



Problem of the Week

Problem C and Solution

What's the Score?

Problem

In gym class, the yellow team and the blue team played soccer. Ali doesn't remember the final score of the game, but she does remember the following.

- There were six goals scored in total.
- Neither team scored more than two goals in a row at any point in the game.
- The blue team won the game.

Determine all the possible final scores and the different ways each score could have been obtained.

Solution

In order to win, the blue team must have scored more goals than the yellow team. Since there were six goals scored in total, the only possibilities for the final scores are $4 - 2$, $5 - 1$, or $6 - 0$ for the blue team.

Next we need to check which of these scores are possible, given that neither team scored more than two goals in a row at any point in the game.

- Is a final score of $6 - 0$ possible?

We can easily eliminate $6 - 0$, since the blue team would have had to score more than two goals in a row.

- Is a final score of $5 - 1$ possible?

This would mean that the blue team scored 5 goals and the yellow team scored 1 goal. Is there a way to arrange these goals so that the blue team didn't score two goals in a row? Let's look at all the possible arrangements, where B represents a goal for the blue team, and Y represents a goal for the yellow team. These are all shown below.

$YBBBBB$, $BYBBBB$, $BBYBBB$, $BBBYBB$, $BBBBYB$, $BBBBBY$

As we can see, in all of these arrangements, the blue team scored more than two goals in a row. Thus, a final score of $5 - 1$ is not possible.



- Is a final score of $4 - 2$ possible?

This would mean that the blue team scored 4 goals and the yellow team scored 2 goals. Is there a way to arrange these goals so that the blue team didn't score two goals in a row? Let's look at all the possible arrangements, where B represents a goal for the blue team, and Y represents a goal for the yellow team.

- Case 1: The yellow team scored their 2 goals in a row. The possible arrangements are shown below.

YYBBBB, BYYBBB, BBYYBB, BBBYYB, BBBBY

In this case, there is only 1 arrangement where neither team scored more than two goals in a row, namely *BBYYBB*.

- Case 2: The yellow team did not score their 2 goals in a row. The possible arrangements are shown below.

YBYBBB, YBBYBB, YBBBYB, YBBBBY, BYBYBB, BYBBYB, BYBBBY, BBYBYB, BBYBBY, BBBYBY

In this case, there are 5 arrangements where neither team scored more than two goals in a row, namely

YBBYBB, BYBYBB, BYBBYB, BBYBYB, and BBYBBY.

Therefore, the only possible final score is $4 - 2$ for the blue team, and it could be obtained in the following six ways.

BBYYBB, YBBYBB, BYBYBB, BYBBYB, BBYBYB, BBYBBY