



Problem of the Week

Problem E and Solution

Sesquicentennial

Problem

This year, Canada celebrated its sesquicentennial, the 150th anniversary of Confederation. Many people had or are having special celebrations to honour this occasion. For one of the gatherings, an invitation was made by overlapping three squares, as shown above. Each of the squares has a positive integer side length. Side AB of the smallest square lies along side AC of the middle square which lies along side AD of the largest square. The area of the middle square not covered by the smallest square is 33 cm^2 . If $BC = CD$, determine all possible side lengths of each square.

Solution

Let $AB = x$ and $BC = y$. Therefore $CD = BC = y$. Also, since the side lengths of the squares are integers, x and y are integers.

The shaded region has area 33. The shaded region is equal to the area of the square with side length AC minus the area of the square with side length AB . Since $AB = x$ and $AC = AB + BC = x + y$, we have

$$\begin{aligned}
 33 &= (\text{area of square with side length } AC) - (\text{area of square with side length } AB) \\
 &= (x + y)^2 - x^2 \\
 &= x^2 + 2xy + y^2 - x^2 \\
 &= 2xy + y^2 \\
 &= y(2x + y)
 \end{aligned}$$

Since x and y are integers, so is $2x + y$. Therefore, $2x + y$ and y are two positive integers that multiply to give 33. Therefore, we must have $y = 1$ and $2x + y = 33$ or $y = 3$ and $2x + y = 11$ or $y = 11$ and $2x + y = 3$ or $y = 33$ and $2x + y = 1$. The last two would imply that $x < 0$, which is not possible. Therefore, $y = 1$ and $2x + y = 33$ or $y = 3$ and $2x + y = 11$.

When $y = 1$ and $2x + y = 33$, it follows that $x = 16$. Then the small square has side length $x = 16 \text{ cm}$, the middle square has side length $x + y = 17 \text{ cm}$, and the largest square has side length $x + 2y = 18 \text{ cm}$.

When $y = 3$ and $2x + y = 11$, it follows that $x = 4$. Then the small square has side length $x = 4 \text{ cm}$, the middle square has side length $x + y = 7 \text{ cm}$, and the largest square has side length $x + 2y = 10 \text{ cm}$.

Therefore, there are two possible sets of squares: $16 \text{ cm} \times 16 \text{ cm}$, $17 \text{ cm} \times 17 \text{ cm}$ and $18 \text{ cm} \times 18 \text{ cm}$ or $4 \text{ cm} \times 4 \text{ cm}$, $7 \text{ cm} \times 7 \text{ cm}$ and $10 \text{ cm} \times 10 \text{ cm}$. Each set of squares satisfies the conditions of the problem.

