Problem of the Week

Problem D and Solution

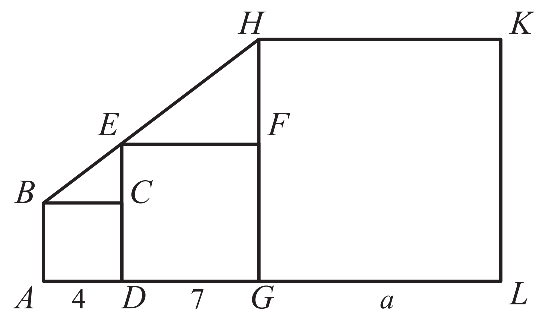
Big, Bigger, Biggest

Problem

Three squares are placed beside each other as shown. The smallest square has side length 4 units, the middle-sized square has side length 7 units, but the side length of the largest square is unknown. However, the top left corner of each of the three squares lies on a straight line. Determine the side length of the largest square.

Solution

Label the vertices as shown on the diagram. Draw line segment BH through E. Let a represent the side length of the larger square.



In Solution 1, we will solve the problem by calculating the slope of BH.

In Solution 2, we will solve the problem using similar triangles.

In Solution 3, we will place the diagram on the xy-plane and solve the problem using analytic geometry.

Solution 1

The slope of a line is equal to its rise divided by its run.

If we look at the line segment from B to E, $BC = 4$ and $CE = DE - DC = 7 - 4 = 3$. Therefore, slope $BE = \frac{CE}{BC} = \frac{3}{4}$.

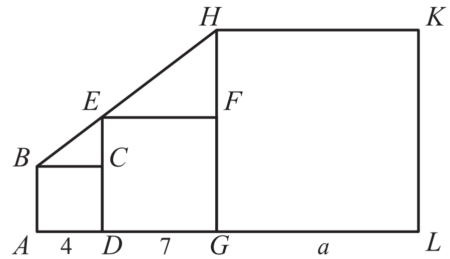
If we look at the line segment from E to H, $EF = 7$ and $FH = GH - GF = a - 7$. Therefore, slope $EH = \frac{FH}{EF} = \frac{a-7}{7}$.

Since B, E and H lie on a straight line, slope BE must equal slope EH. Therefore,

$$\begin{aligned}
\text{slope } BE &= \text{slope } EH \\
\frac{3}{4} &= \frac{a-7}{7} \\
4(a-7) &= 3(7) \\
4a - 28 &= 21 \\
4a &= 49
\end{aligned}$$

$\therefore a = \dfrac{49}{4}$ and the side length of the larger square is $\dfrac{49}{4}$ units.

Solution 2



Consider $\triangle BCE$ and $\triangle EFH$. We will first show that $\triangle BCE \sim \triangle EFH$.

Since $ABCD$ is a square, $\angle BCD = 90^\circ$.
Therefore, $\angle BCE = 180^\circ - \angle BCD = 180^\circ - 90^\circ = 90^\circ$.

Since $DEFG$ is a square, $\angle EFG = 90^\circ$.
Therefore, $\angle EFH = 180^\circ - \angle EFG = 180^\circ - 90^\circ = 90^\circ$.
Thus, $\angle BCE = \angle EFH$.

Since $ABCD$ and $DEFG$ are squares and AG is a straight line, BC is parallel to EF. Therefore, $\angle EBC$ and $\angle HEF$ are corresponding angles and so $\angle EBC = \angle HEF$.

Since the angles in a triangle add to 180°, then we must also have $\angle BEC = \angle EHF$.

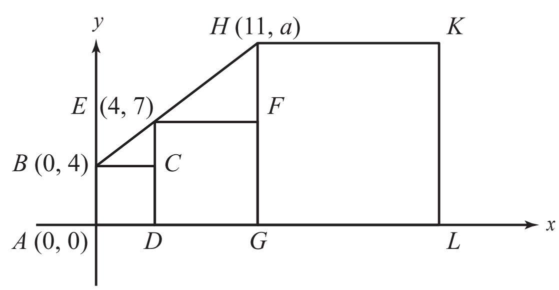
Therefore, $\triangle BCE \sim \triangle EFH$, by Angle−Angle−Angle Triangle Similarity.

Since $\triangle BCE \sim \triangle EFH$, corresponding side lengths are in the same ratio. In particular,

$$\begin{aligned}
\frac{EC}{BC} &= \frac{HF}{EF} \\
\frac{DE - DC}{BC} &= \frac{GH - GF}{EF} \\
\frac{7-4}{4} &= \frac{a-7}{7} \\
\frac{3}{4} &= \frac{a-7}{7} \\
4(a-7) &= 3(7) \\
4a - 28 &= 21 \\
4a &= 49
\end{aligned}$$

$\therefore a = \frac{49}{4}$ and the side length of the larger square is $\frac{49}{4}$ units.

Solution 3



We proceed by placing the diagram on the xy-plane with A at $(0,0)$ and AL along the x-axis.
The coordinates of B are $(0,4)$, the coordinates of E are $(4,7)$, and the coordinates of H are $(11,a)$.
Let's determine the equation of the line through B, E, H.

Since this line passes through $(0,4)$, it has y-intercept 4.
Since the line passes through $(0,4)$ and $(4,7)$, it has slope $= \frac{7-4}{4-0} = \frac{3}{4}$.
Therefore, the equation of the line is $y = \frac{3}{4}x + 4$.

Since $H(11,a)$ lies on this line, substituting $x = 11$ and $y = a$ into $y = \frac{3}{4}x + 4$, we have

$$a = \frac{3}{4}(11) + 4 = \frac{33}{4} + 4 = \frac{33+16}{4} = \frac{49}{4}$$

$\therefore$ the side length of the larger square is $\frac{49}{4}$ units.