Problem of the Week
Problem E and Solution
Another Point of Division

Problem
A square has coordinates $A(0, 0)$, $B(-9, 12)$, $C(3, 21)$ and $D(12, 9)$. The line $l$ passes through $A$ and intersects $CD$ at point $T(r, s)$ splitting the square so that the area of square $ABCD$ is three times the area of $\triangle ATD$. Determine the equation of line $l$.

Solution
Both solutions start by finding the area of square $ABCD$, the area of $\triangle ATD$, the length $AD$ and the length $TD$. We present the common start to both solutions at this point.

Using the distance formula, $AD = \sqrt{(9 - 0)^2 + (12 - 0)^2} = \sqrt{81 + 144} = \sqrt{225} = 15$, since $AD > 0$.

Therefore, the area of square $ABCD = 15^2 = 225$.

Since the area of square $ABCD$ is three times the area of $\triangle ATD$, area($\triangle ATD$) = $\frac{1}{3} \times$ area(square $ABCD$) = $\frac{1}{3} (225) = 75$.

Since $ABCD$ is a square, $\angle ADC = 90^\circ$. Consider $\triangle ATD$. This triangle is a right-triangle with base $AD = 15$ and height $TD$.

Using the formula area = base $\times$ height $\div 2$, $\text{area}(\triangle ATD) = \frac{AD \times TD}{2}$.

$75 = \frac{15 \times TD}{2}$

$\therefore TD = 10$

Solution 1
We now calculate the equation of the line that the segment $CD$ lies on.

Since $D$ has coordinates $(12, 9)$ and $C$ has coordinates $(3, 21)$, this line has slope $\frac{21 - 9}{3 - 12} = \frac{12}{-9} = -\frac{4}{3}$.

Since the line has slope $-\frac{4}{3}$ and point $(3, 21)$ lies on the line, we have $\frac{y - 21}{x - 3} = -\frac{4}{3} \implies 3y - 63 = -4x + 12 \implies 3y = -4x + 75 \implies y = -\frac{4}{3}x + 25$.

Since $T(r, s)$ lies on this line, $s = -\frac{4}{3}r + 25$.

Using the distance formula, since $TD = 10$, we have

$\sqrt{(r - 12)^2 + (s - 9)^2} = 10$

$(r - 12)^2 + (s - 9)^2 = 100$

$(r - 12)^2 + \left(\left(-\frac{4}{3}r + 25\right) - 9\right)^2 = 100$, since $s = -\frac{4}{3}r + 25$

$(r - 12)^2 + \left(\frac{4}{3}r + 16\right)^2 = 100$

$r^2 - 24r + 144 + \frac{16}{9}r^2 - \frac{128}{3}r + 256 = 100$
\[
\begin{align*}
\frac{25}{9}r^2 - \frac{200}{3}r + 300 &= 0 \\
\frac{25}{9}(r^2 - 24r + 108) &= 0 \\
r^2 - 24r + 108 &= 0 \\
(r - 6)(r - 18) &= 0 \\
r &= 6, 18
\end{align*}
\]

But \( r = 18 \) lies outside the square. Therefore, \( r = 6 \) and \( s = -\frac{4}{3}(6) + 25 = -8 + 25 = 17 \).

The line \( l \) passes through \( A(0, 0) \) and \( T(6, 17) \), has \( y \)-intercept 0 and slope \( = \frac{17 - 0}{6 - 0} = \frac{17}{6} \).

Therefore, the equation of line \( l \) is \( y = \frac{17}{6}x \) or \( 17x - 6y = 0 \).

**Solution 2**

Since \( TD = 10 \) and \( CD = 15 \), \( CT = CD - TD = 15 - 10 = 5 \).
\( \triangle TDA \) is right-angled so, using the Pythagorean Theorem,
\[
AT^2 = AD^2 + TD^2 \\
(r - 0)^2 + (s - 0)^2 = 15^2 + 10^2 \\
r^2 + s^2 = 325 \tag{1}
\]

Using the distance formula, we can calculate the length of each of \( CT \) and \( TD \),
\[
CT = \sqrt{(r - 3)^2 + (s - 21)^2} \quad \text{and} \quad TD = \sqrt{(r - 12)^2 + (s - 9)^2}
\]

Squaring both sides and simplifying
\[
CT^2 = r^2 - 6r + 9 + s^2 - 42s + 441 \quad \text{and} \quad TD^2 = r^2 - 24r + 144 + s^2 - 18s + 81
\]
Substituting \( CT = 5 \) and \( TD = 10 \),
\[
5^2 = r^2 + s^2 - 6r - 42s + 450 \quad \text{and} \quad 10^2 = r^2 + s^2 - 24r - 18s + 225
\]
Rearranging
\[
6r + 42s = r^2 + s^2 + 425 \quad \text{and} \quad 24r + 18s = r^2 + s^2 + 125
\]
From (1), \( r^2 + s^2 = 325 \) so
\[
6r + 42s = 325 + 425 \quad \text{and} \quad 24r + 18s = 325 + 125
\]
\[
6r + 42s = 750 \tag{2} \quad \text{and} \quad 24r + 18s = 450 \tag{3}
\]

We now have a system of equations. Equation (3) subtract \( 4 \times \) equation (2) gives \(-150s = -2550\) and \( s = 17 \) follows. Substituting for \( s \) in (2), we obtain \( r = 6 \).

The line \( l \) passes through \( A(0, 0) \) and \( T(6, 17) \), has \( y \)-intercept 0 and slope \( = \frac{17 - 0}{6 - 0} = \frac{17}{6} \).

Therefore, the equation of line \( l \) is \( y = \frac{17}{6}x \) or \( 17x - 6y = 0 \).

**For Further Thought:**

The point \( U \) is on \( CB \) so that the area of \( \triangle ABU \) = the area of \( \triangle UAT \) = the area of \( \triangle ATD \).

Determine the coordinates of \( U \). By finding \( U \) and \( T \), you will have found two line segments, \( AU \) and \( AT \), that divide square \( ABCD \) into three equal areas.