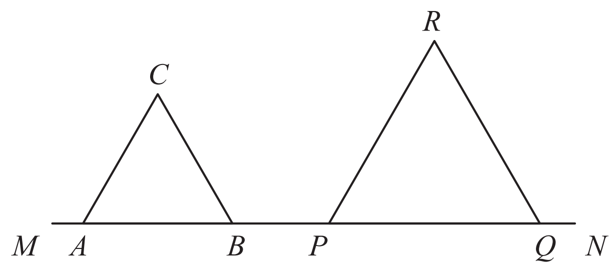
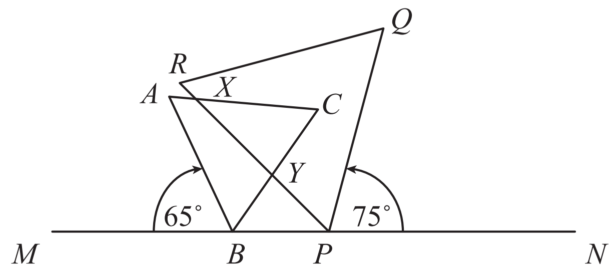
Problem of the Week
Problem C and Solution
Rotate Right, Rotate Left

Problem

Two equilateral triangles, $\triangle ABC$ and $\triangle PQR$, have their bases AB and PQ sitting on line segment MN, as shown.



$\triangle ABC$ is tipped clockwise 65° about point B so that $\angle MBA = 65^\circ$ and point B remains where it is on MN. $\triangle PQR$ is tipped counterclockwise 75° about point P so that $\angle NPQ = 75^\circ$ and point P remains where it is on MN. As a result of tipping the two, $\triangle ABC$ overlaps $\triangle PQR$ such that AC and BC intersect RP at X and Y, respectively. Vertex C of $\triangle ABC$ lies inside $\triangle PQR$. Determine the measure of $\angle CXY$.



Solution

In any equilateral triangle, all sides are equal in length and each angle measures 60°.

Since $\triangle ABC$ and $\triangle PQR$ are equilateral,
$\angle ABC = \angle ACB = \angle CAB = \angle QPR = \angle PRQ = \angle RQP = 60^\circ$.

Since the angles in a straight line sum to 180°, we have
$180^\circ = 65^\circ + \angle ABC + \angle YBP = 65^\circ + 60^\circ + \angle YBP$.
Rearranging, we have $\angle YBP = 180^\circ - 65^\circ - 60^\circ = 55^\circ$.

Similarly, since angles in a straight line sum to 180°, we have
$180^\circ = 75^\circ + \angle QPR + \angle YPB = 75^\circ + 60^\circ + \angle YPB$.
Rearranging, we have $\angle YPB = 180^\circ - 75^\circ - 60^\circ = 45^\circ$.

Since the angles in a triangle sum to 180°, in $\triangle BYP$ we have
$\angle YPB + \angle YBP + \angle BYP = 180^\circ$, and so $45^\circ + 55^\circ + \angle BYP = 180^\circ$.
Rearranging, we have $\angle BYP = 180^\circ - 45^\circ - 55^\circ = 80^\circ$.

When two lines intersect, vertically opposite angles are equal. Since $\angle XYC$ and $\angle BYP$ are vertically opposite angles, we have $\angle XYC = \angle BYP = 80^\circ$.

Again, since angles in a triangle sum to 180°, in $\triangle XYC$ we have
$\angle XYC + \angle XCY + \angle CXY = 180^\circ$. We have already found that $\angle XYC = 80^\circ$, and since $\angle XCY = \angle ACB$, we have $\angle XCY = 60^\circ$. So, $\angle XYC + \angle XCY + \angle CXY = 180^\circ$ becomes $80^\circ + 60^\circ + \angle CXY = 180^\circ$. Rearranging, we have $\angle CXY = 180^\circ - 80^\circ - 60^\circ = 40^\circ$.

Therefore, $\angle CXY = 40^\circ$