24b \) we obtain \( b = \frac{21}{24}a = \frac{7}{8}a \) and from \( 21a = 56c \) we obtain \( c = \frac{21}{56}a = \frac{3}{8}a \). The ratio of the sides in \( \triangle ABC \) is \( a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3 \). Let \( BC = 8x \), \( AC = 7x \), and \( AB = 3x \), \( x > 0 \).

Using the cosine law,

\[AC^2 = AB^2 + CB^2 - 2(AB)(CB) \cos(\angle ABC)\]
\[(7x)^2 = (3x)^2 + (8x)^2 - 2(3x)(8x) \cos(\angle ABC)\]

\[49x^2 = 9x^2 + 64x^2 - 48x^2 \cos(\angle ABC)\]

Dividing by \( x^2 \) since \( x > 0 \), \( 49 = 73 - 48 \cos(\angle ABC) \)

Rearranging, \( 48 \cos(\angle ABC) = 24 \)

\[\cos(\angle ABC) = \frac{1}{2}\]

\[\therefore \angle ABC = 60^\circ\]
In right \( \triangle BPC \),

\[
\frac{PC}{BC} = \sin 60^\circ
\]

\[
BC = \frac{PC}{\sin 60^\circ}
\]

\[
BC = \frac{56}{\sqrt{3}}
\]

\[
BC = \frac{112}{\sqrt{3} \times \sqrt{3}}
\]

\[
BC = \frac{112\sqrt{3}}{3}
\]

But \( BC = 8x \)

\[
\therefore 8x = \frac{112\sqrt{3}}{3}
\]

\[
x = \frac{14\sqrt{3}}{3}
\]

\[
3x = 14\sqrt{3}
\]

\[
7x = \frac{98\sqrt{3}}{3}
\]

The side lengths of \( \triangle ABC \) are \( AB = 3x = 14\sqrt{3} \), \( AC = 7x = \frac{98\sqrt{3}}{3} \) and \( BC = \frac{112\sqrt{3}}{3} \).
Johann Carl Friedrich Gauss was a mathematician who lived from 1777 to 1855. He made major contributions to number theory and algebra, to name just a few. Some of the earliest stories about Gauss deal with determining the sum of sequences of numbers. Our problem today deals with a long sequence of numbers. So, make Gauss proud as you work with this problem.

A sequence consists of 2018 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2018 terms is 27 243.

Determine the sum of the odd numbered terms. That is, determine the sum of every second term starting with the first term and ending with the second last term.

Some helpful information about sequences is included on the next page. You may or may not wish to refer to it.
A sequence is made up of terms like $t_1$, $t_2$, $t_3$, $\cdots$, $t_n$. The subscript indicates the position of the term in the sequence. For example, $t_8$ would represent the term in the eighth position in the sequence and $t_n$ represents the general term in the sequence.

A series is the sum of the terms of a sequence. So $t_1 + t_2 + t_3 + \cdots + t_n$ would represent the sum of the first $n$ terms.

The following information may be helpful in the solution of the problem.

An arithmetic sequence is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3,5,7,9 is an arithmetic sequence with four terms and constant difference 2.

The general term of an arithmetic sequence is $t_n = a + (n - 1)d$, where $a$ is the first term, $d$ is the constant difference and $n$ is the number of terms.

The sum, $S_n$, of the first $n$ terms of an arithmetic sequence can be found using either $S_n = \frac{n}{2}[2a + (n - 1)d]$ or $S_n = n \left( \frac{t_1 + t_n}{2} \right)$, where $t_1$ is the first term of the sequence and $t_n$ is the $n^{th}$ term of the sequence.

The following example is provided to verify the accuracy of the formulas and to illustrate their use.

For the arithmetic sequence 3,5,7,9, $a = t_1 = 3$, $d = 2$, $n = 4$ and $t_n = t_4 = 9$.

$$S_n = 3 + 5 + 7 + 9 = 24$$
$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{4}{2}[2(3) + (3)2] = 2[12] = 24$$
$$S_n = n \left( \frac{t_1 + t_n}{2} \right) = 4 \left( \frac{3 + 9}{2} \right) = 4(6) = 24$$

Gauss developed a specific formula for the sum of the first $n$ positive integers.

$$1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$$

For example, in honour of the current year, the sum of the positive integers from 1 to 2017 is

$$1 + 2 + 3 + \cdots + 2016 + 2017 = \frac{2017(2018)}{2} = 2035153$$
Problem of the Week
Problem E and Solution
Make Gauss Proud

Problem
A sequence consists of 2018 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2018 terms is 27 243. Determine the sum of the odd numbered terms. That is, determine the sum of every second term starting with the first term and ending with the second last term.

Solution
Solution 1
Let \( S_O \) represent the sum of the terms in the odd numbered positions. Half of the terms of the sequence are in odd numbered positions so there are \( 2018 \div 2 = 1009 \) terms in \( S_O \).

\[
S_O = t_1 + t_3 + t_5 + \cdots + t_{2017}
\]

Let \( S_E \) represent the sum of the terms in the even numbered positions. Half of the terms of the sequence are in even numbered positions so there are 1009 terms in \( S_E \).

\[
S_E = t_2 + t_4 + t_6 + \cdots + t_{2018}
\]

Let \( S \) represent the sum of the 2018 terms. \( S = S_O + S_E = 27 243 \) \hspace{1cm} (1)

Since each term after the first term is 1 greater than the term before,

\[
t_2 = t_1 + 1, \quad t_4 = t_3 + 1, \quad t_6 = t_5 + 1, \cdots, \quad t_{2018} = t_{2017} + 1
\]

Now, \( S_E = t_2 + t_4 + t_6 + \cdots + t_{2016} + t_{2018} \), 1009 terms

\[
= (t_1 + 1) + (t_3 + 1) + (t_5 + 1) + \cdots + (t_{2015} + 1) + (t_{2017} + 1)
\]

\[
= (t_1 + t_3 + t_5 + \cdots + t_{2015} + t_{2017}) + 1009(1)
\]

\[ \therefore S_E = S_O + 1009 \] \hspace{1cm} (2)

Substituting \( S_O + 1009 \) for \( S_E \) in (1),

\[
S_O + S_E = 27 243
\]

\[
S_O + S_O + 1009 = 27 243
\]

\[
2S_O = 27 243 - 1009
\]

\[
2S_O = 26 234
\]

\[
S_O = 13 117
\]

Therefore, the sum of the terms in the odd positions in the sequence is 13 117.

Notice that this solution did not require any special formulas.
Solution 2

Let \( t_1 \) represent the first term in the sequence.

Every term in the sequence can be written in terms of \( t_1 \). The second term is 1 more than the first term, the third term is 2 more than the first term, the fourth term is 3 more than the first term, and so on.

\[
t_1 + t_2 + t_3 + t_4 + \cdots + t_{2016} + t_{2017} + t_{2018} = 27243
\]
\[
t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2016) + (t_1 + 2017) = 27243
\]
\[
2018t_1 + (1 + 2 + 3 + \cdots + 2016 + 2017) = 27243
\]

Using the formula for the sum of the first \( n \) positive integers with \( n = 2017 \),
\[
2018t_1 + \frac{2017(2018)}{2} = 27243
\]

Dividing each term by 2018,
\[
t_1 + \frac{2017}{2} = \frac{27243}{2018}
\]
\[
t_1 + 1008.5 = 13.5
\]
\[
t_1 = -995
\]

Now that we know that the first term in the sequence is \(-995\), we know that the original series is
\[
t_1 + t_2 + t_3 + t_4 + \cdots + t_{2016} + t_{2017} + t_{2018}
\]
\[
= t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2016) + (t_1 + 2017)
\]
\[
= -995 - 994 - 993 - 992 - \cdots + 1021 + 1022
\]

We are interested in the sum
\[
-995 - 993 - 991 - 989 - \cdots + 1019 + 1021
\]

This is an arithmetic series with 1009 terms, first term \( t_1 = -995 \), and last term \( t_n = 1021 \). Then, using \( S_n = n \left[ t_1 + t_n \right] \frac{2}{2} \)
\[
-995 - 994 - 993 - 992 - \cdots + 1021 + 1022 = 1009 \left[ \frac{-995 + 1021}{2} \right] = 1009(13) = 13117
\]

Therefore the sum of the odd numbered terms in the sequence is \( 13117 \).

Many other approaches could be taken once the first term is found.
Problem of the Week
Problem E
The Truth of the Matter

Four people, Andy, Barb, Carl and Dana, each said two statements such that:

- one person lied in both statements;
- one person told the truth in both statements; and
- two people told the truth in one statement and a lie in the other statement.

Andy said, “Barb lied once” and “Dana lied twice.”
Barb said, “I never lie” and “Andy never lied.”
Carl said, “Dana lied twice” and “Barb never lied.”
Dana said, “Andy lied twice” and “I never lie.”

Who lied twice? Who never lied? Who lied exactly once?
Problem of the Week
Problem E and Solution
The Truth of the Matter

Problem
Four people, Andy, Barb, Carl and Dana, each said two statements such that:

- one person lied in both statements;
- one person told the truth in both statements; and
- two people told the truth in one statement and a lie in the other statement.

Andy said, “Barb lied once” and “Dana lied twice.” Barb said, “I never lie” and “Andy never lied.” Carl said, “Dana lied twice” and “Barb never lied.” Dana said, “Andy lied twice” and “I never lie.”

Who lied twice? Who never lied? Who lied exactly once?

Solution

1. Who lied twice?

   a) Assume that Andy lied twice. If so, “Barb lied once” is a lie. Therefore, “Barb never lied” or “Barb lied twice.” But “Barb never lied” cannot be true because Barb says, “Andy never lied.” This contradicts our assumption that Andy lied twice. “Barb lied twice” cannot be true since that means Barb and Andy both lied twice and this contradicts the fact that only one person lied twice. Therefore, our assumption that Andy lied twice is false.

   b) Assume that Barb lied twice. If so, “Andy never lied” is a lie. Then, Andy lied twice or Andy lied once. Andy cannot have lied twice since both he and Barb would have lied twice and this contradicts the fact that only one person lied twice. But “Andy lied once” is also false since “Barb lied once” is a lie (we assumed she lied twice) and “Dana lied twice” is a lie because it contradicts the assumption that Barb lied twice (and only one person can lie twice). Therefore, our assumption that Barb lied twice is false.

   c) Assume that Carl lied twice. If so, “Barb never lied” is a lie and “Dana lied twice” is a lie. Since “Barb never lied” is a lie, then she lied twice or she lied once. But if Barb lied twice our assumption that Carl lied twice cannot be true since only one person lied twice.

   If Barb lied once, then “I never lie” must be the lie and “Andy never lied” must be true. But if Andy never lied, then “Dana lied twice” must be true and this contradicts the fact that only one person can lie twice. Therefore, our assumption that Carl lied twice is false.

   We are told that one person lied twice and none of Andy, Barb or Carl lied twice. Therefore, by elimination, Dana is the one who lied twice.
2. Who never lied?

a) Assume that Barb never lied. Then her statement that “Andy never lied” must be true. There are then two people who never lied. This contradicts the fact that only one person never lied. Therefore, our assumption that Barb never lied is false.

b) Assume that Carl never lied. Then his statement that “Barb never lied” must be true. There are then two people who never lied. This contradicts the fact that only one person never lied. Therefore, our assumption that Carl never lied is false.

Dana lied twice. Barb and Carl lied. Therefore, by elimination, Andy is the one who never lied. It then follows that Barb and Carl each make one true statement and tell one lie.

We can now check our results.

Andy never lied. Then his statements are both true. Barb lied once is true and Dana lied twice is true.

Barb lied once. Then one of her statements is true and the other is a lie. Her statement that she never lies is a lie and her statement that Andy never lies is true.

Carl lied once. Then one of his statements is true and the other is a lie. His statement that Dana lied twice is true and his statement that Barb never lied is a lie.

Dana lied twice. Then both of her statements are lies. Andy lied twice is a lie. And her statement that she never lies is a lie.
Problem of the Week
Problem E
Keep On Chuggin’

Two trains of equal length are on parallel tracks. One train is travelling at 40 km/h and the other at 20 km/h. It takes two minutes longer for the trains to completely pass one another when going in the same direction, than when going in opposite directions.

Determine the length of each train.
Problem of the Week

Problem E and Solution

Keep On Chuggin’

Problem

Two trains of equal length are on parallel tracks. One train is travelling at 40 km/h and the other at 20 km/h. It takes two minutes longer for the trains to completely pass one another when going in the same direction, than when going in opposite directions. Determine the length of each train.

Solution

Solution 1

Let \( L \) represent the length, in km, of each train. Let \( t_1 \) represent the time, in hours, required for the fast train to completely pass the slow train when going in the same direction. Let \( t_2 \) represent the time, in hours, required for the fast train to completely pass the slow train when going in opposite directions.

In order to completely pass one another when going in the same direction, the faster train must travel two lengths of the train plus whatever distance the slower train travels. Therefore,

\[
40t_1 = 20t_1 + 2L \\
20t_1 = 2L \\
t_1 = \frac{L}{10}
\]

In order to completely pass one another when going in the opposite direction, the total distance travelled by the two trains must be \( 2L \). Therefore,

\[
40t_2 + 20t_2 = 2L \\
60t_2 = 2L \\
t_2 = \frac{L}{30}
\]

Since it takes two minutes or \( \frac{2}{60} \) hours longer for the trains to completely pass one another when going in the same direction than when going in opposite directions,

\[
t_1 - t_2 = \frac{2}{60} \\
\frac{L}{10} - \frac{L}{30} = \frac{1}{30} \\
Multiplying \ by \ 30: \ 3L - L = 1 \\
2L = 1 \\
L = 0.5
\]

Therefore, the length of each train is 0.5 km.
Solution 2

Let $L$ represent the length, in km, of each train.

When going in the same direction, the faster train is travelling at $40 - 20 = 20$ km/h relative to the slower train. In order to completely pass, the faster train must travel $2L$ km. Therefore, it takes $\frac{2L}{20} = \frac{L}{10}$ hours to completely pass.

When travelling in opposite directions, the faster train is travelling at $40 + 20 = 60$ km/h relative to the slower train. In order to completely pass, the faster train must travel $2L$ km. Therefore, it takes $\frac{2L}{60} = \frac{L}{30}$ hours to completely pass.

Since it takes two minutes or $\frac{2}{60} = \frac{1}{30}$ hours longer for the trains to completely pass one another when going in the same direction than when going in opposite directions,

\[
\frac{L}{10} - \frac{L}{30} = \frac{1}{30}
\]

Multiplying by 30:

\[
3L - L = 1
\]

\[
2L = 1
\]

\[
L = 0.5
\]

Therefore, the length of each train is 0.5 km.
**Solution 3**

Let $L$ represent the length, in km, of each train.

While the trains are travelling in opposite directions, let $y$ km be the distance travelled by the slower train from the time the faster train begins to pass until it completely passes. The slower train travels $y$ km and the faster train travels $(2L - y)$ km. We know that the time travelled will be the same so:

\[
\frac{y}{20} = \frac{2L - y}{40} \quad \frac{2y}{40} = \frac{2L - y}{40} \quad 3y = 2L \quad (1)
\]

While the trains are travelling in the same direction, let $x$ km be the distance travelled by the slower train from the time the faster train begins to pass until it completely passes. The slower train travels $x$ km and the faster train travels $(x + 2L)$ km. We know that the time travelled will be the same so:

\[
\frac{x}{20} = \frac{x + 2L}{40} \quad \frac{2x}{40} = \frac{x + 2L}{40} \quad x = 2L \quad (2)
\]

We know that it takes two minutes or $\frac{2}{60}$ hours longer for the trains to completely pass one another when going in the same direction than when going in opposite directions. So,

\[
\frac{x}{20} - \frac{y}{20} = \frac{2}{60} \quad \frac{x}{20} - \frac{3y}{60} = \frac{2}{60}
\]

Substituting $2L$ for $x$ from (2) and $2L$ for $3y$ from (1),

\[
\frac{2L}{20} - \frac{2L}{60} = \frac{2}{60} \quad \frac{6L}{2L} - \frac{2L}{60} = \frac{2}{60} \quad \frac{60}{4L} = \frac{2}{60} \quad L = 0.5
\]

Therefore, the length of each train is 0.5 km.
Problem of the Week
Problem E
Useful Facts Indeed!

A prime number is a positive integer greater than 1 that has exactly two positive integer factors, 1 and the number itself. A composite number is a positive integer greater than 1 that has more than two positive integer factors. The number 1 is neither prime nor composite.

For some number \(21609d\), with units digit \(d\), \(2^{21609d} - 1\) is a very large prime number.

In fact, the number contains 65 050 digits. The number begins 746 093 103 064 661 343 \(\cdots\) and ends with the units digit 7.

Determine the value of \(d\), the units digit of \(21609d\).

Here are some useful facts which may be helpful in solving this problem:

1. if \(n\) is divisible by 3, then \(2^n - 1\) is divisible by 7; and
2. if \(n\) is divisible by 5, then \(2^n - 1\) is divisible by 31.

One use for very large prime numbers is in the area of cryptography, the study of coding and decoding information so that it can be securely transmitted. This area of study is very important because of its application to areas like online banking, email, and general internet security, to list just a few.
Problem of the Week
Problem E and Solution
Useful Facts Indeed!

Problem \( 2^{21609d} - 1 = 746\ 093\ 103\ 064\ 661\ 343 \cdots 7 \)

Some number 21609d, with units digit \( d \), is a very large prime number. In fact, the number contains 65 050 digits. The number begins 746 093 103 064 661 343 \( \cdots \) and ends with the units digit 7. Determine the value of \( d \), the units digit of 21609d.

Here are some useful facts which may be helpful in solving this problem:

- If \( n \) is divisible by 3, then \( 2^n - 1 \) is divisible by 7; and
- If \( n \) is divisible by 5, then \( 2^n - 1 \) is divisible by 31.

Solution
To start, let’s look for a pattern in the units digit of powers of 2.

\[
\begin{array}{cccc}
2^1 & = & 2 \\
2^2 & = & 4 \\
2^3 & = & 8 \\
2^4 & = & 16 \\
2^5 & = & 32 \\
2^6 & = & 64 \\
2^7 & = & 128 \\
2^8 & = & 256 \\
\end{array}
\]

It appears that the units digit of powers of 2 repeat in the cycle 2, 4, 8, 6. The next four powers of 2, \( 2^9 \), \( 2^{10} \), \( 2^{11} \), and \( 2^{12} \), end with units digits 2, 4, 8, and 6, respectively, as expected. Then \( 2^{216088} \) would end in a 6 since 216088 is divisible by 4. It then follows that \( 2^{216089} \) ends in 2, \( 2^{216090} \) ends in 4 and \( 2^{216091} \) ends in an 8. Since \( 2^{216091} \) ends in an 8, \( 2^{216095} \) and \( 2^{216099} \) also ends in 8.

Then \( 2^{216091} - 1 \), \( 2^{216095} - 1 \) and \( 2^{216099} - 1 \) each end in a 7. Therefore, the only possible values of \( d \) are 1, 5 and 9.

If a number ends in 0 or 5, then it is divisible by 5. If \( d = 5 \) then 216095 is divisible by 5. From the useful facts, it follows that \( 2^{216095} - 1 \) is divisible by 31 and is therefore not a prime number.

If the sum of the digits of a number is divisible by 3, then the number is divisible by three. If \( d = 9 \) then 216099 is divisible by 3, since the sum of the digits of 216099 is 27 which is divisible by 3. From the useful facts, it follows that \( 2^{216099} - 1 \) is divisible by 7 and is therefore not a prime number.

The only possible value for \( d \) is 1 and \( 2^{216091} - 1 \) is a prime number ending in 7. This prime number is from a group of prime numbers called Mersenne primes. This number is the 31st Mersenne prime and it was discovered in September of 1985. For more on Mersenne Primes, check out the Great Internet Mersenne Prime Search (GIMPS) at www.mersenne.org. According to GIMPS, as of January 2016, 49 Mersenne Primes are known. Perhaps you will be part of a team that will discover the next Mersenne Prime. There are prizes awarded when new discoveries are found and verified.

Extension: Can you prove the two useful facts?
Problem of the Week
Problem E
Search and Swap

Randi has a deck consisting of 10 cards. One side of each card is red and the other side of each card has one of the letters A, B, C, D, E, F, G, H, I, or J on it. Each letter occurs exactly once. The cards are shuffled and placed letter-side down on a table from left to right.

Every time Randi looks for a letter, she turns over cards one by one starting with the leftmost card and moving to the right. If a card does not have the letter she is looking for on it, Randi puts it back letter-side down in the same location and continues with the next card. If a card does have the letter she is looking for on it, Randi swaps the locations of this card and the card on its immediate left placing both cards letter-side down. One exception is when she finds the letter she is looking for on the leftmost card. In this case, Randi puts the card back letter-side down in the same location and no swap occurs. Either way, once Randi finds the letter she is looking for, she does not look at any more cards. Also, Randi never remembers the locations of any cards on the table.

For example, suppose Randi is asked to find the letter E and the cards were on the table as shown below.

\[\begin{array}{cccccccc}
G & B & F & J & A & I & E & H & C & D \\
\end{array}\]

Randi would look at each of the first 6 cards and return each of them, letter-side down, to the same place on the table. When she looked at the seventh card and found that it was the E, she would swap the location of the E and the I. So to locate the letter E, Randi looked at 7 cards and the resulting card ordering would be as follows.

\[\begin{array}{cccccccc}
G & B & F & J & A & E & I & H & C & D \\
\end{array}\]

To find another card, Randi must begin her search with the leftmost card. For example, If next she wanted to search for the F, she would look at the G and B and not change their locations. She would then look at the third card, see the F, and swap the locations of the third and second card. The resulting card ordering would be as follows.

\[\begin{array}{cccccccc}
G & F & B & J & A & E & I & H & C & D \\
\end{array}\]

After searching for the E and the F, Randi has looked at a total of 7 + 3 = 10 cards.

If the ten cards begin in some unknown order and Randi searches for each of the ten letters exactly once, what is the maximum possible number of cards that Randi looks at?
Problem

Randi has a deck consisting of 10 cards. One side of each card is red and the other side of each card has one of the letters A, B, C, D, E, F, G, H, I, or J on it. Each letter occurs exactly once. The cards are shuffled and placed letter-side down on a table from left to right.

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If the ten cards begin in some unknown order and Randi searches for each of the ten letters exactly once, what is the maximum possible number of cards that Randi looks at?

Solution

If no swaps were required as a result of finding a card, how many cards would Randi have to look at in total?

At some point she is looking for the card in position 1. She would have to look at 1 card to find it. At some point she is looking for the card in position 2. She would have to look at 2 cards to find it. At some point she is looking for the card in position 3. She would have to look at 3 cards to find it. This continues until at some point she is looking for the card in position 10. She would have to look at 10 cards to find it. To locate all 10 cards, Randi would have to look at $1 + 2 + 3 + \cdots + 10 = 55$ cards.

Since Randi only looks for each letter exactly once, swapping the position of one letter with the position of another letter can only have the effect of increasing the number of cards looked at for the letter on the preceding card by one. The number of cards looked at to find other letters would not be affected. Therefore, swapping can only increase the number of cards looked at (by one) for all but the first search. These means swapping can increase the number of cards looked at by at most 9 in total making the maximum total number of cards looked at equal to $55 + 9 = 64$.

On the next page, an illustration of how this maximum can be achieved is illustrated.
Is 64 an achievable maximum?

Put the cards in order, left to right, from A to J.

\[
\begin{array}{cccccccccc}
A & B & C & D & E & F & G & H & I & J \\
\end{array}
\]

Now search for each letter in order from B to J and search for A last.

Since B is in the second position, we must look at 2 cards to find it. We then swap A and B.

\[
\begin{array}{cccccccccc}
B & A & C & D & E & F & G & H & I & J \\
\end{array}
\]

Since C is in the third position, we must look at 3 cards to find it. We then swap A and C.

\[
\begin{array}{cccccccccc}
B & C & A & D & E & F & G & H & I & J \\
\end{array}
\]

Since D is in the fourth position, we must look at 4 cards to find it. We then swap A and D.

\[
\begin{array}{cccccccccc}
B & C & D & A & E & F & G & H & I & J \\
\end{array}
\]

Since E is in the fifth position, we must look at 5 cards to find it. We then swap A and E.

\[
\begin{array}{cccccccccc}
B & C & D & E & A & F & G & H & I & J \\
\end{array}
\]

Since F is in the sixth position, we must look at 6 cards to find it. We then swap A and F.

\[
\begin{array}{cccccccccc}
B & C & D & E & F & A & G & H & I & J \\
\end{array}
\]

Since G is in the seventh position, we must look at 7 cards to find it. We then swap A and G.

\[
\begin{array}{cccccccccc}
B & C & D & E & F & G & A & H & I & J \\
\end{array}
\]

Since H is in the eighth position, we must look at 8 cards to find it. We then swap A and H.

\[
\begin{array}{cccccccccc}
B & C & D & E & F & G & H & A & I & J \\
\end{array}
\]

Since I is in the ninth position, we must look at 9 cards to find it. We then swap A and I.

\[
\begin{array}{cccccccccc}
B & C & D & E & F & G & H & I & A & J \\
\end{array}
\]

Since J is in the tenth position, we must look at 10 cards to find it. We then swap A and J.

\[
\begin{array}{cccccccccc}
B & C & D & E & F & G & H & I & J & A \\
\end{array}
\]

Finally, since A is in the tenth position, we must look at 10 cards to find it. We then swap A and J (again).

\[
\begin{array}{cccccccccc}
B & C & D & E & F & G & H & I & A & J \\
\end{array}
\]

We have looked at a total of $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 = 64$ cards to locate each of the cards.

On the next page, we look at a possible extension and a connection to Computer Science.
Extension:
Suppose you have $n$ cards, each with something different on them. You lay the cards out in a similar manner to how we handled the 10 different cards. You search for each of the different cards, one at a time. What is the maximum number of cards you must look at in order to locate all of the cards using the search described in the problem?

Connection to Computer Science:

One of the fundamental problems in computer science is how to organize data in order to search within it quickly. There are many ways to do this: using binary trees, splay trees, skip lists, sorted arrays, etc. The technique outlined in this problem is the idea of moving found items closer to the “front,” with the assumption that if we search for something once, it is quite likely that the same item will be searched for again. The transpose (swap) heuristic used by Randi in this problem is one technique for doing this. Other heuristics include move-to-front, which moves a found element to the very front of the list. Moreover, this problem highlights the process of performing worst-case analysis for an algorithm. Computer scientists care about “what is the worst possible input for this algorithm, and how long will it take to execute on that input?” In this question, we are asking about the worst-case performance of the transpose heuristic on a list of size 10.
Problem of the Week
Problem E
Clipping Along

Gwen and Chris are playing a game. They begin with a pile of paperclips, and use the following rules.

1. The two players alternate turns.
2. On any turn, a player can remove 1, 2, 3, 4, or 5 paperclips from the pile.
3. The same number of paperclips cannot be removed on two different turns during the entire game.
4. The last person who is able to play wins, regardless of whether there are any paperclips remaining in the pile after their turn.

For example, if a game begins with 9 paperclips, then the following moves could occur. Gwen removes 2 paperclips, leaving 7 in the pile. Chris removes 5 paperclips, leaving 2 in the pile. Gwen removes 1 paperclip, leaving 1 in the pile. Gwen is now the winner, since Chris cannot remove 1 paperclip. (Gwen already removed 1 paperclip on one of her turns, and the third rule states that a person cannot remove the same number that has already been removed.)

Suppose the game starts with 8 paperclips and Gwen goes first. Find all initial moves that Gwen can make to guarantee that she will win. Justify your answers.
Problem
Gwen and Chris are playing a game. They begin with a pile of paperclips, and use the following rules.

1. The two players alternate turns.
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4. The last person who is able to play wins, regardless of whether there are any paperclips remaining in the pile after their turn.

Suppose the game starts with 8 paperclips and Gwen goes first. Find all initial moves that Gwen can make to guarantee that she will win. Justify your answers.

Solution
Gwen has five options to begin with. She can remove 5, 4, 3, 2, or 1 paperclip. Let’s look at each case.

Case 1 – Gwen removes 5 paperclips on her first turn.
If Gwen removes 5 paperclips then Chris can remove the remaining 3 paperclips on his turn. Chris would win the game. Therefore, Gwen could lose if she starts by removing 5 paperclips.

Case 2 – Gwen removes 4 paperclips on her first turn.
If Gwen removes 4 paperclips, then there are 4 paperclips remaining. Chris cannot remove 4 paperclips or he will violate the third rule. So, Chris can only remove 1, 2 or 3 paperclips.
If Chris removes 1 paperclip, then there are 3 paperclips remaining. Gwen can now remove the remaining 3 paperclips on her second turn to win the game.
If Chris removes 2 paperclips, then there are 2 paperclips remaining. Gwen can now remove 1 paperclip on her second turn. This leaves 1 paperclip which Chris cannot remove without violating the third rule. Gwen wins the game.
If Chris removes 3 paperclips, then there is 1 paperclip remaining. Gwen can now remove the remaining paper clip on her second turn to win the game.
Therefore, no matter what Chris does on his first turn, Gwen can win if she starts the game by removing 4 paperclips.

Case 3 – Gwen removes 3 paperclips on her first turn.
If Gwen removes 3 paperclips then Chris can remove the remaining 5 paperclips to win the game. Therefore, Gwen could lose if she starts by removing 3 paperclips.
Case 4 – Gwen removes 2 paperclips on her first turn.

If Gwen removes 2 paperclips, then there are 6 paperclips remaining. Chris cannot remove 2 paperclips without violating the third rule. So, Chris can remove 1, 3, 4, or 5 paperclips.
If Chris removes 1 paperclip, then Gwen can remove the 5 remaining paperclips on her second turn to win the game.
If Chris removes 3 paperclips, then 3 paperclips remain. Gwen can remove 1 paperclip on her second turn. (Gwen cannot remove 2 or 3 paperclips without violating the third rule.) There are 2 paperclips left. Chris would lose since he cannot remove 1 or 2 paperclips without violating the third rule. Therefore, Gwen wins the game.
If Chris removes 4 paperclips, then 2 paperclips remain. Gwen will remove 1 paperclip on her second turn leaving 1 paperclip. Chris would lose since he cannot remove 1 paperclip without violating the third rule. Therefore, Gwen wins the game.
If Chris removes 5 paperclips, then Gwen can remove the remaining paperclip on her second turn to win the game.
Therefore, no matter what Chris does on his first turn, Gwen can win if she starts the game by removing 2 paperclips.

Case 5 – Gwen removes 1 paperclip on her first turn.

If Gwen removes 1 paperclip, then there are 7 paperclips remaining. Chris can then remove 2, 3, 4, or 5 paperclips.
If Chris removes 2 paperclips, then Gwen can remove the remaining 5 paperclips on her second turn to win the game.
If Chris removes 3 paperclips, then Gwen can remove the remaining 4 paperclips on her second turn to win the game.
If Chris removes 4 paperclips, then Gwen can remove the remaining 3 paperclips on her second turn to win the game.
If Chris removes 5 paperclips, then Gwen can remove the remaining 2 paperclips on her second turn to win the game.
Therefore, no matter what Chris does on his first turn, Gwen can win if she starts the game by removing 1 paperclip.

Therefore, Gwen can win the game if she goes first and starts by removing 1, 2 or 4 paperclips on her first turn.