The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Problem of the Week
Problems and Solutions
2022 - 2023

Problem D (Grade 9/10)

Themes

(Click on a theme name to jump to that section.)

Number Sense (N)

Geometry & Measurement (G)

Algebra (A)

Data Management (D)

Computational Thinking (C)

The problems in this booklet are organized into themes.
A problem often appears in multiple themes.

Number Sense (N)

TAKE ME TO THE COVER

Problem of the Week
Problem D
How Many in the House?

The POTW Theatre has four levels of seating: gold, silver, red, and black.

One night, the manager of the theatre was asked how many patrons are in the theatre. The manager replied that $\frac{1}{6}$ of the patrons in the theatre that night are in the gold seating, $\frac{1}{4}$ of the patrons are in either the red seating or the black seating, there are three times as many patrons in the the silver seating as in the red seating, and there are 138 patrons in the black seating.

How many patrons were in the theatre that night?

Problem of the Week
Problem D and Solution
How Many in the House?

Problem

The POTW Theatre has four levels of seating: gold, silver, red, and black.

One night, the manager of the theatre was asked how many patrons are in the theatre. The manager replied that $\frac{1}{6}$ of the patrons in the theatre that night are in the gold seating, $\frac{1}{4}$ of the patrons are in either the red seating or the black seating, there are three times as many patrons in the the silver seating as in the red seating, and there are 138 patrons in the black seating.

How many patrons were in the theatre that night?

Solution

Let n be the total number of patrons in the theatre that night.
Let g be the number of patrons in the gold seating, s be the number of patrons in the silver seating, r be the number of patrons in the red seating, and b be the number of patrons in the black seating.

Therefore, $n = g + s + r + b$.

Since $\frac{1}{6}$ of the patrons in the theatre are in the gold seating, $g = \frac{1}{6}n$.

Since $\frac{1}{4}$ of the patrons are either in the black seating or the red seating, $r + b = \frac{1}{4}n$.

It is given that $b = 138$. Therefore, $r + b = \frac{1}{4}n$ becomes $r + 138 = \frac{1}{4}n$, or $r = \frac{1}{4}n - 138$.

Since there are three times as many patrons in the silver seating as patrons in the red seating, $s = 3r = 3(\frac{1}{4}n - 138)$.

Substituting these expressions for g, r, and s, and the value for b into $n = g + s + r + b$, we have

$$\begin{aligned}
n &= \left(\frac{1}{6}n\right) + 3\left(\frac{1}{4}n - 138\right) + \left(\frac{1}{4}n - 138\right) + 138 \\
n &= \frac{1}{6}n + \frac{3}{4}n - 414 + \frac{1}{4}n - 138 + 138 \\
n &= \frac{1}{6}n + n - 414 \\
n &= \frac{7}{6}n - 414 \\
\frac{1}{6}n &= 414 \\
n &= 2484
\end{aligned}$$

Therefore, there were 2484 patrons in the theatre that night.

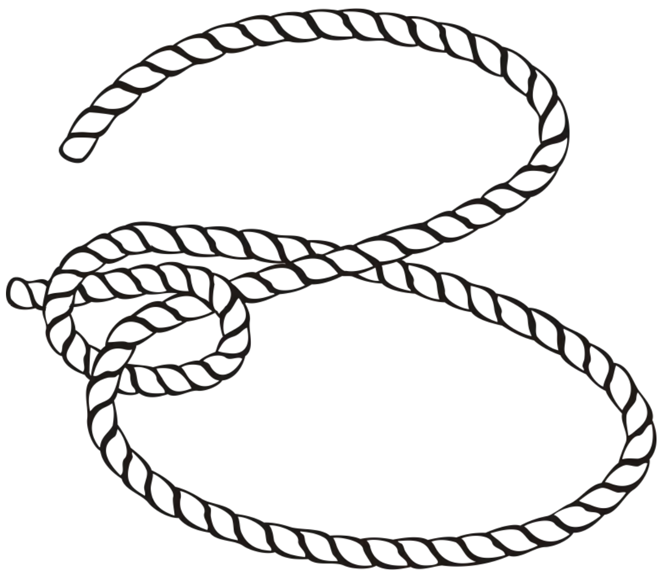
Although it is not required, we could further determine that the number of patrons in the silver seating is 1449, the number of patrons in the gold seating is 414, and the number of patrons in the red seating is 483. We could then use these numbers to verify the given information.

Problem of the Week

Problem D

Three Triangles and a Square

Simon has a rope that is 200 cm long. They cut the rope into four pieces so that one piece can be arranged, with its two ends touching, to form a square, and the three remaining pieces can be arranged, with each having its two ends touching, to form three identical equilateral triangles. If all four shapes have integer side lengths, in cm, determine all possibilities for the side lengths of each triangle and the square.



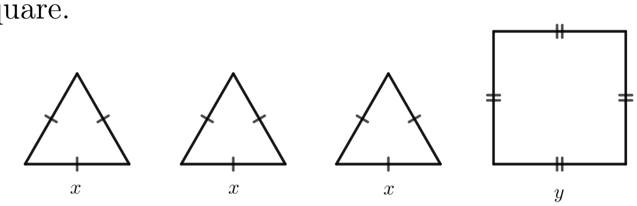
Problem of the Week
Problem D and Solution
Three Triangles and a Square

Problem

Simon has a rope that is 200 cm long. They cut the rope into four pieces so that one piece can be arranged, with its two ends touching, to form a square, and the three remaining pieces can be arranged, with each having its two ends touching, to form three identical equilateral triangles. If all four shapes have integer side lengths, in cm, determine all possibilities for the side lengths of each triangle and the square.

Solution

Let x represent the side length, in cm, of each equilateral triangle and let y represent the side length, in cm, of the square.



The perimeter of each figure is the length of the piece of rope used to form it. For each triangle, the length of rope is $3x$ and for the square the length of rope is $4y$. The total rope used is $3(3x) + 4y = 9x + 4y$. But the length of the rope is 200 cm. Therefore,

$$\begin{aligned} 9x + 4y &= 200 \\ 9x &= 200 - 4y \\ x &= \frac{4(50-y)}{9} \end{aligned}$$

We are given that x and y are integers, and the information given in the problem implies x and y must be positive. Since both x and y are integers, $4(50 - y)$ must be a multiple of 9. But 4 is not divisible by 9, so $50 - y$ must be divisible by 9. There are five positive multiples of 9 between 0 and 50, namely 9, 18, 27, 36, and 45. So $50 - y$ must be equal to 9, 18, 27, 36, or 45. It follows that y is equal to 41, 32, 23, 14, or 5. The corresponding values of x are computed in the table below.

y	$4y$	$200 - 4y$	$x = \frac{200-4y}{9}$
41	164	36	4
32	128	72	8
23	92	108	12
14	56	144	16
5	20	180	20

Thus, there are 5 possibilities. When the side length of the square is 41 cm, the side length of each triangle is 4 cm; when the side length of the square is 32 cm, the side length of each triangle is 8 cm; when the side length of the square is 23 cm, the side length of each triangle is 12 cm; when the side length of the square is 14 cm, the side length of each triangle is 16 cm; and when the side length of the square is 5 cm, the side length of each triangle is 20 cm.

Problem of the Week

Problem D

How Much?

At a fundraiser hosted by a local restaurant, customers can pay as much or as little as they like for a meal, as long as they pay at least \$1. Any profits are donated to a local charity. One evening, the mean (average) price paid per customer was \$55. One more customer walked in and paid \$70 for a meal, bringing the average up to \$56. What is the highest possible price that a customer could have paid for their meal that evening? Do you think the amounts in your solution are reasonable for this situation?

Problem of the Week
Problem D and Solution
How Much?

Problem

At a fundraiser hosted by a local restaurant, customers can pay as much or as little as they like for a meal, as long as they pay at least \$1. Any profits are donated to a local charity. One evening, the mean (average) price paid per customer was \$55. One more customer walked in and paid \$70 for a meal, bringing the average up to \$56. What is the highest possible price that a customer could have paid for their meal that evening? Do you think the amounts in your solution are reasonable for this situation?

Solution

To calculate the mean (average) of a set of values, we first calculate the sum of the values in the set, and then divide that by the number of values in the set. It follows that the sum of the values in the set is equal to their average multiplied by the number of values in the set.

Let n represent the number of customers that evening. The total amount paid by all customers that evening was therefore $56n$. The final customer paid 70 dollars for their meal. Before this customer arrived, there were $(n-1)$ customers and they had paid a total of $(56n-70)$ dollars. At that point, the average price paid per customer was 55 dollars. Using this information, we can write and solve the following equation.

$$\begin{aligned}\frac{56n-70}{n-1} &= 55\\ 56n-70 &= 55(n-1)\\ 56n-70 &= 55n-55\\ n &= 15\end{aligned}$$

Since $n=15$, it follows that there were 15 customers that evening, and the total amount paid by all customers was therefore $56n=56(15)=840$ dollars.

To determine the highest possible price that a customer could have paid for their meal that evening, we will assume that 13 of the customers paid the lowest possible price of \$1. Then the remaining customer would have paid $840-13\times 1-70=757$ dollars.

Therefore, the highest possible price that a customer could have paid for their meal that evening is \$757.

Since this is a fundraiser, \$1 is probably a very small amount and \$757 would be considered a very generous donation for a meal.

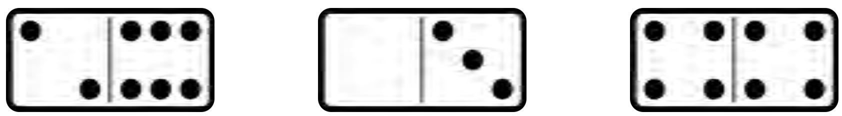
Extension:

How would the answer change if no two customers paid the same amount?

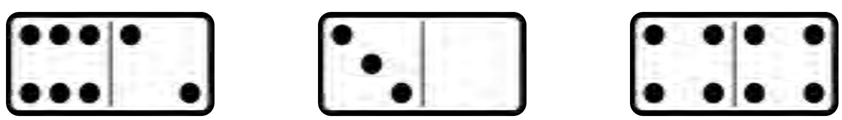
Problem of the Week
Problem D
Missing Tile

A domino tile is a rectangular tile with a line dividing its face into two square ends. Each end is marked with a number of dots (also called pips) or is blank.

The first domino shown below is a $[2, 6]$ domino, since there are 2 pips on its left end and 6 pips on its right end. The second domino shown below is a $[0, 3]$ domino, since there are 0 pips on its left end and 3 pips on its right end. The third domino shown below is a $[4, 4]$ domino, since there are 4 pips on its left end and 4 pips on its right end.



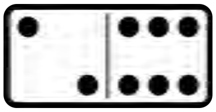
We can also rotate the domino tiles. The first domino shown below is a $[6, 2]$ domino, since there are 6 pips on its left end and 2 pips on its right end. However, since this tile can be obtained by rotating the $[2, 6]$ tile, $[6, 2]$ and $[2, 6]$ represent the same domino. Similarly, the second domino shown below is a $[3, 0]$ domino. Again, note that $[3, 0]$ and $[0, 3]$ represent the same domino.



A 2-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 2, with no two dominoes being the same. A 2-set of dominoes has the following six tiles: $[0, 0]$, $[0, 1]$, $[0, 2]$, $[1, 1]$, $[1, 2]$, $[2, 2]$. Notice that the three dominoes $[1, 0]$, $[2, 0]$, and $[2, 1]$ are not listed because they are the same as the three dominoes $[0, 1]$, $[0, 2]$, and $[1, 2]$.

Similarly, a 9-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 9, with no two dominoes being the same.

Drew and Bennett separate a 9-set of dominoes into two piles. Drew counts all of the pips on the dominoes in the first pile. He counts that there are a total of 213 pips. Bennett counts all of the pips on the dominoes in the second pile. He counts that there are a total of 266 pips. They then realize that one domino is missing from the set. Drew also notes that every domino that has the same number of pips on its left and right ends is accounted for. Which domino is missing from the set?

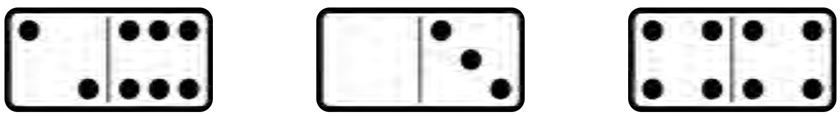


Problem of the Week
Problem D and Solution
Missing Tile

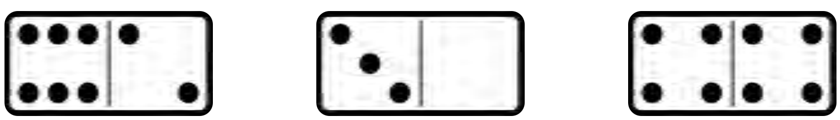
Problem

A domino tile is a rectangular tile with a line dividing its face into two square ends. Each end is marked with a number of dots (also called pips) or is blank.

The first domino shown below is a $[2, 6]$ domino, since there are 2 pips on its left end and 6 pips on its right end. The second domino shown below is a $[0, 3]$ domino, since there are 0 pips on its left end and 3 pips on its right end. The third domino shown below is a $[4, 4]$ domino, since there are 4 pips on its left end and 4 pips on its right end.



We can also rotate the domino tiles. The first domino shown below is a $[6, 2]$ domino, since there are 6 pips on its left end and 2 pips on its right end. However, since this tile can be obtained by rotating the $[2, 6]$ tile, $[6, 2]$ and $[2, 6]$ represent the same domino. Similarly, the second domino shown below is a $[3, 0]$ domino. Again, note that $[3, 0]$ and $[0, 3]$ represent the same domino.



A 2-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 2, with no two dominoes being the same. A 2-set of dominoes has the following six tiles: $[0, 0]$, $[0, 1]$, $[0, 2]$, $[1, 1]$, $[1, 2]$, $[2, 2]$. Notice that the three dominoes $[1, 0]$, $[2, 0]$, and $[2, 1]$ are not listed because they are the same as the three dominoes $[0, 1]$, $[0, 2]$, and $[1, 2]$.

Similarly, a 9-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 9, with no two dominoes being the same.

Drew and Bennett separate a 9-set of dominoes into two piles. Drew counts all of the pips on the dominoes in the first pile. He counts that there are a total of 213 pips. Bennett counts all of the pips on the dominoes in the second pile. He counts that there are a total of 266 pips. They then realize that one domino is missing from the set. Drew also notes that every domino that has the same number of pips on its left and right ends is accounted for. Which domino is missing from the set?

Solution

We first determine which dominoes are in a 9-set of dominoes and calculate the total number of pips on all of the dominoes in the set. In a 9-set of dominoes, the number of pips on each end of a domino tile can range from 0 to 9. Since rotating a domino tile does not change the domino, we orient each domino so that the smaller number is always on the left end of the domino. For each possible number on the left end of the domino, we examine the possible numbers that can occur on the right end of the domino, and then calculate the total number of pips on all dominoes with that number of pips on the left end. We compile this information in a table.

Number on Left End of Domino	Possible Numbers on Right End of Domino	Total Number of Pips on All Dominoes
0	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	$1+2+3+4+5+6+7+8+9=45$
1	1, 2, 3, 4, 5, 6, 7, 8, 9	$9(1)+1+2+3+4+5+6+7+8+9=54$
2	2, 3, 4, 5, 6, 7, 8, 9	$8(2)+2+3+4+5+6+7+8+9=60$
3	3, 4, 5, 6, 7, 8, 9	$7(3)+3+4+5+6+7+8+9=63$
4	4, 5, 6, 7, 8, 9	$6(4)+4+5+6+7+8+9=63$
5	5, 6, 7, 8, 9	$5(5)+5+6+7+8+9=60$
6	6, 7, 8, 9	$4(6)+6+7+8+9=54$
7	7, 8, 9	$3(7)+7+8+9=45$
8	8, 9	$2(8)+8+9=33$
9	9	$1(9)+9=18$

Therefore, the total number of pips on all of the dominoes in a 9-set of dominoes is

$$45+54+60+63+63+60+54+45+33+18=495$$

Now, the total number of pips in the two piles is $213+266=479$. That leaves a total of 16 pips on the missing tile.

In a 9-set of dominoes, the only tiles with a total of 16 pips are $[8,8]$ and $[7,9]$. Since every domino with the same number of pips on its left and right ends is present, then the $[8,8]$ tile is present. Therefore, the missing tile must be the $[7,9]$ tile.

Problem of the Week
Problem D
Let's Dance

The student council at POTW High School is throwing a school dance. They want to give a welcome gift to each Grade 9 student that attends the dance.

Gifts-R-Us charges \$1.00 per gift. However, if they were to purchase the gifts at Gifts-R-Us, they would exceed their budget by \$17.

At Presents-4-U, they only charge \$0.80 per gift. At this price, the student council would have \$5.00 left over in their budget.

Determine the number of gifts the student council is planning to buy.

Problem of the Week
Problem D and Solution
Let's Dance

Problem

The student council at POTW High School is throwing a school dance. They want to give a welcome gift to each Grade 9 student that attends the dance.

Gifts-R-Us charges \$1.00 per gift. However, if they were to purchase the gifts at Gifts-R-Us, they would exceed their budget by \$17.

At Presents-4-U, they only charge \$0.80 per gift. At this price, the student council would have \$5.00 left over in their budget.

Determine the number of gifts the student council is planning to buy.

Solution

Solution 1

Let n represent the number of gifts that the student council is planning to buy.

Since each gift at Gifts-R-Us costs \$1.00, the student council would spend $1 \times n = n$ dollars in total. If the student council were to purchase all of the gifts they want at Gifts-R-Us, they would be short \$17 dollars in their budget. Therefore, the amount they have in their budget is $(n - 17)$ dollars.

Since each gift at Presents-4-U costs \$0.80, the student council would spend $0.8 \times n = 0.8n$ dollars in total. If the student council were to purchase all of the gifts they want at Presents-4-U, they would have \$5 dollars left over in their budget. Therefore, the amount they have in their budget is $(0.8n + 5)$ dollars.

We have two expressions for the amount in their budget, so we can establish the equality $n - 17 = 0.8n + 5$. This simplifies to $0.2n = 22$. After dividing each side by 0.2, we obtain $n = 110$.

Therefore, the student council is planning to buy 110 gifts.

Solution 2

Let n represent the number of gifts that the student council is planning to buy. Let x represent the amount that the student council has budgeted.

Since the difference between the costs of a single gift is $\$1.00 - \$0.80 = \$0.20$, the total cost difference of buying n gifts would be $\$0.2n$.

To purchase from Gifts-R-Us, the student council would need to spend \$17 more than they budgeted. Therefore, they would need $(x + 17)$ dollars. To purchase from Presents-4-U, the student council would need to spend \$5 less than they budgeted. Therefore, they would need $(x - 5)$ dollars. The total cost difference of purchasing n gifts would be $(x + 17) - (x - 5) = 22$ dollars.

We have two expressions for the cost difference and can establish the equality $0.2n = 22$. After dividing each side by 0.2, we obtain $n = 110$.

Therefore, the student council is planning to buy 110 gifts.

Note that in Solution 1 and Solution 2, we were able to solve for the number of gifts without calculating the budget. In Solution 3, we will first calculate the budget and then use that to calculate the number of gifts.

Solution 3

Let n represent the number of gifts that the student council is planning to buy. Let x represent the amount that the student council has budgeted.

Since each gift at Gifts-R-Us costs \$1.00, n gifts would cost $n \times \$1 = \$n$. Also, the student council would need to spend \$17 more than they budgeted. Therefore, we have

$$n = x + 17 \tag{1}$$

Since each gift at Presents-4-U costs \$0.80, n gifts would cost $n \times \$0.8 = \$0.8n$. Also, the student council would need to spend \$5 less than they budgeted. Therefore, we have

$$0.8n = x - 5 \tag{2}$$

Substituting equation (1) into equation (2), we have

$$\begin{aligned} 0.8n &= x - 5 \\ 0.8(x + 17) &= x - 5 \\ 0.8x + 13.6 &= x - 5 \\ 18.6 &= 0.2x \\ x &= 93 \end{aligned}$$

Thus, the student council has budgeted \$93.

Then, using equation (1), we see that $n = x + 17 = 93 + 17 = 110$.

Therefore, the student council is planning to buy 110 gifts.

Problem of the Week
Problem D
Teacher Road Trip 2

To help pass time on a long bus ride, a group of math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first and second teachers each said a non-negative integer, and every teacher after that said the sum of all of the previous terms in the sequence.

For example, if the first teacher said the number 2 and the second teacher said the number 8, then

- the third teacher would say the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher would say the sum of the first, second, and third terms, which is $2 + 8 + 10 = 20$.

How many possible sequences could the teachers have said if the first teacher said the number 3 and another teacher said the number 3072?

Problem of the Week
Problem D and Solution
Teacher Road Trip 2

Problem

To help pass time on a long bus ride, a group of math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first and second teachers each said a non-negative integer, and every teacher after that said the sum of all of the previous terms in the sequence.

For example, if the first teacher said the number 2 and the second teacher said the number 8, then

- the third teacher would say the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher would say the sum of the first, second, and third terms, which is $2 + 8 + 10 = 20$.

How many possible sequences could the teachers have said if the first teacher said the number 3 and another teacher said the number 3072?

Solution

We know how to construct the sequence, and we know that the first term is 3, but where is the term whose value is 3072?

- Could 3072 be the second term?

  If the first two terms are 3 and 3072, then we can calculate the next few terms.

  - The third term would be $3 + 3072 = 3075$.
  - The fourth term would be $3 + 3072 + 3075 = 3075 + 3075 = 2(3075) = 6150$.
  - The fifth term would be $3 + 3072 + 3075 + 6150 = 6150 + 6150 = 2(6150) = 12\,300$.

  We see that we can determine any term beyond the third term by summing all of the previous terms, or we can simply double the term immediately before the required term, since that term is the sum of all the preceding terms. (This also means that if any term after the third term is known, then the preceding term is half the value of that term.)

  Therefore, there is one possible sequence with 3072 as the second term. The first 6 terms of this sequence are $3,\ 3072,\ 3075,\ 6150,\ 12\,300,\ 24\,600$.

- Could 3072 be the third term?

  Yes, since the third term is the sum of the first two terms, and the first term is 3, then the second term would be $3072 - 3 = 3069$ and the first 6 terms of this sequence are $3,\ 3069,\ 3072,\ 6144,\ 12\,288,\ 24\,576$.

- Could 3072 be the fourth term?

  Yes, since the fourth term is even, then we can determine the third term to be half of the fourth term, which is $3072 \div 2 = 1536$, then the second term would be $1536 - 3 = 1533$. The first 6 terms of this sequence are $3,\ 1533,\ 1536,\ 3072,\ 6144,\ 12\,288$.

- Could 3072 be the fifth term?

  To get from the fifth term to the third term we would divide by 2 twice, or we could divide by 4. If the resulting third term is a non-negative integer greater than or equal to 3, then the sequence exists. The third term would be $3072 \div 4 = 768$, and the second term would be $768 - 3 = 765$. Thus the sequence exists and the first 6 terms are $3,\ 765,\ 768,\ 1536,\ 3072,\ 6144$.

We could continue in this way until we discover all possible sequences that are formed according to the given rules with first term 3 and 3072 somewhere in the sequence. However, if we look at the prime factorization of 3072 we see that the highest power of 2 that divides 3072 is 1024 (or 2^{10}), since $3072 = 2^{10} \times 3$. In fact, dividing 3072 by 1024 would produce a third term that would be 3. The second term would then be 0, a non-negative integer, and the resulting sequence would be $3,\ 0,\ 3,\ 6,\ 12,\ 24,\ 48,\ 96,\ 192,\ 384,\ 768,\ 1536,\ 3072,\ 6144, \ldots$.

If we divide 3072 by any integral power of 2 from $2^0 = 1$ to $2^{10} = 1024$, the resulting third term would be an integer greater than or equal to 3, and 3072 would appear in each of these sequences. There are 11 such sequences. The number 3072 would appear somewhere from term 3 to term 13 in the acceptable sequence. However, 3072 can also appear as the second term, so there are a total of 12 possible sequences.

Could 3072 be the fourteenth term? From the fourteenth term to the third term we would need to divide 3072 by 2^{11}. The resulting third term would be $\frac{3}{2}$. This would mean the second term is not an integer and so the sequence is not possible.

Therefore, there are a total of 12 such sequences.

Problem of the Week

Problem D

Digits Multiplied

The digits of any positive integer can be multiplied together to give the *digit product* for the integer. For example, 345 has the digit product of $3 \times 4 \times 5 = 60$. There are many other positive integers that have 60 as a digit product. For example, 2532 and 14 153 both have a digit product of 60. Note that 256 is the smallest positive integer with a digit product of 60.

There are also many positive integers that have a digit product of 2160. Determine the smallest such integer.

Problem of the Week
Problem D and Solution
Digits Multiplied

Problem

The digits of any positive integer can be multiplied together to give the *digit product* for the integer. For example, 345 has the digit product of $3 \times 4 \times 5 = 60$. There are many other positive integers that have 60 as a digit product. For example, 2532 and 14 153 both have a digit product of 60. Note that 256 is the smallest positive integer with a digit product of 60.

There are also many positive integers that have a digit product of 2160. Determine the smallest such integer.

Solution

Let N be the smallest positive integer whose digit product is 2160.

In order to find N, we must find the minimum possible number of digits whose product is 2160. This is because if the integer a has more digits than the integer b, then $a > b$. Once we have determined the digits that form N, then the integer N is formed by writing those digits in increasing order.

Note that the digits of N cannot include 0, or else the digit product of N would be 0.

Also, the digits of N cannot include 1, otherwise we could remove the 1 and obtain an integer with fewer digits (and thus, a smaller integer) with the same digit product. Therefore, the digits of N will be between 2 and 9, inclusive.

Since digits of N multiply to 2160, we can use the prime factorization of 2160 to help determine the digits of N:

$$2160 = 2^4 \times 3^3 \times 5$$

In order for the digit product of N to have a factor of 5, one of the digits of N must equal 5.

The digit product of N must also have a factor of $3^3 = 27$. We cannot find one digit whose product is 27 but we can find two digits whose product is 27. In particular, $27 = 3 \times 9$. Therefore, N could also have the digits 3 and 9.

Then the remaining digits of N must have a product of $2^4 = 16$. We need to find a combination of the smallest number of digits whose product is 16. We cannot have one digit whose product is 16, but we can have two digits whose product is 16. In particular, $16 = 2 \times 8$ and $16 = 4 \times 4$.

Therefore, it is possible for N to have 5 digits. We have seen that this can happen when the digits of N are 5, 3, 9, 2, 8 or 5, 3, 9, 4, 4.

However, notice that the product of 2 and 3 is 6. Therefore, rather than using the digits 5, 3, 9, 2, 8, we can replace the two digits 2 and 3 with the single digit 6. We now have the digits 6, 5, 8, and 9. The smallest integer using these digits is 5689.

It is possible that we can take a factor of 2 from the 8 and a factor of 3 from the 9 to make another $2 \times 3 = 6$. However, the digits will be now be 5, 6, 6, 4, and 3. This means we will have a five-digit number which is larger than than the four-digit number 5689.

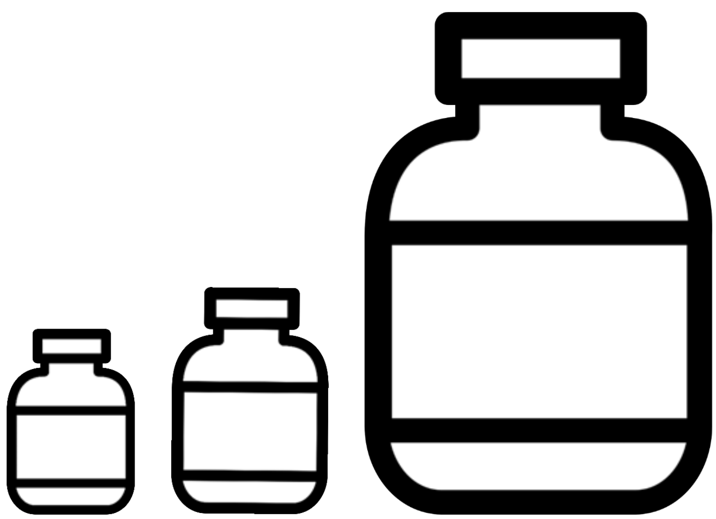
Therefore, the smallest possible integer with a digit product of 2160 is 5689.

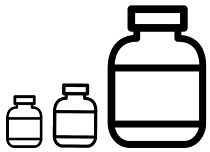
Problem of the Week
Problem D
Dye Refresher

To create an ink refresher for dye-based ink, some crafters will mix pure vegetable glycerine with water to get a mixture that is 12% vegetable glycerine, by volume. Kathy does not have pure vegetable glycerine, but she does have

- a 90 mL mixture that is 10.5% vegetable glycerine,
- a 120 mL mixture that is 30% vegetable glycerine, and
- a 1 L mixture that is 7.5% vegetable glycerine.

Since Kathy is a math teacher, she knows she can use the contents of these three mixtures to create a mixture that is 12% vegetable glycerine, by volume. She combines the contents of the entire 90 mL mixture with the contents of the entire 120 mL mixture, and then adds some of the 1 L mixture. How many millilitres of the 1 L mixture should she add to create a new mixture that is 12% vegetable glycerine, by volume?





Problem of the Week
Problem D and Solution
Dye Refresher

Problem

To create an ink refresher for dye-based ink, some crafters will mix pure vegetable glycerine with water to get a mixture that is 12% vegetable glycerine, by volume. Kathy does not have pure vegetable glycerine, but she does have

- a 90 mL mixture that is 10.5% vegetable glycerine,
- a 120 mL mixture that is 30% vegetable glycerine, and
- a 1 L mixture that is 7.5% vegetable glycerine.

Since Kathy is a math teacher, she knows she can use the contents of these three mixtures to create a mixture that is 12% vegetable glycerine, by volume. She combines the contents of the entire 90 mL mixture with the contents of the entire 120 mL mixture, and then adds some of the 1 L mixture. How many millilitres of the 1 L mixture should she add to create a new mixture that is 12% vegetable glycerine, by volume?

Solution

Let x be the number of millilitres needed from the 1 L mixture.

The 90 mL mixture that is 10.5% vegetable glycerine has $0.105 \times 90 = 9.45$ mL of vegetable glycerine.

The 120 mL mixture that is 30% vegetable glycerine has $0.30 \times 120 = 36$ mL of vegetable glycerine.

In the x mL from the 1 L mixture, there is $0.075 \times x = 0.075x$ mL of vegetable glycerine.

Therefore, the total amount of vegetable glycerine in the new mixture is
$9.45 + 36 + 0.075x = (45.45 + 0.075x)$ mL.

The new mixture contains $90 + 120 + x = (210 + x)$ mL of liquid, of which 12% is vegetable glycerine.
Therefore, $0.12 \times (210 + x) = (25.2 + 0.12x)$ mL of the new mixture is vegetable glycerine.

Since we have shown that the amount of vegetable glycerine in the new mixture is $(45.45 + 0.075x)$ mL and $(25.2 + 0.12x)$ mL, we must have

$$\begin{aligned} 45.45 + 0.075x &= 25.2 + 0.12x \\ 0.075x - 0.12x &= 25.2 - 45.45 \\ -0.045x &= -20.25 \\ x &= 450 \end{aligned}$$

Therefore, she should add 450 mL of the 1 L mixture that is 7.5% vegetable glycerine.

Problem of the Week

Problem D

Counting Ties

There are four intramural softball teams at a school, each named after local wildlife: Squirrels, Chipmunks, Raccoons, and Opossums.

At the end of the season, each team had played every other team exactly four times. A team earns 3 points for a win, 1 point for a tie, and no points for a loss. The total points earned for each team are as follows.

Team Name	Total Number of Points
Squirrels	12
Chipmunks	14
Raccoons	19
Opossums	22

How many of the games played in the season ended in a tie?

Problem of the Week
Problem D and Solution
Counting Ties

Problem

There are four intramural softball teams at a school, each named after local wildlife: Squirrels, Chipmunks, Raccoons, and Opossums. At the end of the season, each team had played every other team exactly four times. A team earns 3 points for a win, 1 point for a tie, and no points for a loss. The total points earned for each team are as follows.

Team Name	Total Number of Points
Squirrels	12
Chipmunks	14
Raccoons	19
Opossums	22

How many of the games played in the season ended in a tie?

Solution

Since each team played every other team four times, each team played $3 \times 4 = 12$ games. Since there are four teams, a total of $\frac{4\times 12}{2} = 24$ games were played. Note that we divided by 2 so that we don't double count games. For example, the Squirrels playing the Chipmunks is the same as the Chipmunks playing the Squirrels.

In games where one team won and one team lost, one team earned 3 points and the other team earned 0 points, so a total of 3 points were awarded. In games that ended in a tie, both teams earned 1 point, so a total of 2 points were awarded.

If there were no ties, then 24 games would result in $24 \times 3 = 72$ points being awarded in total. However, $12 + 14 + 19 + 22 = 67$ points were actually awarded in total. Since a total of 3 points were awarded when there was a win and a total of 2 points were awarded when there was a tie, each tie game adds one fewer point to the total number of points than a game where there was a win. It follows that every point below 72 must represent a tie game. Since $72 - 67 = 5$, there must have been 5 tie games.

Since 24 games were played, $24 - 5 = 19$ games resulted in a win. We should check that there is a combination of wins, ties and losses that satisfies the conditions in the problem. One possibility is shown below.

Team Name	Number of Wins	Number of Ties	Number of Losses	Total Points
Squirrels	3	3	6	12
Chipmunks	3	5	4	14
Raccoons	6	1	5	19
Opossums	7	1	4	22
TOTALS	19	10	19	67

In the table, there are a total of 10 ties. This means that 5 games ended in a tie and a total of 10 points were awarded for ties.

Extension: There are 5 other combinations of wins, ties and losses that satisfy the conditions of the problem. Can you find them all?

Problem of the Week

Problem D

Again and Again and Again

When the fraction $\frac{1}{70\,000\,000}$ is written as a decimal, which digit occurs in the 2023^{rd} place after the decimal point?

Problem of the Week
Problem D and Solution
Again and Again and Again

Problem

When the fraction $\frac{1}{70\,000\,000}$ is written as a decimal, which digit occurs in the 2023rd place after the decimal point?

Solution

Notice that $\frac{1}{70\,000\,000} = \frac{1}{10\,000\,000} \times \frac{1}{7} = 0.000\,000\,1 \times \frac{1}{7}$.

Also, note that $\frac{1}{7} = 0.\overline{142857}$. That is, when $\frac{1}{7}$ is written as a decimal, the digits after the decimal point occur in repeating blocks of the 6 digits 142857.

Therefore,
$\frac{1}{70\,000\,000} = 0.000\,000\,1 \times \frac{1}{7} = 0.000\,000\,1 \times 0.\overline{142\,857} = 0.000\,000\,0\overline{14\,285\,7}$.
That is, when $\frac{1}{70\,000\,000}$ is written as a decimal, the digits after the decimal point will be seven zeros followed by repeating blocks of the six digits 142857.

We see the decimal representation of $\frac{1}{70\,000\,000}$ has the same repetition as that for $\frac{1}{7}$, but the pattern is shifted over 7 places. Since $2023 - 7 = 2016$, the 2023rd digit after the decimal point when $\frac{1}{70\,000\,000}$ is written as a decimal is the same as the 2016th digit after the decimal point when $\frac{1}{7}$ is written as a decimal.

Since $\frac{2016}{6} = 336$, then the 2016th digit after the decimal point occurs after exactly 336 repeating blocks of the 6 digits 142857. Therefore, the 2016th digit is the last digit in the repeating block, which is 7.

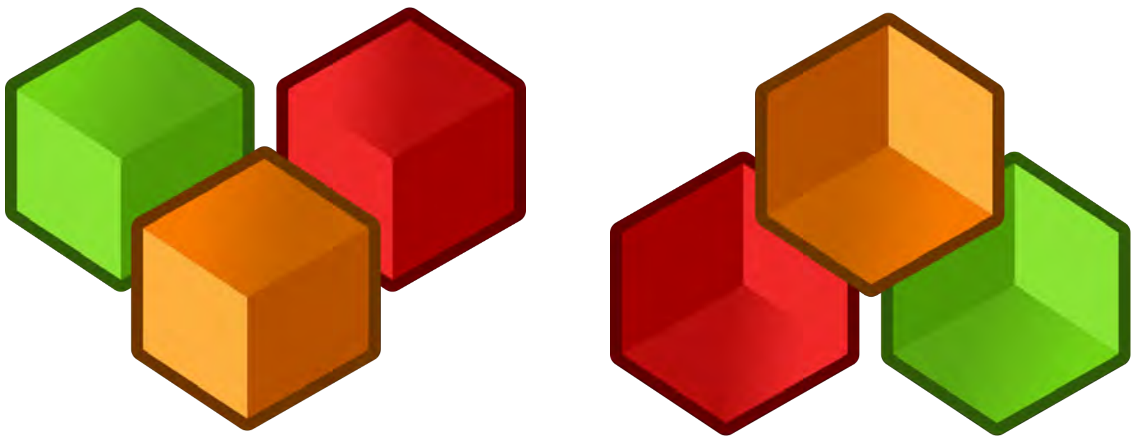
The 2023rd digit after the decimal point in the decimal representation of $\frac{1}{70\,000\,000}$ is the same as the 2016th digit after the decimal point in the decimal representation of $\frac{1}{7}$, and is therefore 7.

Problem of the Week

Problem D

Caen's Cubes

Caen has a cube with a volume of n cm^3. They cut this cube into n smaller cubes, each with a side length of 1 cm. The total surface area of the n smaller cubes is ten times the surface area of Caen's original cube. Determine the side length of Caen's original cube.



Problem of the Week

Problem D and Solution

Caen's Cubes

Problem

Caen has a cube with a volume of n cm^3. They cut this cube into n smaller cubes, each with a side length of 1 cm. The total surface area of the n smaller cubes is ten times the surface area of Caen's original cube. Determine the side length of Caen's original cube.

Solution

Let the side length of Caen's original cube be x cm, where $x > 0$. It follows that $n = x^3$.

Each of the six sides of Caen's original cube has area x^2 cm^2, so the total surface area of the original cube is $6x^2$ cm^2.

Consider one of the smaller cubes. The area of one the six faces is 1 cm^2. So, the surface area of one of these smaller cubes is 6 cm^2. Thus, the total surface area of the n smaller cubes is $6n$ cm^2.

Since the total surface area of the n cubes is ten times the surface area of Caen's original cube, we have

$$6n = 10(6x^2)$$

Dividing both sides by 6, we have

$$n = 10x^2$$

But $n = x^3$, so this tells us that

$$x^3 = 10x^2$$

Since $x > 0$, we have $x^2 > 0$. Dividing both sides by x^2, we find that $x = 10$.

Therefore, the side length of Caen's original cube was 10 cm.

Extension:

If the combined surface area of the n cubes with a side length of 1 cm was Q times the surface area of the original uncut cube, then the side length of the original uncut cube would have been Q cm. Can you see why?

Problem of the Week

Problem D

Two Birds

Katya owns two cockatoos, an older white cockatoo and a younger Galah cockatoo. At present, the sum of the cockatoos' ages is 44 years. In n years, where $n > 0$, the white cockatoo's age will be four times the Galah cockatoo's age. If n is an integer, determine the possible present ages of each cockatoo.

Problem of the Week
Problem D and Solution
Two Birds

Problem

Katya owns two cockatoos, an older white cockatoo and a younger Galah cockatoo. At present, the sum of the cockatoos' ages is 44 years. In n years, where $n > 0$, the white cockatoo's age will be four times the Galah cockatoo's age. If n is an integer, determine the possible present ages of each cockatoo.

Solution

Let g represent the present age of the Galah cockatoo and w represent the present age of the white cockatoo. Since the sum of their present ages is 44, we have $g + w = 44$ or $w = 44 - g$.

In n years, the Galah cockatoo will be $(g + n)$ years old and the white cockatoo will be $(44 - g + n)$ years old. At that time the white cockatoo will be four times older than the Galah cockatoo. Therefore,

$$\begin{aligned} 4(g+n) &= 44 - g + n \\ 4g + 4n &= 44 - g + n \\ 5g + 3n &= 44 \\ g &= \frac{44 - 3n}{5} \end{aligned}$$

We are looking for integer values of n so that $44 - 3n$ is divisible by 5.

When $n = 3$, $g = \frac{44-3n}{5} = \frac{44-3(3)}{5} = \frac{35}{5} = 7$. When $g = 7$, $w = 44 - g = 44 - 7 = 37$.

When $n = 8$, $g = \frac{44-3n}{5} = \frac{44-3(8)}{5} = \frac{20}{5} = 4$. When $g = 4$, $w = 44 - g = 44 - 4 = 40$.

When $n = 13$, $g = \frac{44-3n}{5} = \frac{44-3(13)}{5} = \frac{5}{5} = 1$. When $g = 1$, $w = 44 - g = 44 - 1 = 43$.

When $n = 18$, $g = \frac{44-3n}{5} = \frac{44-3(18)}{5} = \frac{-10}{5} = -2$. Since $g < 0$, $n = 16$ does not produce a valid age for the Galah cockatoo. No higher value of n would produce a value of $g > 0$.

No integer values of n between 0 and 18, other than 3, 8, and 13, produce a multiple of 5 when substituted into $44 - 3n$.

If today the white cockatoo is 37 and the Galah cockatoo is 7, then in 3 years the white cockatoo will be 40 and the Galah cockatoo will be 10. The white cockatoo will be four times older than the Galah cockatoo since $4 \times 10 = 40$.

If today the white cockatoo is 40 and the Galah cockatoo is 4, then in 8 years the white cockatoo will be 48 and the Galah cockatoo will be 12. The white cockatoo will be four times older than the Galah cockatoo since $4 \times 12 = 48$.

If today the white cockatoo is 43 and the Galah cockatoo is 1, then in 13 years the white cockatoo will be 56 and the Galah cockatoo will be 14. The white cockatoo will be four times older than the Galah cockatoo since $4 \times 14 = 56$.

Therefore, the possible present ages for the white cockatoo and Galah cockatoo are 37 and 7, or 40 and 4, or 43 and 1.

Problem of the Week
Problem D
Missing the Fives II

Bobbi lists the positive integers, in order, excluding all multiples of 5. Her resulting list is

$$1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, \ldots$$

If the nth integer in Bobbi's list is 2023, what is the value of n?

Problem of the Week
Problem D and Solution
Missing the Fives II

Problem

Bobbi lists the positive integers, in order, excluding all multiples of 5. Her resulting list is

$$1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, \ldots$$

If the nth integer in Bobbi's list is 2023, what is the value of n?

Solution

Solution 1

Note that 2023 is two integers before 2025, which is a multiple of 5.
Beginning at 1, 2025 is the 405th multiple of 5, since $\frac{2025}{5} = 405$. That is, the integers from 1 to 2025 contain 405 groups of 5 integers.
Each of these 405 groups contain one integer that is a multiple of 5, and so Bobbi leaves out 406 integers (including 2024) in the list of all integers from 1 to 2025.
If the nth integer in Bobbi's list is 2023, then $n = 2025 - 406 = 1619$.

Solution 2

Note that 2023 is two integers before 2025, which is a multiple of 5.
Beginning at 1, 2025 is the 405th multiple of 5, since $\frac{2025}{5} = 405$. That is, the integers from 1 to 2025 contain 405 groups of 5 integers.
In Bobbi's list of integers, she leaves out the integers that are multiples of 5, and so in every group of five integers, Bobbi lists four of these integers. However, she also does not list 2024. Thus, if the nth integer in Bobbi's list is 2023, then
$n = 405 \times 4 - 1 = 1619$.

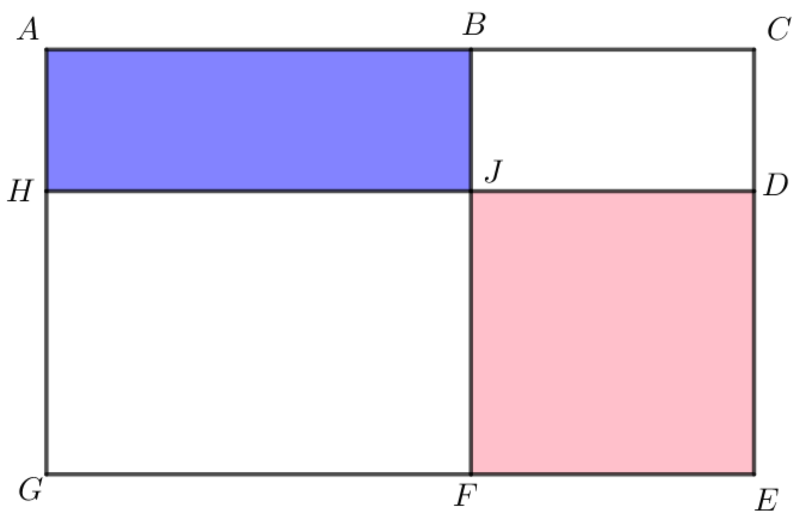
Problem of the Week

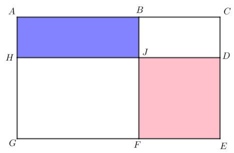
Problem D

Find the Largest Area

Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$. Note that the diagram is just an illustration and is not intended to be to scale.





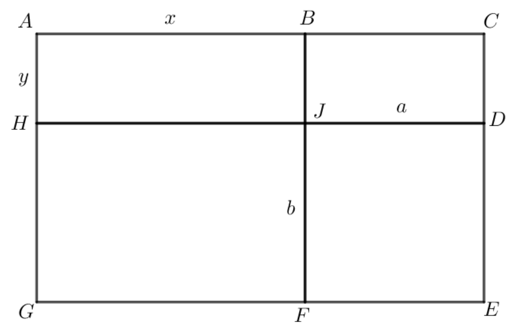
Problem of the Week
Problem D and Solution
Find the Largest Area

Problem

Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$.

Solution



Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$. Then,

$$\begin{aligned} HJ &= GF = AB = x \\ BJ &= CD = AH = y \\ BC &= FE = JD = a \\ HG &= DE = JF = b \end{aligned}$$

Thus, we have

$$\begin{aligned} \text{area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG) \\ &= 6 + ya + 15 + xb \\ &= 21 + ya + xb \end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

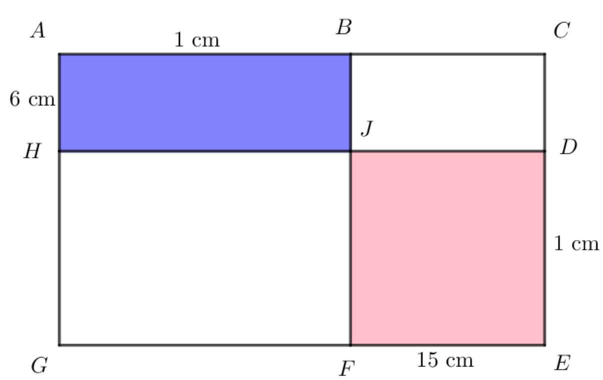
Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

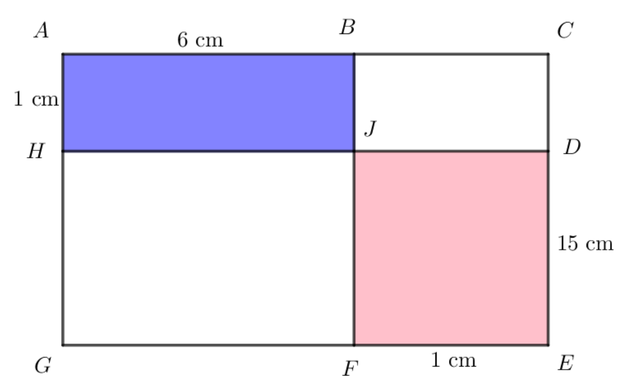
To maximize the area, we need to pick the values of x, y, a, and b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

- **Case 1:** $x = 1$ cm, $y = 6$ cm, $a = 1$ cm, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42$ cm^2.
- **Case 2:** $x = 1$ cm, $y = 6$ cm, $a = 15$ cm, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112$ cm^2.
- **Case 3:** $x = 1$ cm, $y = 6$ cm, $a = 3$ cm, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44$ cm^2.
- **Case 4:** $x = 1$ cm, $y = 6$ cm, $a = 5$ cm, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54$ cm^2.
- **Case 5:** $x = 2$ cm, $y = 3$ cm, $a = 1$, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54$ cm^2.
- **Case 6:** $x = 2$ cm, $y = 3$ cm, $a = 15$, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68$ cm^2.
- **Case 7:** $x = 2$ cm, $y = 3$ cm, $a = 3$, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40$ cm^2.
- **Case 8:** $x = 2$ cm, $y = 3$ cm, $a = 5$, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42$ cm^2.

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram given in the problem was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.

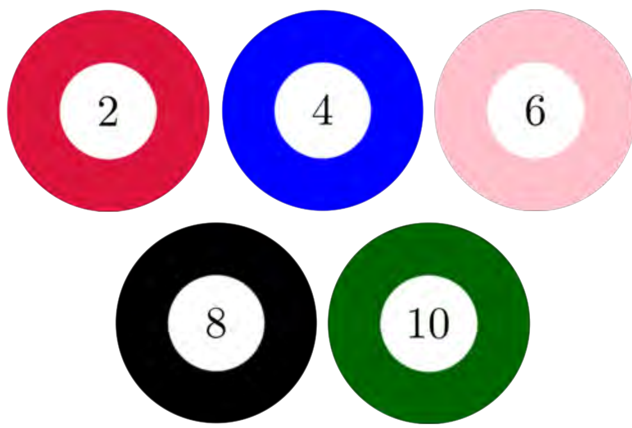


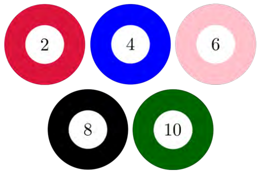


Problem of the Week
Problem D
No Power

Five balls are placed in a bag. Each ball is labelled with a 2, 4, 6, 8, or 10, with no ball having the same label as any other. Adeleke randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Then Bo randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Finally, Carlos randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag.

Determine the probability that the product of the three recorded integers is not a power of 2.





Problem of the Week
Problem D and Solution
No Power

Problem
Five balls are placed in a bag. Each ball is labelled with a 2, 4, 6, 8, or 10, with no ball having the same label as any other. Adeleke randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Then Bo randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Finally, Carlos randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Determine the probability that the product of the three recorded integers is not a power of 2.

Solution
Solution 1

One way to solve this problem is to list out all of the possible choices, calculate the product for each choice, and then count the number of products that are not a power of 2. If we did so, we would find that there are 125 possible choices. Of these, 98 result in a product that is not a power of 2. Therefore, the probability that the product is not a product of 2 is $\frac{98}{125}$. In Solutions 2 and 3, we will see more efficient ways to calculate this probability.

Solution 2

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2. So, to determine the number of choices that result in a product that is not a power of 2, we will count the number of choices that result in a product that is a power of 2, and subtract this from the total number of choices.

Since Adeleke, Bo, and Carlos each have five possible integers they can choose, there are $5 \times 5 \times 5 = 125$ possible choices of integers. For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three chosen integers must be a power of 2. There are three balls labelled with a power of 2, namely, 2, 4, and 8. Therefore, the number of choices that result in a power of 2 is $3 \times 3 \times 3 = 27$.

Since there are 27 choices that give a product that is a power of 2, there must be $125 - 27 = 98$ choices that give a product that is not a power of 2. Therefore, the probability that the product is not a power of 2 is $\frac{98}{125}$.

Solution 3

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2. If p is the probability that the product is a power of 2 and q is the probability that the product is not a power of 2, then $p + q = 1$. Therefore, we can calculate q by calculating p and noting that $q = 1 - p$.

For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three integers must be a power of 2. There are three balls labelled with a power of 2, namely, 2, 4, and 8. Thus, the probability of randomly choosing a ball with a label that is power of 2 is $\frac{3}{5}$. Since Adeleke, Bo, and Carlos choose their integers independently, then the probability that each chooses a power of 2 is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$. In other words, $p = \frac{27}{125}$, and so $q = 1 - p = 1 - \frac{27}{125} = \frac{98}{125}$. Therefore, the probability that the product is not a power of 2 is $\frac{98}{125}$.

Problem of the Week

Problem D

Layover Between the Trips

A plane travels from Calgary, AB to Grande Prairie, AB. The total flight time, including takeoff and landing, is 1 hour and 40 minutes. The return flight takes the same route and time. The average speed for these two flights is 500 km/h.

After a brief layover in Grande Prairie, the average speed of this entire round trip (including the two flights and the layover in between) becomes 425 km/h. How long was the layover?

Problem of the Week

Problem D and Solution

Layover Between the Trips

Problem

A plane travels from Calgary, AB to Grande Prairie, AB. The total flight time, including takeoff and landing, is 1 hour and 40 minutes. The return flight takes the same route and time. The average speed for these two flights is 500 km/h.

After a brief layover in Grande Prairie, the average speed of this entire round trip (including the two flights and the layover in between) becomes 425 km/h. How long was the layover?

Solution

Let t be the length of the layover, in hours.

The plane travels from Calgary to Grande Prairie in 1 hour 40 minutes at a speed of 500 km/h. Using the formula distance = speed × time, the distance from Calgary to Grande Prairie must be $500\frac{\text{km}}{\text{h}} \times 1\frac{2}{3}\text{ h} = 500 \times \frac{5}{3} = \frac{2500}{3}$ km.

Therefore, for the two-way trip, the plane travels $2 \times \frac{2500}{3} = \frac{5000}{3}$ km.

The length of time of the entire two-way trip is the time of the two flights plus the layover time. Therefore, the total length of time of the trip is $\frac{5}{3} + \frac{5}{3} + t = \frac{10}{3} + t$ hours.

Since the average speed of the entire two-way trip is 425 km/h, using the formula distance = speed × time, we have

$$\begin{aligned}
\frac{5000}{3} &= 425 \times \left(\frac{10}{3} + t\right) \\
\frac{10}{3} + t &= \frac{5000}{3 \times 425} \\
t &= \frac{200}{51} - \frac{10}{3} \\
&= \frac{200}{51} - \frac{170}{51} \\
&= \frac{10}{17}
\end{aligned}$$

Therefore, the layover was $\frac{10}{17}$ hours, or approximately 35 minutes.

Problem of the Week
Problem D
What's in That Square?

Fourteen squares are placed in a row forming the grid below. Each square is to be filled with a positive integer, according to the following rules.

1. The product of any four integers in adjacent squares is 120.
2. Integers may appear more than once in the grid.

Four of the squares are already filled with a positive integer, as shown. Determine all possible values of x.

		2			4			x			3		

Problem of the Week
Problem D and Solution
What's in That Square?

Problem

Fourteen squares are placed in a row forming the grid below. Each square is to be filled with a positive integer, according to the following rules.

1. The product of any four integers in adjacent squares is 120.
2. Integers may appear more than once in the grid.

Four of the squares are already filled with a positive integer, as shown. Determine all possible values of x.

		2			4			x			3		

Solution

In both solutions, let a_1 be the positive integer in the first square, a_2 the positive integer in the second square, a_3 be the positive integer in the third square, a_4 the positive integer in the fourth square, and so on.

Solution 1

Consider squares 3 to 6. Since the product of any four adjacent integers is 120, we have $2 \times a_4 \times a_5 \times 4 = 120$. Therefore, $a_4 \times a_5 = \frac{120}{2\times 4} = 15$. Since a_4 and a_5 are positive integers, there are four possibilities: $a_4 = 1$ and $a_5 = 15$, or $a_4 = 15$ and $a_5 = 1$, or $a_4 = 3$ and $a_5 = 5$, or $a_4 = 5$ and $a_5 = 3$.

In each of the four cases, we will have $a_7 = 2$. We can see why by considering squares 4 to 7. We have $a_4 \times a_5 \times 4 \times a_7 = 120$, or $15 \times 4 \times a_7 = 120$, since $a_4 \times a_5 = 15$. Therefore, $a_7 = \frac{120}{15\times 4} = 2$.

- Case 1: $a_4 = 1$ and $a_5 = 15$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $15 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{15\times 4\times 2} = 1$.
  Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $4 \times 2 \times 1 \times x = 120$, or $x = \frac{120}{4\times 2} = 15$.
  Let's check that $x = 15$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
  Consider squares 9 to 12. If $x = 15$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{15\times 3} = \frac{8}{3}$. But a_{10} and a_{11} must both be integers, so is not possible for $a_{10} \times a_{11} = \frac{8}{3}$. Therefore, it must not be possible for $a_4 = 1$ and $a_5 = 15$, and so we find that there is no solution for x in this case.

- Case 2: $a_4 = 15$ and $a_5 = 1$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $1 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{4\times 2} = 15$.

Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times15} = 1$.
Let's check that $x = 1$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
Consider squares 7 to 10. Since $a_7 = 2$, $a_8 = 15$, and $x = 1$, then $a_{10} = \frac{120}{2\times15\times1} = 4$.
Similarly, $a_{11} = \frac{120}{15\times1\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 1 \times 4 \times 2 \times 3 = 24 \neq 120$.
Therefore, it is not possible for $a_4 = 15$ and $a_5 = 1$. There is no solution for x in this case.

- Case 3: $a_4 = 3$ and $a_5 = 5$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $5 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{5\times4\times2} = 3$.
  Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times3} = 5$.
  Let's check that $x = 5$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
  Consider squares 7 to 10. Since $a_7 = 2$, $a_8 = 3$, and $x = 5$, then $a_{10} = \frac{120}{2\times3\times5} = 4$.
  Similarly, $a_{11} = \frac{120}{3\times5\times4} = 2$. Then we have $x \times a_{10} \times a_{11} \times a_{12} = 5 \times 4 \times 2 \times a_{12} = 120$, so $a_{12} = \frac{120}{5\times4\times2} = 3$. Therefore, the condition that $a_{12} = 3$ is satisfied in the case where $a_4 = 3$ and $a_5 = 5$. If we continue to fill out the entries in the squares, we obtain the entries shown in the diagram below.

5	4	2	3	5	4	2	3	5	4	2	3	5	4

  We see that $x = 5$ is a possible solution. However, is it the only solution? We have one final case to check.

- Case 4: $a_4 = 5$ and $a_5 = 3$
  Consider squares 5 to 8. We have $a_5 \times 4 \times a_7 \times a_8 = 120$, or $3 \times 4 \times 2 \times a_8 = 120$, or $a_8 = \frac{120}{3\times4\times2} = 5$.
  Next, consider squares 6 to 9. We have $4 \times a_7 \times a_8 \times x = 120$, or $x = \frac{120}{4\times2\times5} = 3$.
  Let's check that $x = 3$ satisfies the only other condition in the problem that we have not yet used, that is $a_{12} = 3$.
  Consider squares 9 to 12. If $x = 3$ and $a_{12} = 3$, then $a_{10} \times a_{11} = \frac{120}{3\times3} = \frac{40}{3}$. But a_{10} and a_{11} must both be integers, so it is not possible for $a_{10} \times a_{11} = \frac{40}{3}$. Therefore, it must not be possible for $a_4 = 5$ and $a_5 = 3$, and so we find that there is no solution for x in this case.

Therefore, the only possible value for x is $x = 5$.

Solution 2

You may have noticed a pattern for the a_i's in Solution 1. We will explore this pattern.

Since the product of any four adjacent integers is 120, $a_1a_2a_3a_4 = a_2a_3a_4a_5 = 120$. Since both sides are divisible by $a_2a_3a_4$, and each is a positive integer, then $a_1 = a_5$.
Similarly, $a_2a_3a_4a_5 = a_3a_4a_5a_6 = 120$, and so $a_2 = a_6$.
In general, $a_na_{n+1}a_{n+2}a_{n+3} = a_{n+1}a_{n+2}a_{n+3}a_{n+4}$, and so $a_n = a_{n+4}$.
We can use this along with the given information to fill out the entries in the squares as follows:

x	4	2	3	x	4	2	3	x	4	2	3	x	4

Therefore, $4 \times 2 \times 3 \times x = 120$ and so $x = \frac{120}{4\times2\times3} = 5$.

Problem of the Week

Problem D

How Many Fives?

The product of the first seven positive integers is equal to

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Mathematicians will write this product as 7!. This is read as "7 factorial". So, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

This factorial notation can be used with any positive integer. For example, $11! = 11 \times 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 39\,916\,800$. The three dots "$\cdots$" represent the product of the integers between 9 and 3.

Suppose $N = 1000!$. That is,

$$N = 1000! = 1000 \times 999 \times 998 \times 997 \times \cdots \times 3 \times 2 \times 1$$

Note that N is divisible by 5, 25, 125, and 625. Each of these factors is a power of 5. That is, $5 = 5^1$, $25 = 5^2$, $125 = 5^3$, and $625 = 5^4$.

Determine the largest power of 5 that divides N.

$5^{??}$

Problem of the Week
Problem D and Solution
How Many Fives?

Problem

The product of the first seven positive integers is equal to

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Mathematicians will write this product as 7!. This is read as "7 factorial". So, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

This factorial notation can be used with any positive integer. For example, $11! = 11 \times 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 39\,916\,800$. The three dots "$\cdots$" represent the product of the integers between 9 and 3.

Suppose $N = 1000!$. That is,

$$N = 1000! = 1000 \times 999 \times 998 \times 997 \times \cdots \times 3 \times 2 \times 1$$

Note that N is divisible by 5, 25, 125, and 625. Each of these factors is a power of 5. That is, $5 = 5^1$, $25 = 5^2$, $125 = 5^3$, and $625 = 5^4$.

Determine the largest power of 5 that divides N.

Solution

Solution 1

In order to determine the largest power of 5 that divides N, we need to count the number of times the factor 5 appears in the prime factorization of N.

Since N is equal to the product of the integers from 1 to 1000, let's first look at which of these integers are divisible by 5. The integers from 1 to 1000 that are divisible by 5 are $5, 10, 15, 20, \ldots, 990, 995, 1000$. That is, a total of $\frac{1000}{5} = 200$ integers from 1 to 1000 are divisible by 5.

Each integer that is a multiple of 25 will add an additional factor of 5, since $25 = 5 \times 5$. There are $\frac{1000}{25} = 40$ integers from 1 to 1000 that are multiples of 25. These integers give another 40 factors of 5 bringing the total to $200 + 40 = 240$.

Each integer that is a multiple of 125 will add an additional factor of 5. This is because $125 = 5 \times 5 \times 5$, and two of the factors have already been counted when we looked at 5 and 25. There are $\frac{1000}{125} = 8$ integers from 1 to 1000 that are multiples of 125. These integers give another 8 factors of 5 bringing the total to $240 + 8 = 248$.

Each integer that is a multiple of 625 will add an additional factor of 5. This is because $625 = 5 \times 5 \times 5 \times 5$, and three of the factors have already been counted when we looked at 5, 25 and 125. There is 1 integer from 1 to 1000 that is a multiple of 625, namely, 625. This integer gives another factor of 5 bringing the total to $248 + 1 = 249$.

The next power of 5 is $5^5 = 3125 > 1000$, so we have counted all factors of 5 in 1000!.

Thus, the prime factorization of N contains exactly 249 factors of 5. Therefore, the largest power of 5 that divides N is 5^{249}.

Solution 2

There are many similarities between Solution 1 and the following solution. In this solution we will divide out factors of 5 until there are none left.

1. In the integers from 1 to 1000, there are $\frac{1000}{5} = 200$ integers that are divisible by 5, namely, $5, 10, 15, \ldots, 990, 995, 1000$. If we divide each of these integers by 5, we obtain the second list $1, 2, 3, \ldots, 198, 199, 200$.

2. This second list contains $\frac{200}{5} = 40$ integers that are divisible by 5, namely, $5, 10, 15, \ldots, 190, 195, 200$. If we divide each of these integers by 5, we obtain the third list $1, 2, 3, \ldots, 38, 39, 40$.

3. This third list contains $\frac{40}{5} = 8$ integers that are divisible by 5, namely, $5, 10, 15, 20, 25, 30, 35, 40$. If we divide each of these integers by 5, we obtain the fourth list $1, 2, 3, 4, 5, 6, 7, 8$.

4. This fourth list contains 1 integer that is divisible by 5, namely the integer 5.

In total, there are $200 + 40 + 8 + 1 = 249$ factors of 5 in $1000!$. Therefore, the largest power of 5 that divides N is 5^{249}.

An interpretation of what has happened is in order. When we created the first list of multiples of 5, we discovered that there were 200 integers from 1 to 1000 that are divisible by 5. When we created the second list of multiples of 5, we were actually counting the 40 integers from 1 to 1000 that are divisible by 25. When we created the third list of multiples of 5, we were actually counting the 8 integers from 1 to 1000 that are divisible by 125. And finally, when we created the fourth list of multiples of 5, we were actually counting the 1 integer from 1 to 1000 that is divisible by 625.

Problem of the Week
Problem D
All Mixed Up

A large bowl contains a mixture of Himalayan Pink Salt and common salt. When 1 kg of common salt is added to the bowl, the ratio, by mass, of Himalayan Pink Salt to common salt becomes $1 : 2$. When 1 kg of Himalayan Pink Salt is added to the new mixture, the ratio becomes $2 : 3$. Find the ratio of Himalayan Pink Salt to common salt in the original mixture.

Problem of the Week
Problem D and Solution
All Mixed Up

Problem
A large bowl contains a mixture of Himalayan Pink Salt and common salt. When 1 kg of common salt is added to the bowl, the ratio, by mass, of Himalayan Pink Salt to common salt becomes 1 : 2. When 1 kg of Himalayan Pink Salt is added to the new mixture, the ratio becomes 2 : 3. Find the ratio of Himalayan Pink Salt to common salt in the original mixture.

Solution

Let h be the amount of Himalayan Pink Salt, in kgs, in the original mixture.
Let c be the amount of common salt, in kgs, in the original mixture.

When 1 kg of common salt is added, the ratio of Himalayan Pink Salt to common salt is 1 : 2. Therefore,

$$\frac{h}{c+1} = \frac{1}{2}$$

Simplifying, we obtain $c + 1 = 2h$ and $c = 2h - 1$ follows.

When 1 kg of Himalayan Pink Salt is added to the new mixture, the ratio becomes 2 : 3. Therefore,

$$\frac{h+1}{c+1} = \frac{2}{3}$$

Since $c = 2h - 1$, we have

$$\begin{aligned}
\frac{h+1}{(2h-1)+1} &= \frac{2}{3} \\
\frac{h+1}{2h} &= \frac{2}{3} \\
2(2h) &= 3(h+1) \\
4h &= 3h + 3 \\
h &= 3
\end{aligned}$$

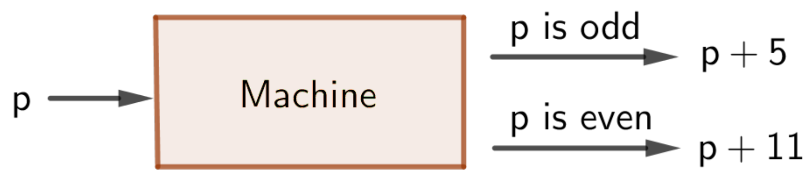
Substituting $h = 3$ in $c = 2h - 1$, we obtain $c = 2(3) - 1 = 5$.

Therefore, there was originally 3 kgs of Himalayan Pink Salt in the bowl and 5 kgs of common salt. Thus, the ratio of Himalayan Pink Salt to common salt in the original mixture was 3 : 5.

Problem of the Week
Problem D
Machine Math

A positive integer p is input into a machine. If p is odd, the machine outputs the integer $p + 5$. If p is even, the machine outputs the integer $p + 11$. This process can be repeated using each successive output as the next input. For example, if the input is $p = 1$ and the machine is used three times, the final output is 22.

If the input is $p = 2023$ and the machine is used 101 times, find the final output.

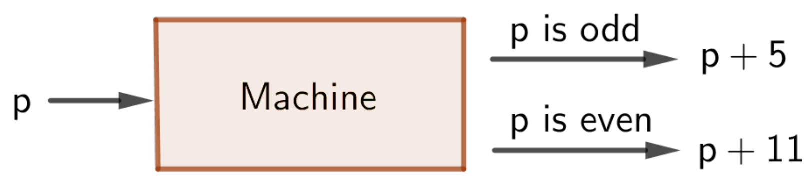


Problem of the Week
Problem D and Solution
Machine Math

Problem

A positive integer p is input into a machine. If p is odd, the machine outputs the integer $p + 5$. If p is even, the machine outputs the integer $p + 11$. This process can be repeated using each successive output as the next input. For example, if the input is $p = 1$ and the machine is used three times, the final output is 22.

If the input is $p = 2023$ and the machine is used 101 times, find the final output.



Solution

If p is odd, the output is $p + 5$, which is even because it is the sum of two odd integers. If p is even, the output is $p + 11$, which is odd, because it is the sum of an even integer and an odd integer.

Starting with $p = 2023$ and using the machine 2 times, we obtain $2023 + 5 = 2028$ and then $2028 + 11 = 2039$.

Starting with 2039 and using the machine 2 times, we obtain $2039 + 5 = 2044$ and then $2044 + 11 = 2055$.

Starting with an odd integer and using the machine 2 times, the net result is always adding 16 to the input, because the odd input generates a first output that is 5 larger (and so even) and a second output that is 11 larger than the first output. This generates a net result that is $5 + 11 = 16$ larger than the input.

Therefore, using the machine 96 more times (that is, repeating the 2 steps a total of 48 more times) we add 16 a total of 48 more times to obtain the output $2055 + 48 \times 16 = 2823$. To this point, the machine has been used 100 times.

The next time the machine is used, the output is $2823 + 5 = 2828$.

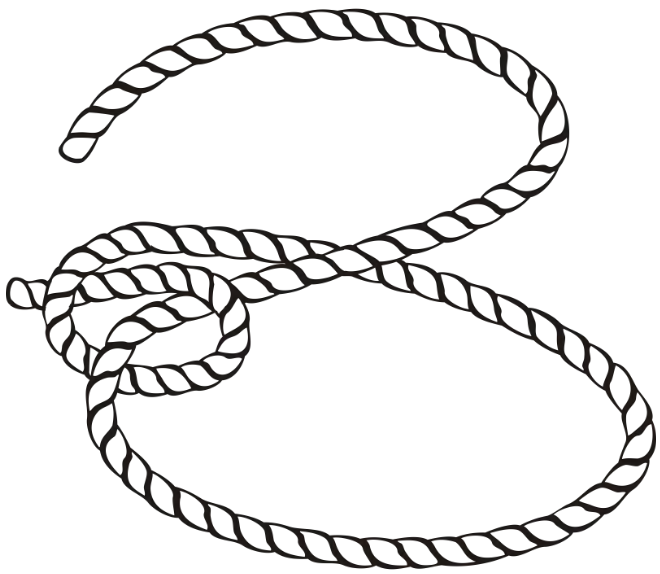
Thus, the final output after the machine is used 101 times is 2828.

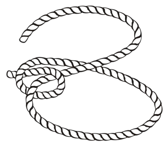
Geometry & Measurement (G)

TAKE ME TO THE COVER

Problem of the Week
Problem D
Three Triangles and a Square

Simon has a rope that is 200 cm long. They cut the rope into four pieces so that one piece can be arranged, with its two ends touching, to form a square, and the three remaining pieces can be arranged, with each having its two ends touching, to form three identical equilateral triangles. If all four shapes have integer side lengths, in cm, determine all possibilities for the side lengths of each triangle and the square.





Problem of the Week

Problem D and Solution

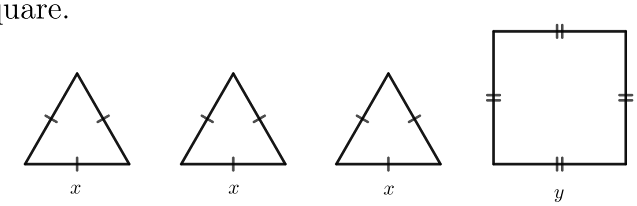
Three Triangles and a Square

Problem

Simon has a rope that is 200 cm long. They cut the rope into four pieces so that one piece can be arranged, with its two ends touching, to form a square, and the three remaining pieces can be arranged, with each having its two ends touching, to form three identical equilateral triangles. If all four shapes have integer side lengths, in cm, determine all possibilities for the side lengths of each triangle and the square.

Solution

Let x represent the side length, in cm, of each equilateral triangle and let y represent the side length, in cm, of the square.



The perimeter of each figure is the length of the piece of rope used to form it. For each triangle, the length of rope is $3x$ and for the square the length of rope is $4y$. The total rope used is $3(3x) + 4y = 9x + 4y$. But the length of the rope is 200 cm. Therefore,

$$\begin{aligned} 9x + 4y &= 200 \\ 9x &= 200 - 4y \\ x &= \frac{4(50-y)}{9} \end{aligned}$$

We are given that x and y are integers, and the information given in the problem implies x and y must be positive. Since both x and y are integers, $4(50 - y)$ must be a multiple of 9. But 4 is not divisible by 9, so $50 - y$ must be divisible by 9. There are five positive multiples of 9 between 0 and 50, namely 9, 18, 27, 36, and 45. So $50 - y$ must be equal to 9, 18, 27, 36, or 45. It follows that y is equal to 41, 32, 23, 14, or 5. The corresponding values of x are computed in the table below.

y	$4y$	$200-4y$	$x = \frac{200-4y}{9}$
41	164	36	4
32	128	72	8
23	92	108	12
14	56	144	16
5	20	180	20

Thus, there are 5 possibilities. When the side length of the square is 41 cm, the side length of each triangle is 4 cm; when the side length of the square is 32 cm, the side length of each triangle is 8 cm; when the side length of the square is 23 cm, the side length of each triangle is 12 cm; when the side length of the square is 14 cm, the side length of each triangle is 16 cm; and when the side length of the square is 5 cm, the side length of each triangle is 20 cm.

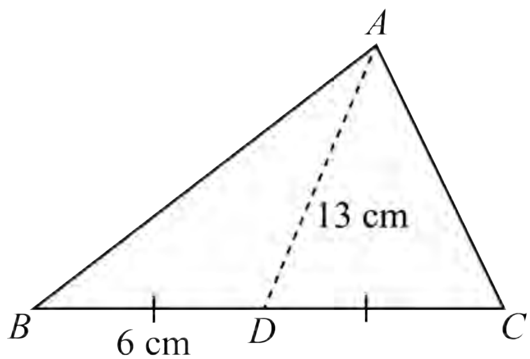
Problem of the Week

Problem D

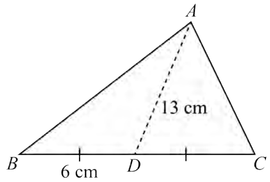
There are Two Sides

A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle ABC$, a median is drawn from vertex A, meeting side BC at point D. The length of BD is 6 cm and the length of the median AD is 13 cm.



The area of $\triangle ABC$ is 72 cm^2. Determine the lengths of sides AB and AC.



Problem of the Week

Problem D and Solution

There are Two Sides

Problem

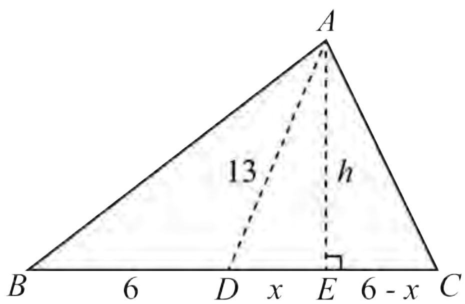
A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle ABC$, a median is drawn from vertex A, meeting side BC at point D. The length of BD is 6 cm and the length of the median AD is 13 cm.

The area of $\triangle ABC$ is 72 cm^2. Determine the lengths of sides AB and AC.

Solution

First we will draw the altitude from vertex A, meeting side BC at point E. Let h be the length of the altitude AE. Let x be the length of DE. Since AD is a median, $DC = BD = 6$. Since E is on DC and the length of DE is x, the length of EC is $6 - x$.



We know the area of $\triangle ABC$ is 72 cm^2. Also, since $BD = DC = 6$ cm, it follows that $BC = 12$ cm. Thus,

$$\begin{aligned}\frac{BC \times AE}{2} &= 72\\ \frac{12h}{2} &= 72\\ h &= 12\end{aligned}$$

Since $\triangle AED$ is right-angled, we can use the Pythagorean Theorem as follows.

$$\begin{aligned}DE^2 + AE^2 &= AD^2\\ x^2 + 12^2 &= 13^2\\ x^2 &= 13^2 - 12^2\\ x^2 &= 169 - 144 = 25\end{aligned}$$

Since $x > 0$, it follows that $x = 5$ cm. Thus, $BE = 6 + x = 6 + 5 = 11$ cm, and $EC = 6 - x = 6 - 5 = 1$ cm.

Since $\triangle AEB$ is right-angled, we can use the Pythagorean Theorem as follows.

$$\begin{aligned} AE^2 + BE^2 &= AB^2 \\ 12^2 + 11^2 &= AB^2 \\ AB^2 &= 144 + 121 = 265 \end{aligned}$$

Since $AB > 0$, it follows that $AB = \sqrt{265}$ cm.

Since $\triangle AEC$ is right-angled, we can use the Pythagorean Theorem as follows.

$$\begin{aligned} AE^2 + EC^2 &= AC^2 \\ 12^2 + 1^2 &= AC^2 \\ AC^2 &= 144 + 1 = 145 \end{aligned}$$

Since $AC > 0$, it follows that $AC = \sqrt{145}$ cm.

Therefore, the lengths of sides AB and AC are $\sqrt{265}$ cm and $\sqrt{145}$ cm, respectively. These are approximately equal to 16.3 cm and 12.0 cm.

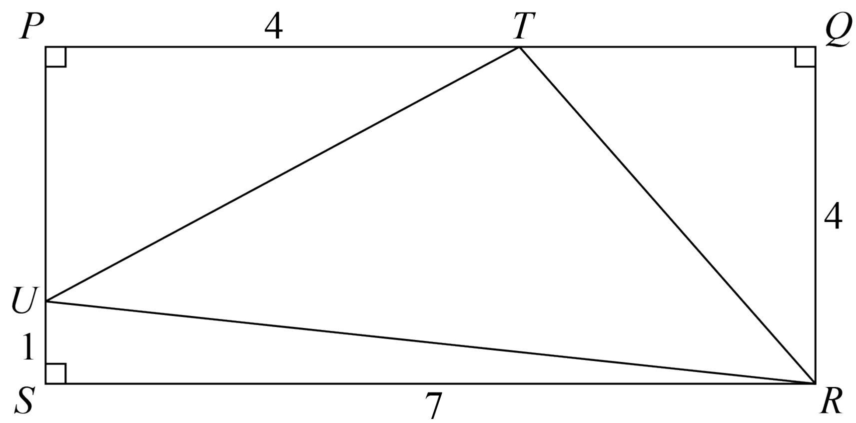
Problem of the Week

Problem D

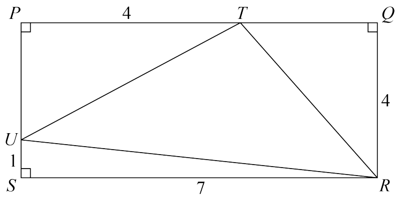
Many Ways to Get There

Rectangle $PQRS$ has $QR = 4$ and $RS = 7$. $\triangle TRU$ is inscribed in rectangle $PQRS$ with T on PQ such that $PT = 4$, and U on PS such that $SU = 1$.

Determine the value of $\angle RUS + \angle PUT$.



There are many ways to solve this problem. After you have solved it, see if you can solve it a different way.



Problem of the Week

Problem D and Solution

Many Ways to Get There

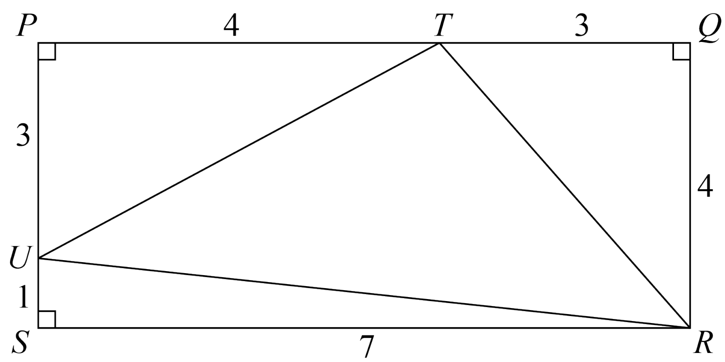
Problem

Rectangle $PQRS$ has $QR = 4$ and $RS = 7$. $\triangle TRU$ is inscribed in rectangle $PQRS$ with T on PQ such that $PT = 4$, and U on PS such that $SU = 1$.

Determine the value of $\angle RUS + \angle PUT$. There are many ways to solve this problem. After you have solved it, see if you can solve it a different way.

Solution

Since $PQRS$ is a rectangle, $PQ = RS$, so $TQ = 3$. Similarly $PS = QR$, so $PU = 3$.



We will now present three different solutions. The first uses the Pythagorean Theorem, the second uses congruent triangles, and the third uses basic trigonometry.

Solution 1

Since $\triangle UPT$ has a right angle at P, we can apply the Pythagorean Theorem to find that $UT^2 = PU^2 + PT^2 = 3^2 + 4^2 = 25$. Therefore, $UT = 5$, since $UT > 0$.

Similarly, since $\triangle TQR$ has a right angle at Q, we can apply the Pythagorean Theorem to find that $TR = 5$.

Since $\triangle RSU$ has a right angle at S, we can apply the Pythagorean Theorem to find that $UR^2 = RS^2 + SU^2 = 7^2 + 1^2 = 50$ and so $UR = \sqrt{50}$, since $UR > 0$.

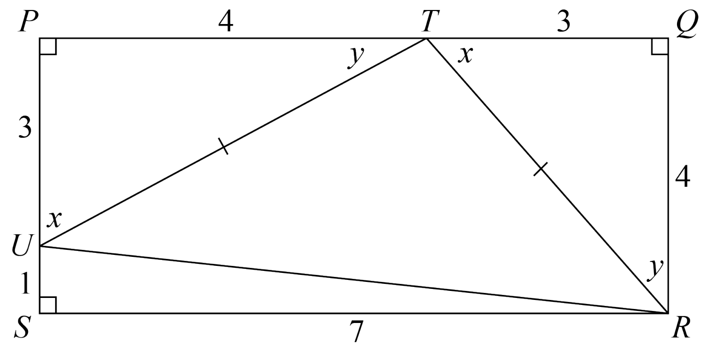
In $\triangle TRU$, notice that $UT^2 + TR^2 = 5^2 + 5^2 = 25 + 25 = 50 = UR^2$. Therefore, $\triangle TRU$ is a right-angled triangle, with $\angle UTR = 90^\circ$. Also, since $UT = TR = 5$, $\triangle TRU$ is an isosceles right-angled triangle, and so $\angle TUR = \angle TRU = 45^\circ$.

The angles in a straight line sum to 180°, so we have $\angle RUS + \angle TUR + \angle PUT = 180^\circ$.

Since $\angle TUR = 45^\circ$, this becomes $\angle RUS + 45^\circ + \angle PUT = 180^\circ$, and so $\angle RUS + \angle PUT = 180^\circ - 45^\circ = 135^\circ$. Therefore, $\angle RUS + \angle PUT = 135^\circ$.

Solution 2

Looking at $\triangle UPT$ and $\triangle TQR$, we have $PT = QR = 4$, $PU = TQ = 3$, and $\angle UPT = \angle TQR = 90^\circ$. Therefore $\triangle UPT \cong \triangle TQR$ by side-angle-side triangle congruency. From the triangle congruency, it follows that $UT = TR$, $\angle QTR = \angle PUT$, and $\angle TRQ = \angle PTU$. Let $\angle QTR = \angle PUT = x$ and $\angle TRQ = \angle PTU = y$.



Since the angles in a triangle sum to 180°, in right-angled $\triangle UPT$, $\angle PUT + \angle PTU = 90^\circ$. That is, $x + y = 90^\circ$.

Since the angles in a straight line sum to 180°, $\angle PTU + \angle UTR + \angle QTR = 180^\circ$. That is, $y + \angle UTR + x = 180^\circ$. Substituting $x + y = 90^\circ$, we obtain $90^\circ + \angle UTR = 180^\circ$, and $\angle UTR = 90^\circ$ follows.

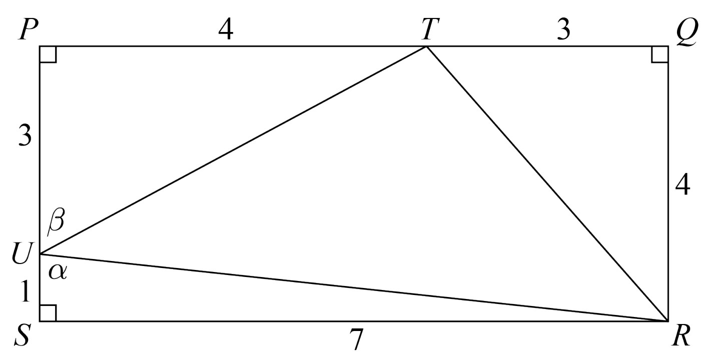
Since $UT = TR$ and $\angle UTR = 90^\circ$, $\triangle TRU$ is an isosceles right-angled triangle and so $\angle TUR = \angle TRU = 45^\circ$.

The angles in a straight line sum to 180°, so we have $\angle RUS + \angle TUR + \angle PUT = 180^\circ$.

Since $\angle TUR = 45^\circ$, this becomes $\angle RUS + 45^\circ + \angle PUT = 180^\circ$, and so $\angle RUS + \angle PUT = 180^\circ - 45^\circ = 135^\circ$. Therefore, $\angle RUS + \angle PUT = 135^\circ$.

Solution 3

Let $\angle RUS = \alpha$ and $\angle PUT = \beta$.



Using basic trigonometry, from right-angled $\triangle RSU$, we have $\tan\alpha = \frac{7}{1} = 7$, and so $\alpha = \tan^{-1}(7)$. Similarly, from right-angled $\triangle UPT$, we have $\tan\beta = \frac{4}{3}$, and so $\beta = \tan^{-1}\left(\frac{4}{3}\right)$.

Then $\angle RUS + \angle PUT = \alpha + \beta = \tan^{-1}(7) + \tan^{-1}\left(\frac{4}{3}\right) = 135^\circ$.

Therefore, $\angle RUS + \angle PUT = 135^\circ$.

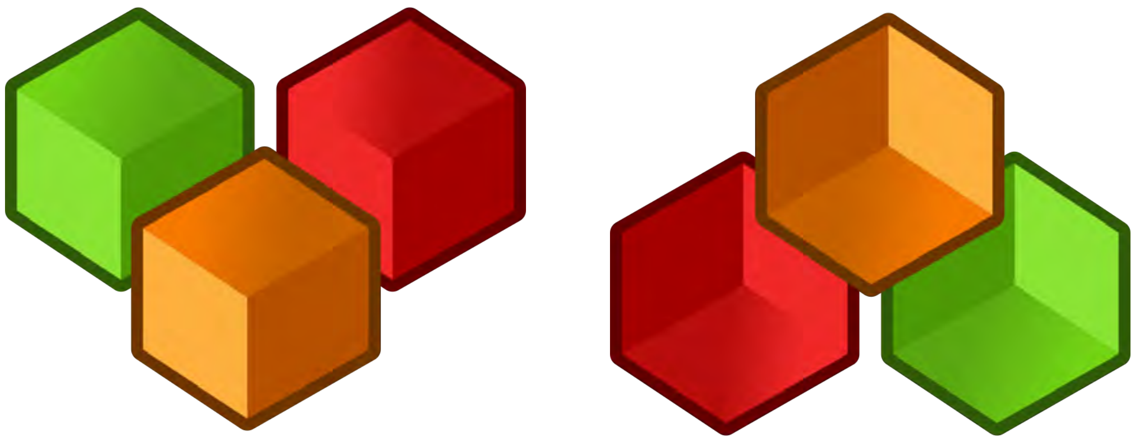
This third solution is very efficient and concise. However, some of the beauty is lost as a result of this direct approach.

Problem of the Week

Problem D

Caen's Cubes

Caen has a cube with a volume of n cm^3. They cut this cube into n smaller cubes, each with a side length of 1 cm. The total surface area of the n smaller cubes is ten times the surface area of Caen's original cube. Determine the side length of Caen's original cube.



Problem of the Week
Problem D and Solution
Caen's Cubes

Problem

Caen has a cube with a volume of n cm^3. They cut this cube into n smaller cubes, each with a side length of 1 cm. The total surface area of the n smaller cubes is ten times the surface area of Caen's original cube. Determine the side length of Caen's original cube.

Solution

Let the side length of Caen's original cube be x cm, where $x > 0$. It follows that $n = x^3$.

Each of the six sides of Caen's original cube has area x^2 cm^2, so the total surface area of the original cube is $6x^2$ cm^2.

Consider one of the smaller cubes. The area of one the six faces is 1 cm^2. So, the surface area of one of these smaller cubes is 6 cm^2. Thus, the total surface area of the n smaller cubes is $6n$ cm^2.

Since the total surface area of the n cubes is ten times the surface area of Caen's original cube, we have

$$6n = 10(6x^2)$$

Dividing both sides by 6, we have

$$n = 10x^2$$

But $n = x^3$, so this tells us that

$$x^3 = 10x^2$$

Since $x > 0$, we have $x^2 > 0$. Dividing both sides by x^2, we find that $x = 10$.

Therefore, the side length of Caen's original cube was 10 cm.

EXTENSION:

If the combined surface area of the n cubes with a side length of 1 cm was Q times the surface area of the original uncut cube, then the side length of the original uncut cube would have been Q cm. Can you see why?

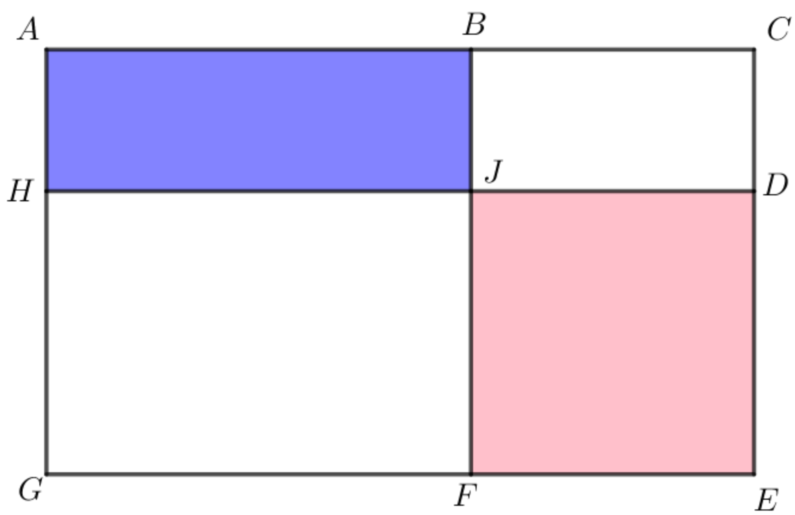
Problem of the Week

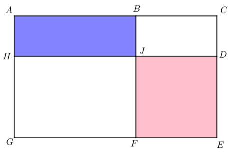
Problem D

Find the Largest Area

Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$. Note that the diagram is just an illustration and is not intended to be to scale.





Problem of the Week
Problem D and Solution
Find the Largest Area

Problem

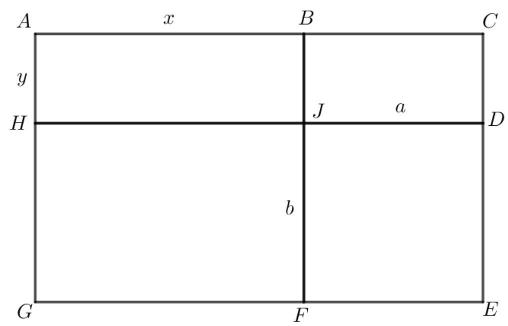
Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$.

Solution

Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$. Then,

$$HJ = GF = AB = x$$
$$BJ = CD = AH = y$$
$$BC = FE = JD = a$$
$$HG = DE = JF = b$$



Thus, we have

$$\begin{aligned}\text{area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG)\\ &= 6 + ya + 15 + xb\\ &= 21 + ya + xb\end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

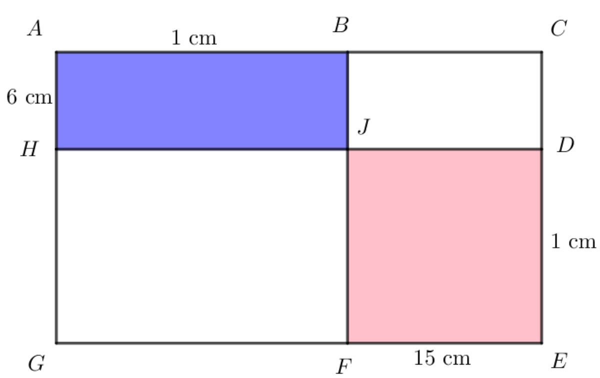
Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

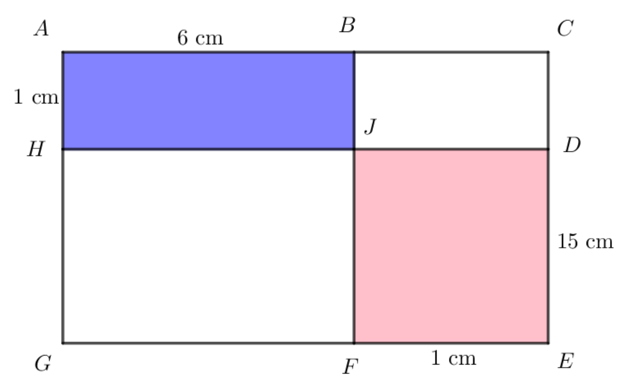
To maximize the area, we need to pick the values of x, y, a, and b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

- **Case 1:** $x = 1$ cm, $y = 6$ cm, $a = 1$ cm, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42$ cm^2.
- **Case 2:** $x = 1$ cm, $y = 6$ cm, $a = 15$ cm, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112$ cm^2.
- **Case 3:** $x = 1$ cm, $y = 6$ cm, $a = 3$ cm, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44$ cm^2.
- **Case 4:** $x = 1$ cm, $y = 6$ cm, $a = 5$ cm, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54$ cm^2.
- **Case 5:** $x = 2$ cm, $y = 3$ cm, $a = 1$, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54$ cm^2.
- **Case 6:** $x = 2$ cm, $y = 3$ cm, $a = 15$, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68$ cm^2.
- **Case 7:** $x = 2$ cm, $y = 3$ cm, $a = 3$, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40$ cm^2.
- **Case 8:** $x = 2$ cm, $y = 3$ cm, $a = 5$, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42$ cm^2.

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram given in the problem was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.



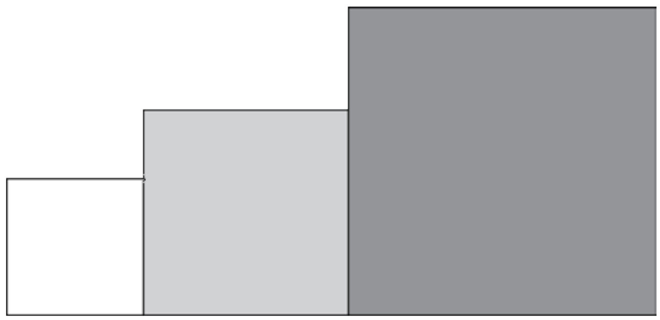


Problem of the Week

Problem D

The Largest Square

Three squares are placed side by side with the smallest square on the left and the largest square on the right. The bottom sides of the three squares form a horizontal line.



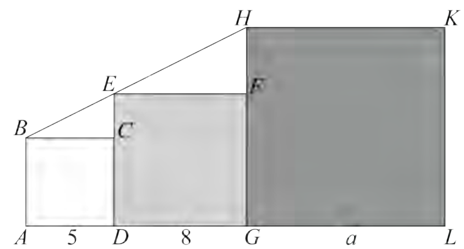
The side length of the smallest square is 5 units, and the side length of the medium-sized square is 8 units. If the top-left corner of each square all lie on a straight line, determine the side length of the largest square.

Problem of the Week
Problem D and Solution
The Largest Square

Problem
Three squares are placed side by side with the smallest square on the left and the largest square on the right. The bottom sides of the three squares form a horizontal line. The side length of the smallest square is 5 units, and the side length of the medium-sized square is 8 units. If the top-left corner of each square all lie on a straight line, determine the side length of the largest square.

Solution

First we draw a line segment connecting the top-left corner of each square and label the vertices as shown in the diagram. Let a represent the side length of the largest square.



From here we present three different solutions.

In Solution 1, we solve the problem by calculating the slope of BH. In Solution 2, we solve the problem using similar triangles. In Solution 3, we place the diagram on the xy-plane and solve the problem using analytic geometry.

Solution 1

The slope of a line is equal to its rise divided by its run. If we look at the line segment from B to E, $BC = 5$ and $CE = DE - DC = 8 - 5 = 3$. Therefore, slope $BE = \frac{CE}{BC} = \frac{3}{5}$.

If we look at the line segment from E to H, $EF = 8$ and $FH = GH - GF = a - 8$. Therefore, slope $EH = \frac{FH}{EF} = \frac{a-8}{8}$.

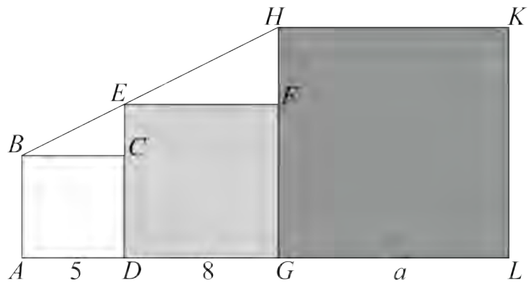
Since B, E, and H lie on a straight line, the slope of BE must equal the slope of EH. Therefore,

$$\begin{aligned}
\frac{3}{5} &= \frac{a-8}{8} \\
5(a-8) &= 3(8) \\
5a - 40 &= 24 \\
5a &= 64 \\
a &= \frac{64}{5}
\end{aligned}$$

Therefore, the side length of the largest square is $\dfrac{64}{5}$ units.

Solution 2

Consider $\triangle BCE$ and $\triangle EFH$. We will first show that $\triangle BCE \sim \triangle EFH$.



Since $ABCD$ is a square, $\angle BCD = 90^\circ$. Therefore, $\angle BCE = 180^\circ - \angle BCD = 180^\circ - 90^\circ = 90^\circ$. Since $DEFG$ is a square, $\angle EFG = 90^\circ$. Therefore, $\angle EFH = 180^\circ - \angle EFG = 180^\circ - 90^\circ = 90^\circ$. Thus, $\angle BCE = \angle EFH$.

Since $ABCD$ and $DEFG$ are squares and AG is a straight line, BC is parallel to EF. Therefore, $\angle EBC$ and $\angle HEF$ are corresponding angles and so $\angle EBC = \angle HEF$.

Since the angles in a triangle add to 180°, then we must also have $\angle BEC = \angle EHF$.

Therefore, $\triangle BCE \sim \triangle EFH$, by Angle-Angle-Angle Triangle Similarity.

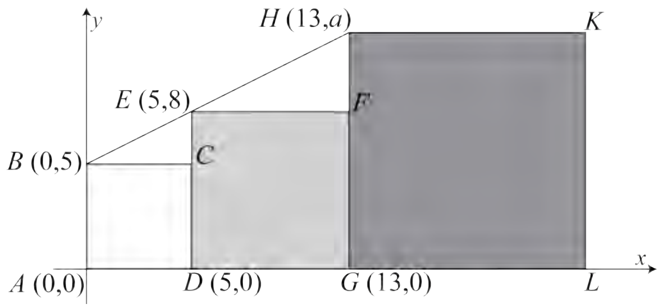
Since $\triangle BCE \sim \triangle EFH$, corresponding side lengths are in the same ratio. In particular,

$$\begin{aligned}
\frac{EC}{BC} &= \frac{HF}{EF} \\
\frac{DE - DC}{BC} &= \frac{GH - GF}{EF} \\
\frac{8-5}{5} &= \frac{a-8}{8} \\
\frac{3}{5} &= \frac{a-8}{8} \\
5(a-8) &= 3(8) \\
5a - 40 &= 24 \\
5a &= 64 \\
a &= \frac{64}{5}
\end{aligned}$$

Therefore, the side length of the largest square is $\frac{64}{5}$ units.

Solution 3

We start by placing the diagram on the xy-plane with A at $(0, 0)$ and AL along the x-axis.



The coordinates of B are $(0, 5)$, the coordinates of D are $(5, 0)$, the coordinates of E are $(5, 8)$, the coordinates of G are $(13, 0)$, and the coordinates of H are $(13, a)$.

Let's determine the equation of the line through B, E, and H.

Since this line passes through $(0, 5)$, it has y-intercept 5. Since the line passes through $(0, 5)$ and $(5, 8)$, it has a slope of $\frac{8-5}{5-0} = \frac{3}{5}$. Therefore, the equation of the line through B, E, and H is $y = \frac{3}{5}x + 5$.

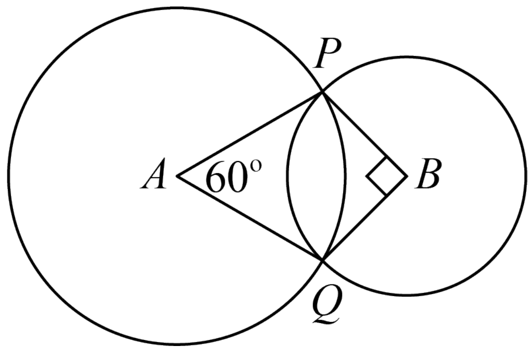
Since $H(13, a)$ lies on this line, substituting $x = 13$ and $y = a$ into $y = \frac{3}{5}x + 5$ gives

$$a = \frac{3}{5}(13) + 5 = \frac{39}{5} + 5 = \frac{39 + 25}{5} = \frac{64}{5}$$

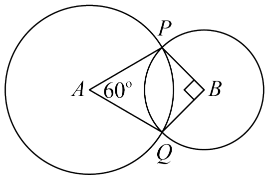
Therefore, the side length of the largest square is $\dfrac{64}{5}$ units.

Problem of the Week
Problem D
The Other Area

Two circles, one with centre A and one with centre B, intersect at points P and Q such that $\angle PAQ = 60^\circ$ and $\angle PBQ = 90^\circ$.



If the area of the circle with centre A is $48\,\mathrm{m}^2$, what is the area of the circle with centre B?

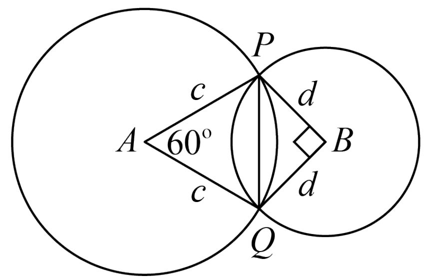


Problem of the Week
Problem D and Solution
The Other Area

Problem

Two circles, one with centre A and one with centre B, intersect at points P and Q such that $\angle PAQ = 60^\circ$ and $\angle PBQ = 90^\circ$. If the area of the circle with centre A is $48\,\text{m}^2$, what is the area of the circle with centre B?

Solution



Let c be the radius of the circle with centre A, in metres, and d be the radius of the circle with centre B, in metres. Then join P to Q.

We will determine the length of PQ in terms of c and then in terms of d in order to find a relationship between c and d.

Consider $\triangle APQ$. Since $AP = AQ = c$, $\triangle APQ$ is isosceles and so $\angle APQ = \angle AQP$. Since $\angle PAQ = 60^\circ$, $\angle APQ = \angle AQP = \frac{180^\circ - 60^\circ}{2} = 60^\circ$. Therefore, $\triangle APQ$ is equilateral and $PQ = AP = AQ = c$.

Consider $\triangle BPQ$. We are given that $\angle PBQ = 90^\circ$. Therefore, $\triangle BPQ$ is a right-angled triangle. The Pythagorean theorem tells us that
$PQ^2 = BP^2 + BQ^2 = d^2 + d^2 = 2d^2$.

We have $PQ = c$ and $PQ^2 = 2d^2$. Therefore, $c^2 = 2d^2$.

The area of the circle with centre B and radius d is πd^2.

The area of the circle with centre A and radius c is πc^2. We know this area is equal to $48\,\text{m}^2$. Then,

$$\begin{aligned}48 &= \pi c^2\\ 48 &= \pi(2d^2)\\ 48 &= 2\pi d^2\\ 24 &= \pi d^2\end{aligned}$$

Therefore, the area of the circle with centre B is $24\,\text{m}^2$.

Problem of the Week

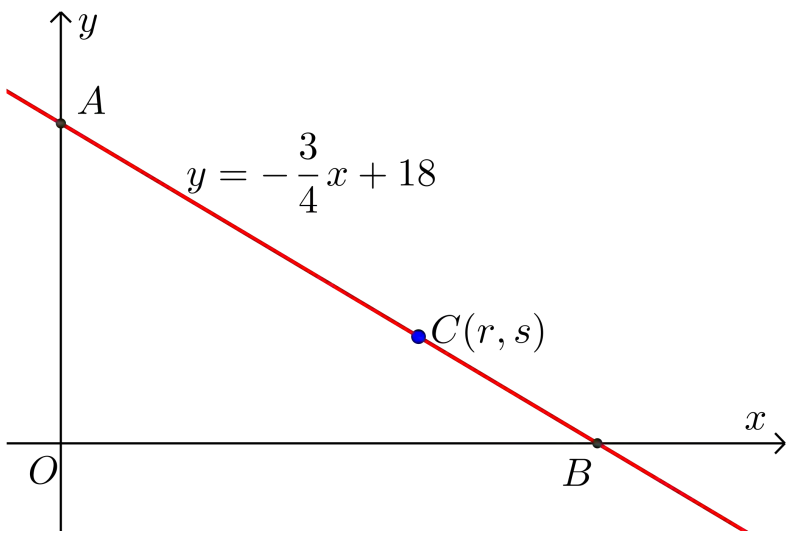
Problem D

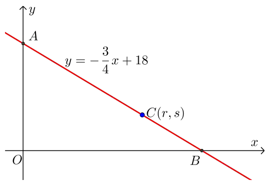
Can You C It?

The line with equation $y = -\frac{3}{4}x + 18$ crosses the positive x-axis at point B and the positive y-axis at point A. The origin, O, and points A and B form the vertices of a triangle.

Point $C(r, s)$ lies on the line segment AB such that the area of $\triangle AOB$ is three times the area of $\triangle COB$.

Determine the values of r and s.





Problem of the Week
Problem D and Solution
Can You C It?

Problem

The line with equation $y = -\frac{3}{4}x + 18$ crosses the positive x-axis at point B and the positive y-axis at point A. The origin, O, and points A and B form the vertices of a triangle.

Point $C(r, s)$ lies on the line segment AB such that the area of $\triangle AOB$ is three times the area of $\triangle COB$.

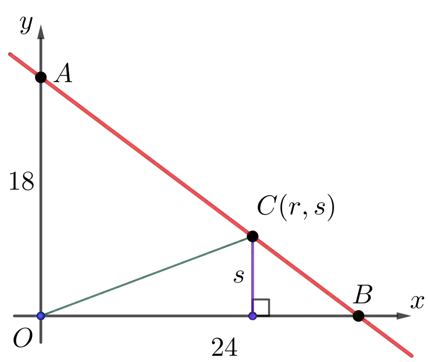
Determine the values of r and s.

Solution

The equation of the line is written in the form $y = mx + b$, where b is the y-intercept of the line. Thus, the y-intercept of the line with equation $y = -\frac{3}{4} + 18$ is 18, and $OA = 18$.

To determine the x-intercept of the line, we set $y = 0$ to obtain $0 = -\frac{3}{4}x + 18$. Solving, we have $\frac{3}{4}x = 18$, and so $x = 24$. Thus, $OB = 24$.

We drop a perpendicular from C to OB. The base of $\triangle COB$ is $OB = 24$, and since C has y-coordinate s, the height of $\triangle COB$ is s.



We now present two solutions to the problem.

Solution 1:

Since $\triangle AOB$ is a right-angled triangle with base $OB = 24$ and height $OA = 18$, using the formula area $= \frac{\text{base}\times\text{height}}{2}$, we have area of $\triangle AOB = \frac{24\times18}{2} = 216$.

Since the area of $\triangle AOB$ is three times the area of $\triangle COB$,
area of $\triangle COB = \frac{1}{3}(\text{area of } \triangle AOB) = \frac{1}{3}(216) = 72$.
Thus, $\triangle COB$ has area 72, base $OB = 24$, and height s.

Using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have

$$\begin{aligned} \text{area of } \triangle COB &= \frac{OB \times s}{2} \\ 72 &= \frac{24 \times s}{2} \\ 72 &= 12s \\ s &= 6 \end{aligned}$$

Since $C(r, s)$ lies on the line with equation $y = -\frac{3}{4}x + 18$ and $s = 6$, we have

$$\begin{aligned} 6 &= -\frac{3}{4}r + 18 \\ \frac{3}{4}r &= 12 \\ r &= 16 \end{aligned}$$

Therefore, $r = 16$ and $s = 6$.

Solution 2:

$\triangle AOB$ and $\triangle COB$ have the same base, OB. If two triangles have the same base, then the areas of the triangles are proportional to the heights of the triangles.

Since the area of $\triangle AOB$ is three times the area of $\triangle COB$, then the height of $\triangle AOB$ is three times the height of $\triangle COB$. In other words, the height of $\triangle COB$ is $\frac{1}{3}$ the height of $\triangle AOB$.

We know that $\triangle AOB$ has height $OA = 18$ and $\triangle COB$ has height s. Therefore, $s = \frac{1}{3}(OA) = \frac{1}{3}(18) = 6$.
Since $C(r, s)$ lies on the line with equation $y = -\frac{3}{4}x + 18$ and $s = 6$, we have

$$\begin{aligned} 6 &= -\frac{3}{4}r + 18 \\ \frac{3}{4}r &= 12 \\ r &= 16 \end{aligned}$$

Therefore, $r = 16$ and $s = 6$.

Notice that in the second solution, it was actually unnecessary to find the length of OB, as this was never used.

Extension:

Can you find the coordinates of point D on line segment AB so that the area of $\triangle AOD$ is equal to the area of $\triangle COB$, thus creating three triangles of equal area? How are the points A, D, C, and B related?

Algebra (A)

TAKE ME TO THE COVER

Problem of the Week

Problem D

How Much?

At a fundraiser hosted by a local restaurant, customers can pay as much or as little as they like for a meal, as long as they pay at least \$1. Any profits are donated to a local charity. One evening, the mean (average) price paid per customer was \$55. One more customer walked in and paid \$70 for a meal, bringing the average up to \$56. What is the highest possible price that a customer could have paid for their meal that evening? Do you think the amounts in your solution are reasonable for this situation?

Problem of the Week
Problem D and Solution
How Much?

Problem

At a fundraiser hosted by a local restaurant, customers can pay as much or as little as they like for a meal, as long as they pay at least \$1. Any profits are donated to a local charity. One evening, the mean (average) price paid per customer was \$55. One more customer walked in and paid \$70 for a meal, bringing the average up to \$56. What is the highest possible price that a customer could have paid for their meal that evening? Do you think the amounts in your solution are reasonable for this situation?

Solution

To calculate the mean (average) of a set of values, we first calculate the sum of the values in the set, and then divide that by the number of values in the set. It follows that the sum of the values in the set is equal to their average multiplied by the number of values in the set.

Let n represent the number of customers that evening. The total amount paid by all customers that evening was therefore $56n$. The final customer paid 70 dollars for their meal. Before this customer arrived, there were $(n-1)$ customers and they had paid a total of $(56n-70)$ dollars. At that point, the average price paid per customer was 55 dollars. Using this information, we can write and solve the following equation.

$$\begin{aligned} \frac{56n-70}{n-1} &= 55 \\ 56n-70 &= 55(n-1) \\ 56n-70 &= 55n-55 \\ n &= 15 \end{aligned}$$

Since $n = 15$, it follows that there were 15 customers that evening, and the total amount paid by all customers was therefore $56n = 56(15) = 840$ dollars.

To determine the highest possible price that a customer could have paid for their meal that evening, we will assume that 13 of the customers paid the lowest possible price of \$1. Then the remaining customer would have paid $840 - 13 \times 1 - 70 = 757$ dollars.

Therefore, the highest possible price that a customer could have paid for their meal that evening is \$757.

Since this is a fundraiser, \$1 is probably a very small amount and \$757 would be considered a very generous donation for a meal.

Extension:

How would the answer change if no two customers paid the same amount?

Problem of the Week

Problem D

Let's Dance

The student council at POTW High School is throwing a school dance. They want to give a welcome gift to each Grade 9 student that attends the dance.

Gifts-R-Us charges \$1.00 per gift. However, if they were to purchase the gifts at Gifts-R-Us, they would exceed their budget by \$17.

At Presents-4-U, they only charge \$0.80 per gift. At this price, the student council would have \$5.00 left over in their budget.

Determine the number of gifts the student council is planning to buy.

Problem of the Week
Problem D and Solution
Let's Dance

Problem

The student council at POTW High School is throwing a school dance. They want to give a welcome gift to each Grade 9 student that attends the dance.

Gifts-R-Us charges \$1.00 per gift. However, if they were to purchase the gifts at Gifts-R-Us, they would exceed their budget by \$17.

At Presents-4-U, they only charge \$0.80 per gift. At this price, the student council would have \$5.00 left over in their budget.

Determine the number of gifts the student council is planning to buy.

Solution

Solution 1

Let n represent the number of gifts that the student council is planning to buy.

Since each gift at Gifts-R-Us costs \$1.00, the student council would spend $1 \times n = n$ dollars in total. If the student council were to purchase all of the gifts they want at Gifts-R-Us, they would be short \$17 dollars in their budget. Therefore, the amount they have in their budget is $(n - 17)$ dollars.

Since each gift at Presents-4-U costs \$0.80, the student council would spend $0.8 \times n = 0.8n$ dollars in total. If the student council were to purchase all of the gifts they want at Presents-4-U, they would have \$5 dollars left over in their budget. Therefore, the amount they have in their budget is $(0.8n + 5)$ dollars.

We have two expressions for the amount in their budget, so we can establish the equality $n - 17 = 0.8n + 5$. This simplifies to $0.2n = 22$. After dividing each side by 0.2, we obtain $n = 110$.

Therefore, the student council is planning to buy 110 gifts.

Solution 2

Let n represent the number of gifts that the student council is planning to buy. Let x represent the amount that the student council has budgeted.

Since the difference between the costs of a single gift is $\$1.00 - \$0.80 = \$0.20$, the total cost difference of buying n gifts would be $\$0.2n$.

To purchase from Gifts-R-Us, the student council would need to spend \$17 more than they budgeted. Therefore, they would need $(x + 17)$ dollars. To purchase from Presents-4-U, the student council would need to spend \$5 less than they budgeted. Therefore, they would need $(x - 5)$ dollars. The total cost difference of purchasing n gifts would be $(x + 17) - (x - 5) = 22$ dollars.

We have two expressions for the cost difference and can establish the equality $0.2n = 22$. After dividing each side by 0.2, we obtain $n = 110$.

Therefore, the student council is planning to buy 110 gifts.

Note that in Solution 1 and Solution 2, we were able to solve for the number of gifts without calculating the budget. In Solution 3, we will first calculate the budget and then use that to calculate the number of gifts.

Solution 3

Let n represent the number of gifts that the student council is planning to buy. Let x represent the amount that the student council has budgeted.

Since each gift at Gifts-R-Us costs \$1.00, n gifts would cost $n \times \$1 = \$n$. Also, the student council would need to spend \$17 more than they budgeted. Therefore, we have

$$n = x + 17 \tag{1}$$

Since each gift at Presents-4-U costs \$0.80, n gifts would cost $n \times \$0.8 = \$0.8n$. Also, the student council would need to spend \$5 less than they budgeted. Therefore, we have

$$0.8n = x - 5 \tag{2}$$

Substituting equation (1) into equation (2), we have

$$\begin{aligned} 0.8n &= x - 5 \\ 0.8(x + 17) &= x - 5 \\ 0.8x + 13.6 &= x - 5 \\ 18.6 &= 0.2x \\ x &= 93 \end{aligned}$$

Thus, the student council has budgeted \$93.

Then, using equation (1), we see that $n = x + 17 = 93 + 17 = 110$.

Therefore, the student council is planning to buy 110 gifts.

Problem of the Week
Problem D
Teacher Road Trip 2

To help pass time on a long bus ride, a group of math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first and second teachers each said a non-negative integer, and every teacher after that said the sum of all of the previous terms in the sequence.

For example, if the first teacher said the number 2 and the second teacher said the number 8, then

- the third teacher would say the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher would say the sum of the first, second, and third terms, which is $2 + 8 + 10 = 20$.

How many possible sequences could the teachers have said if the first teacher said the number 3 and another teacher said the number 3072?

Problem of the Week
Problem D and Solution
Teacher Road Trip 2

Problem

To help pass time on a long bus ride, a group of math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first and second teachers each said a non-negative integer, and every teacher after that said the sum of all of the previous terms in the sequence.

For example, if the first teacher said the number 2 and the second teacher said the number 8, then

- the third teacher would say the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher would say the sum of the first, second, and third terms, which is $2 + 8 + 10 = 20$.

How many possible sequences could the teachers have said if the first teacher said the number 3 and another teacher said the number 3072?

Solution

We know how to construct the sequence, and we know that the first term is 3, but where is the term whose value is 3072?

- Could 3072 be the second term?

  If the first two terms are 3 and 3072, then we can calculate the next few terms.

  - The third term would be $3 + 3072 = 3075$.
  - The fourth term would be $3 + 3072 + 3075 = 3075 + 3075 = 2(3075) = 6150$.
  - The fifth term would be $3 + 3072 + 3075 + 6150 = 6150 + 6150 = 2(6150) = 12\,300$.

  We see that we can determine any term beyond the third term by summing all of the previous terms, or we can simply double the term immediately before the required term, since that term is the sum of all the preceding terms. (This also means that if any term after the third term is known, then the preceding term is half the value of that term.)

  Therefore, there is one possible sequence with 3072 as the second term. The first 6 terms of this sequence are $3,\ 3072,\ 3075,\ 6150,\ 12\,300,\ 24\,600$.

- Could 3072 be the third term?

  Yes, since the third term is the sum of the first two terms, and the first term is 3, then the second term would be $3072 - 3 = 3069$ and the first 6 terms of this sequence are $3,\ 3069,\ 3072,\ 6144,\ 12\,288,\ 24\,576$.

- Could 3072 be the fourth term?

  Yes, since the fourth term is even, then we can determine the third term to be half of the fourth term, which is $3072 \div 2 = 1536$, then the second term would be $1536 - 3 = 1533$. The first 6 terms of this sequence are $3,\ 1533,\ 1536,\ 3072,\ 6144,\ 12\,288$.

- Could 3072 be the fifth term?

  To get from the fifth term to the third term we would divide by 2 twice, or we could divide by 4. If the resulting third term is a non-negative integer greater than or equal to 3, then the sequence exists. The third term would be $3072 \div 4 = 768$, and the second term would be $768 - 3 = 765$. Thus the sequence exists and the first 6 terms are $3,\ 765,\ 768,\ 1536,\ 3072,\ 6144$.

We could continue in this way until we discover all possible sequences that are formed according to the given rules with first term 3 and 3072 somewhere in the sequence. However, if we look at the prime factorization of 3072 we see that the highest power of 2 that divides 3072 is 1024 (or 2^{10}), since $3072 = 2^{10} \times 3$. In fact, dividing 3072 by 1024 would produce a third term that would be 3. The second term would then be 0, a non-negative integer, and the resulting sequence would be $3,\ 0,\ 3,\ 6,\ 12,\ 24,\ 48,\ 96,\ 192,\ 384,\ 768,\ 1536,\ 3072,\ 6144, \ldots$.

If we divide 3072 by any integral power of 2 from $2^0 = 1$ to $2^{10} = 1024$, the resulting third term would be an integer greater than or equal to 3, and 3072 would appear in each of these sequences. There are 11 such sequences. The number 3072 would appear somewhere from term 3 to term 13 in the acceptable sequence. However, 3072 can also appear as the second term, so there are a total of 12 possible sequences.

Could 3072 be the fourteenth term? From the fourteenth term to the third term we would need to divide 3072 by 2^{11}. The resulting third term would be $\frac{3}{2}$. This would mean the second term is not an integer and so the sequence is not possible.

Therefore, there are a total of 12 such sequences.

Problem of the Week

Problem D

Two Birds

Katya owns two cockatoos, an older white cockatoo and a younger Galah cockatoo. At present, the sum of the cockatoos' ages is 44 years. In n years, where $n > 0$, the white cockatoo's age will be four times the Galah cockatoo's age. If n is an integer, determine the possible present ages of each cockatoo.

Problem of the Week

Problem D and Solution

Two Birds

Problem

Katya owns two cockatoos, an older white cockatoo and a younger Galah cockatoo. At present, the sum of the cockatoos' ages is 44 years. In n years, where $n > 0$, the white cockatoo's age will be four times the Galah cockatoo's age. If n is an integer, determine the possible present ages of each cockatoo.

Solution

Let g represent the present age of the Galah cockatoo and w represent the present age of the white cockatoo. Since the sum of their present ages is 44, we have $g + w = 44$ or $w = 44 - g$.

In n years, the Galah cockatoo will be $(g + n)$ years old and the white cockatoo will be $(44 - g + n)$ years old. At that time the white cockatoo will be four times older than the Galah cockatoo. Therefore,

$$\begin{aligned} 4(g + n) &= 44 - g + n \\ 4g + 4n &= 44 - g + n \\ 5g + 3n &= 44 \\ g &= \frac{44 - 3n}{5} \end{aligned}$$

We are looking for integer values of n so that $44 - 3n$ is divisible by 5.

When $n = 3$, $g = \frac{44-3n}{5} = \frac{44-3(3)}{5} = \frac{35}{5} = 7$. When $g = 7$, $w = 44 - g = 44 - 7 = 37$.

When $n = 8$, $g = \frac{44-3n}{5} = \frac{44-3(8)}{5} = \frac{20}{5} = 4$. When $g = 4$, $w = 44 - g = 44 - 4 = 40$.

When $n = 13$, $g = \frac{44-3n}{5} = \frac{44-3(13)}{5} = \frac{5}{5} = 1$. When $g = 1$, $w = 44 - g = 44 - 1 = 43$.

When $n = 18$, $g = \frac{44-3n}{5} = \frac{44-3(18)}{5} = \frac{-10}{5} = -2$. Since $g < 0$, $n = 16$ does not produce a valid age for the Galah cockatoo. No higher value of n would produce a value of $g > 0$.

No integer values of n between 0 and 18, other than 3, 8, and 13, produce a multiple of 5 when substituted into $44 - 3n$.

If today the white cockatoo is 37 and the Galah cockatoo is 7, then in 3 years the white cockatoo will be 40 and the Galah cockatoo will be 10. The white cockatoo will be four times older than the Galah cockatoo since $4 \times 10 = 40$.

If today the white cockatoo is 40 and the Galah cockatoo is 4, then in 8 years the white cockatoo will be 48 and the Galah cockatoo will be 12. The white cockatoo will be four times older than the Galah cockatoo since $4 \times 12 = 48$.

If today the white cockatoo is 43 and the Galah cockatoo is 1, then in 13 years the white cockatoo will be 56 and the Galah cockatoo will be 14. The white cockatoo will be four times older than the Galah cockatoo since $4 \times 14 = 56$.

Therefore, the possible present ages for the white cockatoo and Galah cockatoo are 37 and 7, or 40 and 4, or 43 and 1.

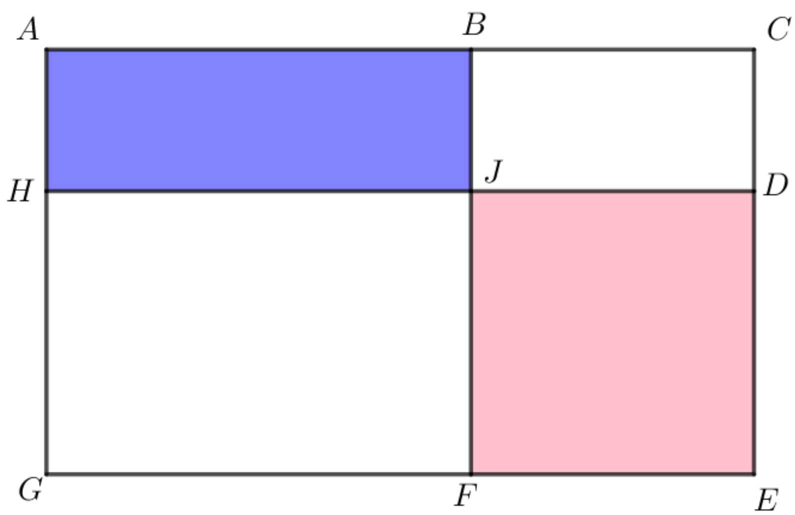
Problem of the Week

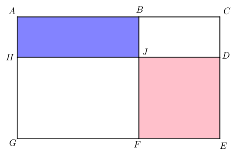
Problem D

Find the Largest Area

Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$. Note that the diagram is just an illustration and is not intended to be to scale.





Problem of the Week
Problem D and Solution
Find the Largest Area

Problem

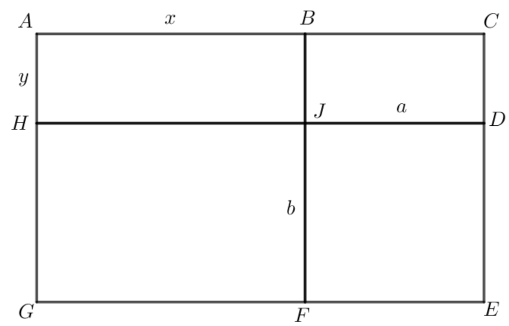
Rectangle $ACEG$ has B on AC and F on EG such that BF is parallel to CE. Also, D is on CE and H is on AG such that HD is parallel to AC, and BF intersects HD at J. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$.

Solution

Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.
Then,

$$HJ = GF = AB = x$$
$$BJ = CD = AH = y$$
$$BC = FE = JD = a$$
$$HG = DE = JF = b$$



Thus, we have

$$\begin{aligned}\text{area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG)\\ &= 6 + ya + 15 + xb\\ &= 21 + ya + xb\end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

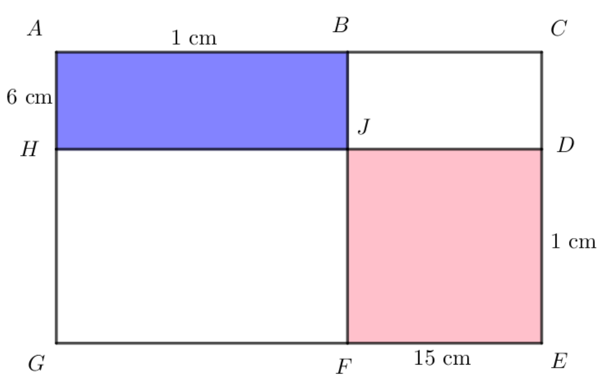
Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

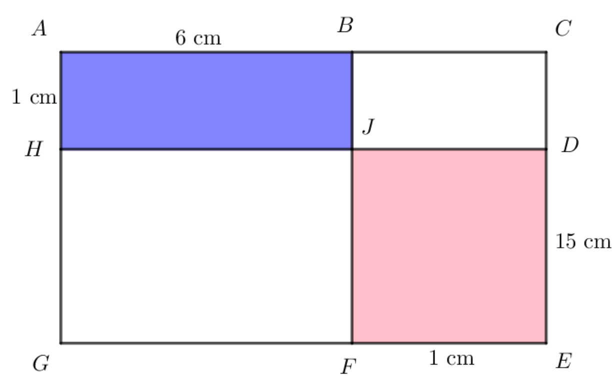
To maximize the area, we need to pick the values of x, y, a, and b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

- **Case 1:** $x = 1$ cm, $y = 6$ cm, $a = 1$ cm, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42$ cm^2.
- **Case 2:** $x = 1$ cm, $y = 6$ cm, $a = 15$ cm, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112$ cm^2.
- **Case 3:** $x = 1$ cm, $y = 6$ cm, $a = 3$ cm, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44$ cm^2.
- **Case 4:** $x = 1$ cm, $y = 6$ cm, $a = 5$ cm, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54$ cm^2.
- **Case 5:** $x = 2$ cm, $y = 3$ cm, $a = 1$, $b = 15$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54$ cm^2.
- **Case 6:** $x = 2$ cm, $y = 3$ cm, $a = 15$, $b = 1$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68$ cm^2.
- **Case 7:** $x = 2$ cm, $y = 3$ cm, $a = 3$, $b = 5$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40$ cm^2.
- **Case 8:** $x = 2$ cm, $y = 3$ cm, $a = 5$, $b = 3$ cm
  Then area$(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42$ cm^2.

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram given in the problem was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.





Problem of the Week
Problem D
All Mixed Up

A large bowl contains a mixture of Himalayan Pink Salt and common salt. When 1 kg of common salt is added to the bowl, the ratio, by mass, of Himalayan Pink Salt to common salt becomes $1 : 2$. When 1 kg of Himalayan Pink Salt is added to the new mixture, the ratio becomes $2 : 3$. Find the ratio of Himalayan Pink Salt to common salt in the original mixture.

Problem of the Week
Problem D and Solution
All Mixed Up

Problem
A large bowl contains a mixture of Himalayan Pink Salt and common salt. When 1 kg of common salt is added to the bowl, the ratio, by mass, of Himalayan Pink Salt to common salt becomes 1 : 2. When 1 kg of Himalayan Pink Salt is added to the new mixture, the ratio becomes 2 : 3. Find the ratio of Himalayan Pink Salt to common salt in the original mixture.

Solution

Let h be the amount of Himalayan Pink Salt, in kgs, in the original mixture.
Let c be the amount of common salt, in kgs, in the original mixture.

When 1 kg of common salt is added, the ratio of Himalayan Pink Salt to common salt is 1 : 2. Therefore,

$$\frac{h}{c+1} = \frac{1}{2}$$

Simplifying, we obtain $c + 1 = 2h$ and $c = 2h - 1$ follows.

When 1 kg of Himalayan Pink Salt is added to the new mixture, the ratio becomes 2 : 3. Therefore,

$$\frac{h+1}{c+1} = \frac{2}{3}$$

Since $c = 2h - 1$, we have

$$\begin{aligned}
\frac{h+1}{(2h-1)+1} &= \frac{2}{3} \\
\frac{h+1}{2h} &= \frac{2}{3} \\
2(2h) &= 3(h+1) \\
4h &= 3h+3 \\
h &= 3
\end{aligned}$$

Substituting $h = 3$ in $c = 2h - 1$, we obtain $c = 2(3) - 1 = 5$.

Therefore, there was originally 3 kgs of Himalayan Pink Salt in the bowl and 5 kgs of common salt. Thus, the ratio of Himalayan Pink Salt to common salt in the original mixture was 3 : 5.

Problem of the Week

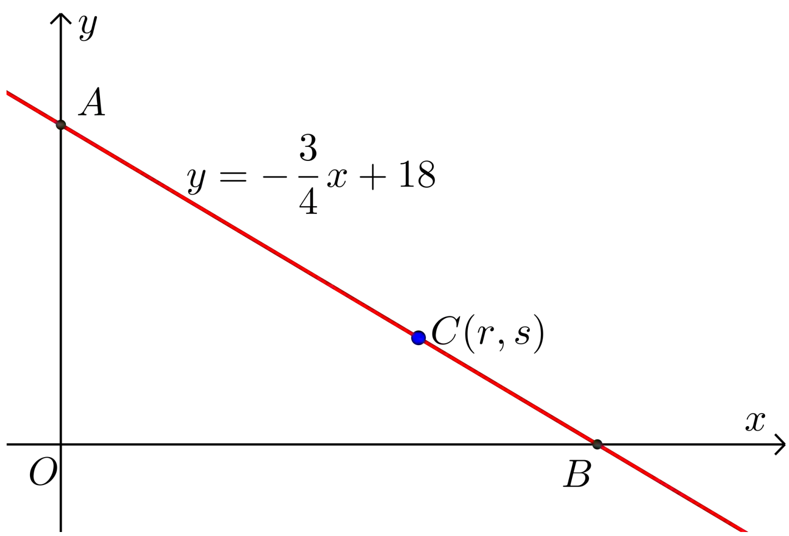
Problem D

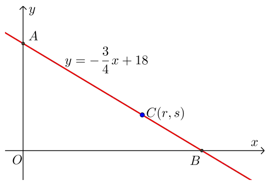
Can You C It?

The line with equation $y = -\frac{3}{4}x + 18$ crosses the positive x-axis at point B and the positive y-axis at point A. The origin, O, and points A and B form the vertices of a triangle.

Point $C(r, s)$ lies on the line segment AB such that the area of $\triangle AOB$ is three times the area of $\triangle COB$.

Determine the values of r and s.





Problem of the Week
Problem D and Solution
Can You C It?

Problem

The line with equation $y = -\frac{3}{4}x + 18$ crosses the positive x-axis at point B and the positive y-axis at point A. The origin, O, and points A and B form the vertices of a triangle.

Point $C(r, s)$ lies on the line segment AB such that the area of $\triangle AOB$ is three times the area of $\triangle COB$.

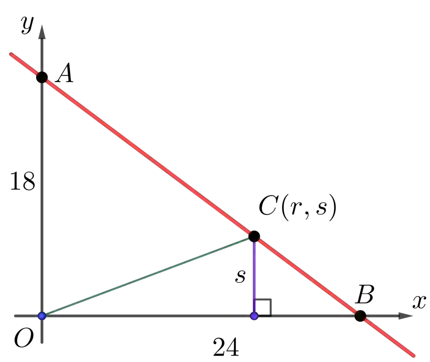
Determine the values of r and s.

Solution

The equation of the line is written in the form $y = mx + b$, where b is the y-intercept of the line. Thus, the y-intercept of the line with equation $y = -\frac{3}{4} + 18$ is 18, and $OA = 18$.

To determine the x-intercept of the line, we set $y = 0$ to obtain $0 = -\frac{3}{4}x + 18$. Solving, we have $\frac{3}{4}x = 18$, and so $x = 24$. Thus, $OB = 24$.

We drop a perpendicular from C to OB. The base of $\triangle COB$ is $OB = 24$, and since C has y-coordinate s, the height of $\triangle COB$ is s.



We now present two solutions to the problem.

Solution 1:

Since $\triangle AOB$ is a right-angled triangle with base $OB = 24$ and height $OA = 18$, using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have area of $\triangle AOB = \frac{24 \times 18}{2} = 216$.

Since the area of $\triangle AOB$ is three times the area of $\triangle COB$,
area of $\triangle COB = \frac{1}{3}(\text{area of } \triangle AOB) = \frac{1}{3}(216) = 72$.
Thus, $\triangle COB$ has area 72, base $OB = 24$, and height s.

Using the formula area $= \frac{\text{base}\times\text{height}}{2}$, we have

$$\begin{aligned}\text{area of }\triangle COB &= \frac{OB\times s}{2}\\ 72 &= \frac{24\times s}{2}\\ 72 &= 12s\\ s &= 6\end{aligned}$$

Since $C(r,s)$ lies on the line with equation $y=-\frac{3}{4}x+18$ and $s=6$, we have

$$\begin{aligned}6 &= -\frac{3}{4}r+18\\ \frac{3}{4}r &= 12\\ r &= 16\end{aligned}$$

Therefore, $r=16$ and $s=6$.

Solution 2:

$\triangle AOB$ and $\triangle COB$ have the same base, OB. If two triangles have the same base, then the areas of the triangles are proportional to the heights of the triangles.

Since the area of $\triangle AOB$ is three times the area of $\triangle COB$, then the height of $\triangle AOB$ is three times the height of $\triangle COB$. In other words, the height of $\triangle COB$ is $\frac{1}{3}$ the height of $\triangle AOB$.

We know that $\triangle AOB$ has height $OA=18$ and $\triangle COB$ has height s. Therefore, $s=\frac{1}{3}(OA)=\frac{1}{3}(18)=6$.
Since $C(r,s)$ lies on the line with equation $y=-\frac{3}{4}x+18$ and $s=6$, we have

$$\begin{aligned}6 &= -\frac{3}{4}r+18\\ \frac{3}{4}r &= 12\\ r &= 16\end{aligned}$$

Therefore, $r=16$ and $s=6$.

Notice that in the second solution, it was actually unnecessary to find the length of OB, as this was never used.

Extension:

Can you find the coordinates of point D on line segment AB so that the area of $\triangle AOD$ is equal to the area of $\triangle COB$, thus creating three triangles of equal area? How are the points A, D, C, and B related?

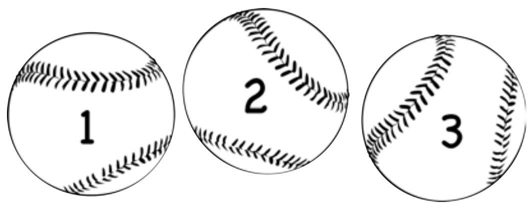
Data Management (D)

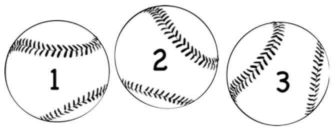
TAKE ME TO THE COVER

Problem of the Week
Problem D
The Baseball Game

Ivy has created a game for her school's math fair. She put three baseballs, numbered 1, 2, and 3, into a bag. Without looking, a player will randomly draw a baseball from the bag, record its number, and then put the baseball back into the bag. They will do this two more times and then calculate the sum of the three numbers recorded. If the sum is less than 8, the player will win a prize.

What is the probability that a player will win a prize when they play this game once?





Problem of the Week
Problem D and Solution
The Baseball Game

Problem

Ivy has created a game for her school's math fair. She put three baseballs, numbered 1, 2, and 3, into a bag. Without looking, a player will randomly draw a baseball from the bag, record its number, and then put the baseball back into the bag. They will do this two more times and then calculate the sum of the three numbers recorded. If the sum is less than 8, the player will win a prize.

What is the probability that a player will win a prize when they play this game once?

Solution

In order to determine the probability, we must determine the number of ways three baseballs whose sum is less than 8 can be drawn from the bag, and then divide by the total number of ways three baseballs can be drawn from the bag.

First, let's determine the total number of ways three baseballs can be drawn from the bag. The baseballs are replaced after each draw, so each time a baseball is drawn from the bag it could be numbered 1, 2, or 3. Since three draws are made and there are three possible outcomes per draw, there are $3 \times 3 \times 3 = 27$ possible ways to draw three baseballs from the bag.

We provide two solutions to this problem. In Solution 1, we take a direct approach to counting the number of ways a sum of less than 8 can be obtained. In Solution 2, our approach is indirect. We count the number of ways a sum of 8 or more can be obtained, and subtract this number from 27 to obtain the desired sum. In this problem it is actually easier to count the desired sum in this indirect way.

Solution 1

Let's determine how many of the 27 draws result in a sum that is less than 8 by systematically looking at the possible selections.

- Ball 1 is drawn three times. In this case, the sum will be $1 + 1 + 1 = 3 < 8$. This can be done only 1 way: $1, 1, 1$.
- Ball 1 is drawn twice and ball 2 is drawn once. In this case, the sum will be $1 + 1 + 2 = 4 < 8$. This can be done 3 ways: $1, 1, 2$ or $1, 2, 1$ or $2, 1, 1$.
- Ball 1 is drawn twice and ball 3 is drawn once. In this case, the sum will be $1 + 1 + 3 = 5 < 8$. This can be done 3 ways: $1, 1, 3$ or $1, 3, 1$ or $3, 1, 1$.
- Ball 1 is drawn once and ball 2 is drawn twice. In this case, the sum will be $1 + 2 + 2 = 5 < 8$. This can be done 3 ways: $1, 2, 2$ or $2, 1, 2$ or $2, 2, 1$.
- Ball 1 is drawn once and ball 3 is drawn twice. In this case, the sum will be $1 + 3 + 3 = 7 < 8$. This can be done 3 ways: $1, 3, 3$ or $3, 1, 3$ or $3, 3, 1$.

- Ball 1 is drawn once, ball 2 is drawn once, and ball 3 is drawn once. In this case the sum will be $1 + 2 + 3 = 6 < 8$. This can be done 6 ways: $1, 2, 3$ or $1, 3, 2$ or $2, 1, 3$ or $2, 3, 1$ or $3, 1, 2$ or $3, 2, 1$.
- Ball 2 is drawn three times. In this case the sum will be $2 + 2 + 2 = 6 < 8$. This can be done only 1 way: $2, 2, 2$.
- Ball 2 is drawn twice and ball 3 is drawn once. In this case the sum will be $2 + 2 + 3 = 7 < 8$. This can be done 3 ways: $2, 2, 3$ or $2, 3, 2$ or $3, 2, 2$.
- Ball 2 is drawn once and ball 3 is drawn twice. In this case the sum will be $2 + 3 + 3 = 8$, which is not less than 8.
- Ball 3 is drawn three times. In this case the sum will be $3 + 3 + 3 = 9$, which is not less than 8.

We see that there are $1 + 3 + 3 + 3 + 3 + 6 + 1 + 3 = 23$ ways to draw three baseballs so that the sum of the numbers recorded is less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$, or approximately 85%.

Solution 2

Let's determine how many of the 27 draws result in a sum that is 8 or more. Since the maximum sum is 9, we need to count the number of ways the sum can be 8 or 9.

- The sum is 8. The only way to do this is to draw ball 2 once and ball 3 twice. This can be done 3 ways: $2, 3, 3$ or $3, 2, 3$ or $3, 3, 2$.
- The sum is 9. This can be done only 1 way: $3, 3, 3$.

We see that there are $3 + 1 = 4$ ways to draw three baseballs so that the sum of the numbers recorded is 8 or 9. Therefore, of the 27 outcomes, $27 - 4 = 23$ give a sum less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$, or approximately 85%.

The indirect approach used in the second solution is definitely more efficient!

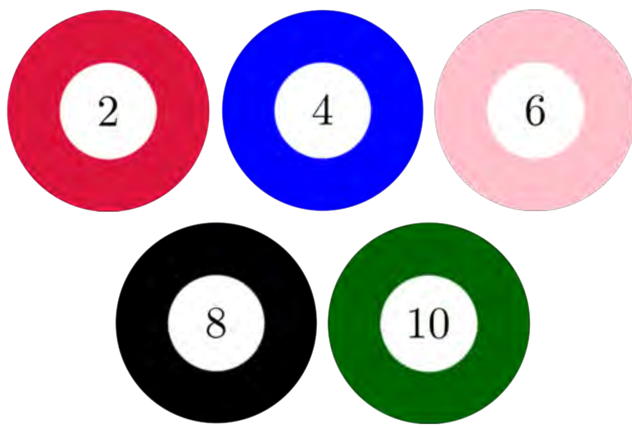
Extension:

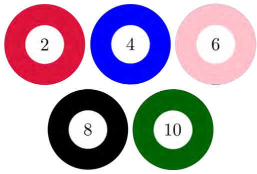
Ivy's game is *unfair* since the probability of obtaining a sum less than 8 is $\frac{23}{27}$ or 85% while the probability of obtaining a sum of 8 or higher is $\frac{4}{27}$ or 15%. In a fair game, we want the probability of winning to be the same as the probability of losing. Can you modify Ivy's game to make it fair?

Problem of the Week
Problem D
No Power

Five balls are placed in a bag. Each ball is labelled with a 2, 4, 6, 8, or 10, with no ball having the same label as any other. Adeleke randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Then Bo randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Finally, Carlos randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag.

Determine the probability that the product of the three recorded integers is not a power of 2.





Problem of the Week
Problem D and Solution
No Power

Problem

Five balls are placed in a bag. Each ball is labelled with a 2, 4, 6, 8, or 10, with no ball having the same label as any other. Adeleke randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Then Bo randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Finally, Carlos randomly chooses a ball, records the integer on the ball, and replaces the ball into the bag. Determine the probability that the product of the three recorded integers is not a power of 2.

Solution

Solution 1

One way to solve this problem is to list out all of the possible choices, calculate the product for each choice, and then count the number of products that are not a power of 2. If we did so, we would find that there are 125 possible choices. Of these, 98 result in a product that is not a power of 2. Therefore, the probability that the product is not a product of 2 is $\frac{98}{125}$. In Solutions 2 and 3, we will see more efficient ways to calculate this probability.

Solution 2

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2. So, to determine the number of choices that result in a product that is not a power of 2, we will count the number of choices that result in a product that is a power of 2, and subtract this from the total number of choices.

Since Adeleke, Bo, and Carlos each have five possible integers they can choose, there are $5 \times 5 \times 5 = 125$ possible choices of integers. For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three chosen integers must be a power of 2. There are three balls labelled with a power of 2, namely, 2, 4, and 8. Therefore, the number of choices that result in a power of 2 is $3 \times 3 \times 3 = 27$.

Since there are 27 choices that give a product that is a power of 2, there must be $125 - 27 = 98$ choices that give a product that is not a power of 2. Therefore, the probability that the product is not a power of 2 is $\frac{98}{125}$.

Solution 3

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2. If p is the probability that the product is a power of 2 and q is the probability that the product is not a power of 2, then $p + q = 1$. Therefore, we can calculate q by calculating p and noting that $q = 1 - p$.

For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three integers must be a power of 2. There are three balls labelled with a power of 2, namely, 2, 4, and 8. Thus, the probability of randomly choosing a ball with a label that is power of 2 is $\frac{3}{5}$. Since Adeleke, Bo, and Carlos choose their integers independently, then the probability that each chooses a power of 2 is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$. In other words, $p = \frac{27}{125}$, and so $q = 1 - p = 1 - \frac{27}{125} = \frac{98}{125}$. Therefore, the probability that the product is not a power of 2 is $\frac{98}{125}$.

Problem of the Week

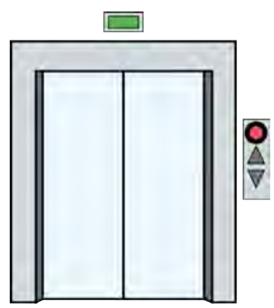
Problem D

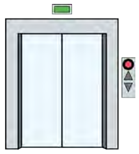
The Elevator

There are six people in an elevator. The sum of all six of their ages is 190 and the median age is 22. From youngest to oldest, the names of the people in the elevator are Ashish, Brook, Calista, Dipak, Enid, and Freyja.

The elevator stops and Ashish and Enid get off. The mean (average) age of the remaining four people in the elevator is then 30. The elevator then stops again and Brook and Calista get off. The mean age of the remaining two people in the elevator is then 40.

If Ashish is 18 years old, and each person's age is a different positive integer, how old is Freyja?





Problem of the Week

Problem D and Solution

The Elevator

Problem

There are six people in an elevator. The sum of all six of their ages is 190 and the median age is 22. From youngest to oldest, the names of the people in the elevator are Ashish, Brook, Calista, Dipak, Enid, and Freyja.

The elevator stops and Ashish and Enid get off. The mean (average) age of the remaining four people in the elevator is then 30. The elevator then stops again and Brook and Calista get off. The mean age of the remaining two people in the elevator is then 40.

If Ashish is 18 years old, and each person's age is a different positive integer, how old is Freyja?

Solution

Let A, B, C, D, E, and F represent the ages of Ashish, Brook, Calista, Dipak, Enid, and Freyja, respectively.

Since the sum of all six ages is 190, it follows that $A + B + C + D + E + F = 190$.

After Ashish and Enid get off the elevator, the mean age of the remaining four people is 30. Thus,

$$\begin{aligned} \frac{B+C+D+F}{4} &= 30 \\ B+C+D+F &= 120 \end{aligned}$$

Thus, after Ashish and Enid leave, the sum of the ages of the people in the elevator is reduced by $190 - 120 = 70$. It follows that $A + E = 70$.

After Brook and Calista get off the elevator, the mean age of the remaining two people is 40. Thus,

$$\begin{aligned} \frac{D+F}{2} &= 40 \\ D+F &= 80 \end{aligned}$$

Thus, after Brook and Calista left, the sum of the ages of the people in the elevator reduced by $120 - 80 = 40$. It follows that $B + C = 40$.

We are told that Ashish is 18 years old, so $A = 18$. Since $A + E = 70$, it follows that $E = 70 - 18 = 52$.

Since there are six people in total, the median age will be halfway between the two ages in the middle positions when they are arranged in increasing order. Thus, the median will be halfway between C and D. Since each age is a different positive integer, C must be less than the median. Thus, $C < 22$. Since $B + C = 40$ and $B < C$, the only possibility is $C = 21$ and $B = 19$.

We know that $C = 21$, and the median is halfway between C and D. Since the median is 22, we can conclude that $D = 23$. Then, since $D + F = 80$, it follows that $F = 80 - 23 = 57$. Thus, Freyja is 57 years old.

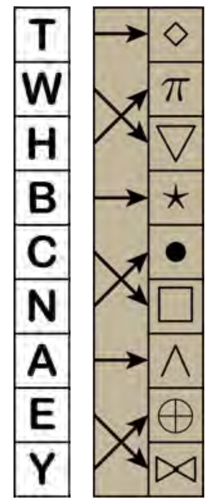
Computational Thinking (C)

TAKE ME TO THE COVER

Problem of the Week

Problem D

Scavenger Hunt



Kavi is creating a scavenger hunt for his younger siblings where the clues are all names of places in Canada. To make it more challenging, he encrypts each clue using a homemade cipher machine. He starts with two strips of paper, each with a length equal to the circumference of a paper tube. One paper contains a column of letters and the other contains a column of symbols with an arrow pointing to each symbol, as shown. He wraps each strip of paper around his tube and tapes them so that the paper with the letters can rotate around the tube but the paper with the symbols is fixed in place. The paper with the letters is called the *left rotor* and the paper with the arrows and symbols is called the *right rotor*.

Kavi follows the steps below to encrypt his clues using his cipher machine.

1. Rotate the left rotor so that the letter T points to the symbol $\diamond$. This is the "start position".
2. Encrypt the first letter in the message by following the arrow from the letter to the symbol. For example, the letter W would be encrypted as $\bigtriangledown$.
3. Rotate the left rotor up one position and encrypt the second letter in the message. For example, the letter A would be encrypted as $\bullet$.
4. Rotate the left rotor up two positions and encrypt the third letter in the message. For example, the letter W would be encrypted as $\bowtie$.
5. Rotate the left rotor up three positions and encrypt the fourth letter in the message. For example, the letter A would be encrypted as $\diamond$.
6. Continue the procedure of rotating the left rotor up n positions and encrypting the $(n+1)^{\text{th}}$ letter in the message until all letters in the message have been encrypted.

Kavi's clue "WAWA" would therefore be encrypted as $\bigtriangledown\bullet\bowtie\diamond$.

Follow the steps to encrypt Kavi's clue "BATCHAWANABAY".

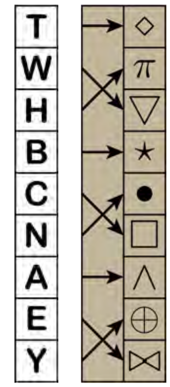
Problem of the Week

Problem D and Solution

Scavenger Hunt

Problem

Kavi is creating a scavenger hunt for his younger siblings where the clues are all names of places in Canada. To make it more challenging, he encrypts each clue using a homemade cipher machine. He starts with two strips of paper, each with a length equal to the circumference of a paper tube. One paper contains a column of letters and the other contains a column of symbols with an arrow pointing to each symbol, as shown. He wraps each strip of paper around his tube and tapes them so that the paper with the letters can rotate around the tube but the paper with the symbols is fixed in place. The paper with the letters is called the *left rotor* and the paper with the arrows and symbols is called the *right rotor*.



Kavi follows the steps below to encrypt his clues using his cipher machine.

1. Rotate the left rotor so that the letter T points to the symbol $\diamond$. This is the "start position".
2. Encrypt the first letter in the message by following the arrow from the letter to the symbol. For example, the letter W would be encrypted as $\bigtriangledown$.
3. Rotate the left rotor up one position and encrypt the second letter in the message. For example, the letter A would be encrypted as $\bullet$.
4. Rotate the left rotor up two positions and encrypt the third letter in the message. For example, the letter W would be encrypted as $\bowtie$.
5. Rotate the left rotor up three positions and encrypt the fourth letter in the message. For example, the letter A would be encrypted as $\diamond$.
6. Continue the procedure of rotating the left rotor up n positions and encrypting the $(n+1)^{\text{th}}$ letter in the message until all letters in the message have been encrypted.

Kavi's clue "WAWA" would therefore be encrypted as $\bigtriangledown \bullet \bowtie \diamond$.

Follow the steps to encrypt Kavi's clue "BATCHAWANABAY".

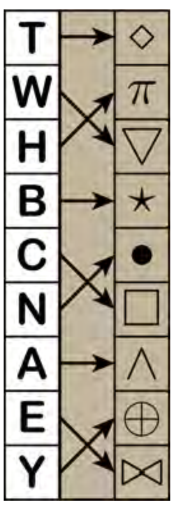
Solution

Let the start position of the left rotor be Position 0. If the left rotor is in Position 1, the letters have moved 1 position up from the start position. If the left rotor is in Position 2, the letters have moved 2 positions up from the start position. Since there are 9 letters on the rotor, it follows that there are only 9 positions that the left rotor can be in. After moving 9 positions up, the letters will be back in the start position, or Position 0. Thus, after moving 9 positions or more, we can determine the position number of the left rotor by subtracting multiples of 9 from the total number of positions moved until we obtain a position number between 0 and 8. We do this in the following table.

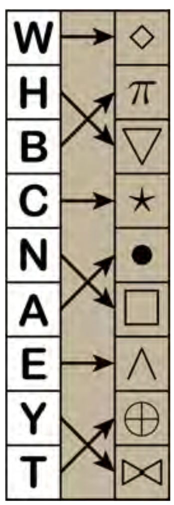
Letter to Encrypt	Number of Positions Moved Before Encryption	Position Number
B	0	0
A	$0+1=1$	1
T	$1+2=3$	3
C	$3+3=6$	6
H	$6+4=10$	$10-9=1$
A	$10+5=15$	$15-9=6$
W	$15+6=21$	$21-18=3$
A	$21+7=28$	$28-27=1$
N	$28+8=36$	$36-36=0$
A	$36+9=45$	$45-45=0$
B	$45+10=55$	$55-54=1$
A	$55+11=66$	$66-63=3$
Y	$66+12=78$	$78-72=6$

It turns out that we need only four positions of the left rotor, namely Positions $0,\ 1,\ 3$, and 6. These are shown below.

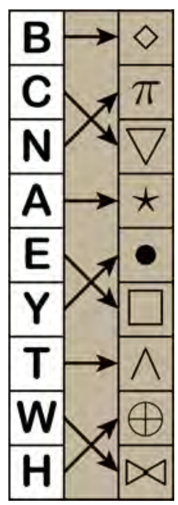
Position 0



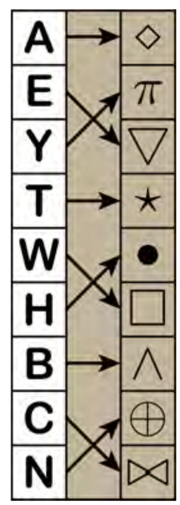
Position 1



Position 3



Position 6



To encrypt the first letter of the clue “BATCHAWANABAY”, the left rotor is in Position 0 and the B is encrypted as $\star$. To encrypt the second letter, the left rotor is in Position 1 and the A is encrypted as $\bullet$. In this way, we can encrypt the clue “BATCHAWANABAY” as
$\star \bullet \wedge \bowtie \bigtriangledown \diamond \bowtie \bullet \bullet \wedge \pi \star \pi$.

Extension:

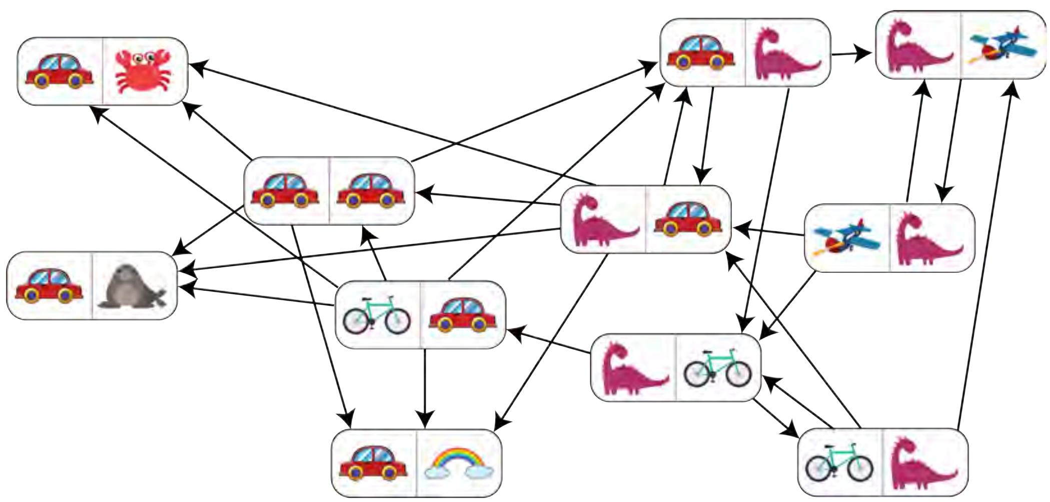
In our example, there were 9 letters on the left rotor and 9 symbols on the right rotor. The cycle of positions used on the left rotor caused only Positions $0,\ 1,\ 3$, and 6 of the left rotor to be used. Is there a size of rotor that would require all positions of the rotor to be used in the encryption process? Experiment with a few different sizes.

Problem of the Week
Problem D
Arranging Cards

Delphine has cards that each contain two pictures; one on the left side of the card and one on the right side of the card. Delphine arranges some of these cards in a row according to the following rules.

1. The picture on the right side of any card in the row is the same as the picture on the left side of the card to its right.
2. Cards can not be rotated.

The following diagram shows all of Delphine's cards. Arrows out of a card indicate the possible card(s) that could be placed to its right.



By following the rules, what is the maximum number of cards Delphine can arrange in a row?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

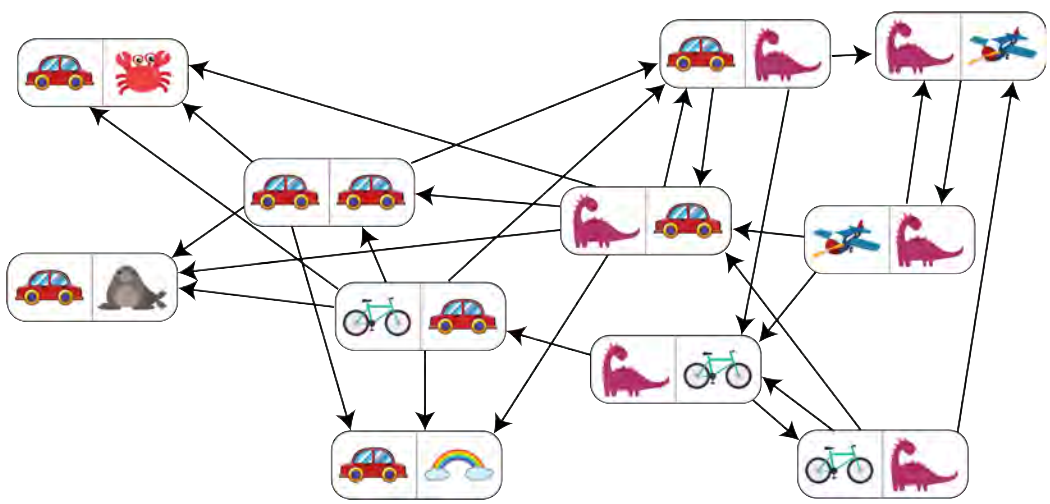
Problem of the Week
Problem D and Solution
Arranging Cards

Problem

Delphine has cards that each contain two pictures; one on the left side of the card and one on the right side of the card. Delphine arranges some of these cards in a row according to the following rules.

1. The picture on the right side of any card in the row is the same as the picture on the left side of the card to its right.
2. Cards can not be rotated.

The following diagram shows all of Delphine's cards. Arrows out of a card indicate the possible card(s) that could be placed to its right.



By following the rules, what is the maximum number of cards Delphine can arrange in a row?

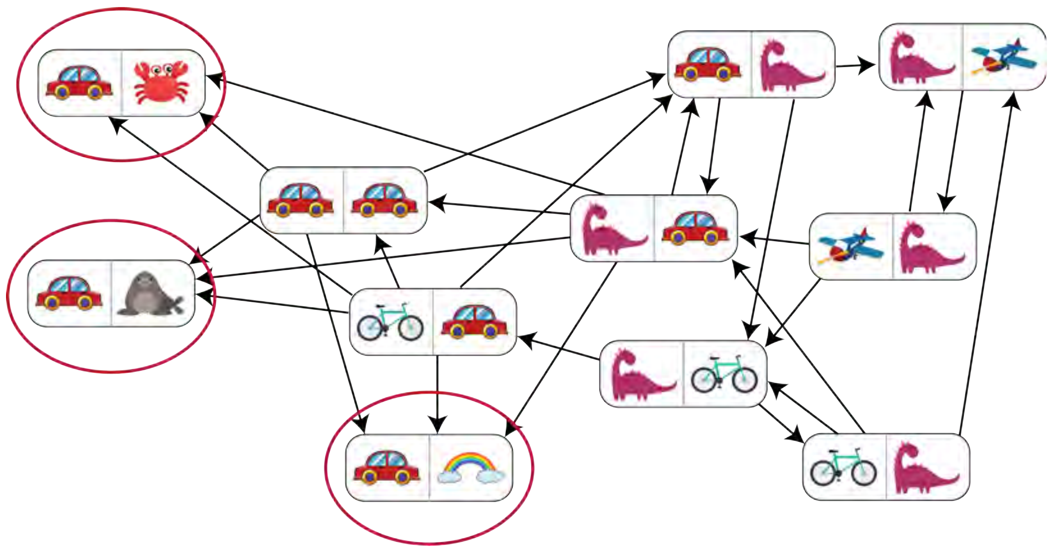
This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

By following the rules, Delphine can arrange 9 cards in a row. One example of a row of 9 cards is shown below.

Note that it is possible to find other rows of 9 cards.

To determine whether or not we can arrange more than 9 cards in a row, look at the three circled cards in the diagram.



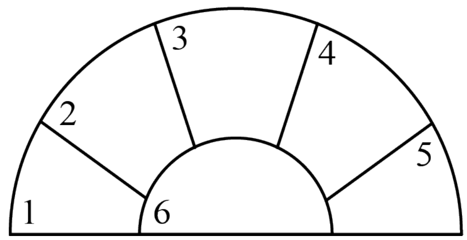
In the diagram, there are no arrows going out of any of these three cards because the picture on the right side of each of the cards is not on the left side of any other card. It follows that if any of these cards are used, they must be the rightmost card in the row. However any row that Delphine creates can contain only one rightmost card, so at least two of these cards cannot be used. Therefore, the maximum number of cards Delphine can arrange in a row is 9.

Problem of the Week

Problem D

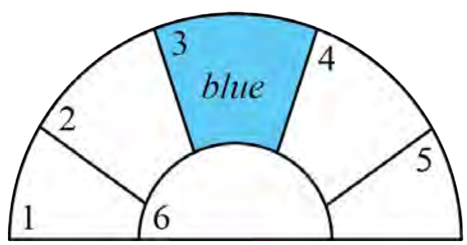
Adding Some Colour 2

Finn and Vidya play a game where they take turns colouring regions in the diagram shown red or blue. On their turn, each player colours a region in the diagram that is not bordering another region of the same colour.



After some number of turns, it won't be possible to colour any more regions, and the game will be over. The winner is the player who coloured the last region.

Finn went first. On his turn, he coloured region 3 blue, so after his turn the diagram is coloured as follows.

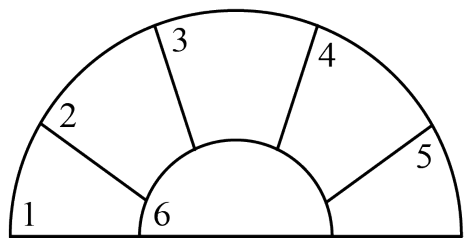


It is now Vidya's turn and there are five remaining regions. Determine all possibilities for the colour Vidya should use and the region she should choose in order to guarantee that she wins the game, regardless of what Finn does on his remaining turns.

Problem of the Week
Problem D and Solution
Adding Some Colour 2

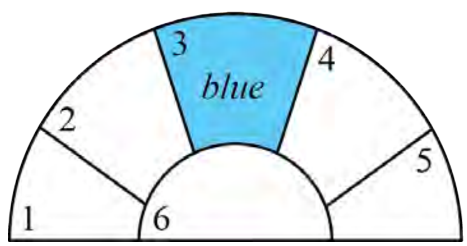
Problem

Finn and Vidya play a game where they take turns colouring regions in the diagram shown red or blue. On their turn, each player colours a region in the diagram that is not bordering another region of the same colour.



After some number of turns, it won't be possible to colour any more regions, and the game will be over. The winner is the player who coloured the last region.

Finn went first. On his turn, he coloured region 3 blue, so after his turn the diagram is coloured as follows.



It is now Vidya's turn and there are five remaining regions. Determine all possibilities for the colour Vidya should use and the region she should choose in order to guarantee that she wins the game, regardless of what Finn does on his remaining turns.

Solution

If Vidya colours region 6 red on her first turn, then she will be guaranteed to win the game, regardless of what Finn does on his remaining turns. First we will show why this is true, and then we will show why all the other possible moves will not guarantee a win for Vidya.

If Vidya colours region 6 red, then the only possible remaining moves are to colour region 1 blue or to colour region 5 blue. Since these moves don't affect each other, Finn will colour one of these regions and Vidya will colour the other and win the game.

The other possible moves for Vidya are to colour region 1 or 5 blue, or to colour region 1, 2, 4, or 5 red.

- If Vidya coloured region 1 blue, then Finn could colour region 4 red. Then the only possible remaining moves would be to colour region 2 red or to colour region 5 blue. Since these moves don't affect each other, Vidya would colour one of these regions and Finn would colour the other and win the game.
- If Vidya coloured region 5 blue, then Finn could colour region 2 red. Then the only possible remaining moves would be to colour region 4 red or to colour region 1 blue. Since these moves don't affect each other, Vidya would colour one of these regions and Finn would colour the other and win the game.
- If Vidya coloured region 1 red, then Finn could colour region 5 red and win the game.
- If Vidya coloured region 5 red, then Finn could colour region 1 red and win the game.
- If Vidya coloured region 2 red, then Finn could colour region 5 blue. Then the only possible remaining moves would be to colour region 4 red or to colour region 1 blue. Since these moves don't affect each other, Vidya would colour one of these regions and Finn would colour the other and win the game.
- If Vidya coloured region 4 red, then Finn could colour region 1 blue. Then the only possible remaining moves would be to colour region 2 red or to colour region 5 blue. Since these moves don't affect each other, Vidya would colour one of these regions and Finn would colour the other and win the game.

Therefore, colouring region 6 red is the only move Vidya can do in order to guarantee that she wins the game, regardless of what Finn does on his remaining turns.