Problem of the Week
Problem E and Solution
No Bills

Problem
Given an unlimited supply of Loonies, Toonies and Quarters, in how many different ways is it possible to make a total of exactly $100?

Solution
We will break into cases based on how many $2 coins we can have. For each case, we will count the number of possibilities for the number of $1 and 25¢ coins.

The maximum number of $2 coins we can have is 50, since $2 \times 50 = $100. If we have 50 $2 coins, then we do not need any $1 or 25¢ coins. Therefore, there is only one way to make a total of $100 if there are 50 $2 coins.

Suppose we have 49 $2 coins. Since $2 \times 49 = $98, to reach a total of $100, we would need two $1 and no 25¢ coins, or one $1 and four 25¢ coins, or no $1 and eight 25¢ coins. Therefore, there are 3 different ways to make a total of $100 if there are 49 $2 coins.

Suppose we have 48 $2 coins. Since $2 \times 48 = $96, to reach a total of $100, we would need four $1 and no 25¢ coins, or three $1 and four 25¢ coins, or two $1 and eight 25¢ coins, or one $1 and twelve 25¢ coins, or no $1 and sixteen 25¢ coins. Therefore, there are 5 different ways to make a total of $100 if there are 48 $2 coins.

We start to see a pattern. When we reduce the number of $2 coins by one, the number of possible combinations using that many $2 coins increases by 2. This is because we can increase the number of $1 coins by 1 or 2, so there are two new possibilities.

When there are 47 $2 coins, there are 7 possible ways to make a total of $100.
When there are 46 $2 coins, there are 9 possible ways to make a total of $100, and so on.
When there is one $2 coin, there are 99 different ways to make up the difference of $98 (you can use 0 to 98 $1 coins).
When there are no $2 coins, there are 101 different ways to get a total of $100 (you can use 0 to 100 $1 coins). Therefore, the number of different ways to make a total of exactly $100 is

\[
\begin{align*}
1 + 3 + 5 + 7 + 9 + \ldots + 99 + 101 \\
= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \ldots + 98 + 99 + 100 + 101 - (2 + 4 + 6 + 8 + \ldots + 98 + 100) \\
&= (1 + 2 + 3 + \ldots + 100 + 101) - 2(1 + 2 + 3 + 4 + \ldots + 50) \\
&= \frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) \\
&= 101(51) - 50(51) \\
&= 2601
\end{align*}
\]

Therefore, there are 2601 different combinations of coins that can be used to make $100.
Extending the ideas

Let’s look at the end of the previous computation another way.

\[
1 + 3 + 5 + 7 + 9 + \ldots + 99 + 101 \\
= \frac{101(102)}{2} - 2 \left( \frac{50(51)}{2} \right) \quad \text{(using the formula for the sum of the first } n \text{ positive integers)} \\
= 101(51) - 50(51) \quad \text{(simplify)} \\
= 51(101 - 50) \quad \text{(common factor 51 from both terms)} \\
= 51(51) \quad \text{(simplify)} \\
= 51^2
\]

How many odd integers are in the list 1 to 101?

From 1 to 101, there are 101 integers.
This list contains the even integers, 2 to 100, 50 in total.
Therefore, there are \(101 - 50 = 51\) odd integers from 1 to 101.

Is it a coincidence that the sum of the first 51 odd positive integers is \(51^2\)? Is the sum of the first 1000 odd positive integers \(1000^2\)? Is the sum of the first \(n\) odd positive integers \(n^2\)?

We will develop a formula for the sum of the first \(n\) odd positive integers.

We saw in the problem statement that the sum of the first \(n\) positive integers is

\[
1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}
\]

Every odd positive integer can be written in the form \(2n - 1\), where \(n\) is an integer \(\geq 1\). When \(n = 1\), \(2n - 1 = 2(1) - 1 = 1\); when \(n = 2\), \(2n - 1 = 2(2) - 1 = 3\), and so on. So the 51st odd positive integer is \(2(51) - 1 = 101\), as we determined above. The \(n\)th odd positive integer is \(2n - 1\). Let’s consider the sum of the first \(n\) odd positive integers. That is,

\[
1 + 3 + 5 + 7 + \ldots + (2n - 3) + (2n - 1)
\]

\[
= 1 + 2 + 3 + 4 + 5 + \ldots + (2n - 3) + (2n - 2) + (2n - 1) + 2n - (2 + 4 + 6 + \ldots + (2n - 2) + 2n) \quad \text{(add and subtract the even numbers from 2 to } 2n) \\
= (1 + 2 + 3 + 4 + \ldots + 2n) - (2 + 4 + 6 + 8 + \ldots + (2n - 2) + 2n) \\
= (1 + 2 + 3 + 4 + \ldots + 2n) - 2(1 + 2 + 3 + \ldots + n) \quad \text{(factor out a 2 from the even numbers)} \\
= \frac{2n(2n + 1)}{2} - 2 \left( \frac{n(n + 1)}{2} \right) \quad \text{(using the formula for the sum of the first } n \text{ integers)} \\
= n(2n + 1) - n(n + 1) \quad \text{(simplify)} \\
= 2n^2 + n - n^2 - n \quad \text{(simplify)} \\
= n^2
\]

Therefore, the sum of the first \(n\) odd positive integers is \(n^2\).

For further thought.

Can you develop a formula for the sum of the first \(n\) even positive integers?