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Problem of the Week
Problem C and Solution
Six Zeros

Problem

The product of the first seven positive integers is equal to

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Mathematicians will write this product as 7!. This is read as "7 factorial". So, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

This factorial notation can be used with any positive integer. For example, $11! = 11 \times 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 39\,916\,800$. The three dots "$\cdots$" represent the product of the integers between 9 and 3.

In general, for a positive integer n, $n!$ is equal to the product of the positive integers from 1 to n.

Find the smallest positive integer n such that $n!$ ends in exactly six zeros.

Solution

We start by examining the first few factorials:

$$\begin{aligned}
1! &= 1\\
2! &= 2 \times 1 = 2\\
3! &= 3 \times 2 \times 1 = 6\\
4! &= 4 \times 3 \times 2 \times 1 = 24\\
5! &= 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{120}\\
6! &= 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times 5! = 6(120) = 720\\
7! &= 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 7 \times 6! = 7(720) = 5040\\
8! &= 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7! = 8(5040) = 40\,320\\
9! &= 9 \times (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 9 \times 8! = 9(40\,320) = 362\,880\\
10! &= 10 \times (9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 10 \times 9! = 10(362\,880) = \mathbf{3\,628\,800}
\end{aligned}$$

These numbers are getting very large and soon will not fit on the display of a standard calculator. So, let's look at what is going on.

We observe that 5! ends in 0 and 10! ends in 00. Notice that the number of zeros at the end of the number increased by one at each of 5! and at 10!. Why is this?

A zero is added to the end of a positive integer when we multiply by 10. Multiplying a number by 10 is the same as multiplying a number by 2 and then by 5, or by 5 and then by 2, since $2 \times 5 = 10$ and $5 \times 2 = 10$. We must determine the next time we multiply by 2 and 5 (in some order), to know the next time the number of zeros at the end of the number increases again. Every time we multiply by an even positive integer we are multiplying by at least one more 2. In the integers from 1 to n, there are less multiples of 5. So, each multiple of 5 will affect the number of zeros at the end of the product.

Multiplying by 11, 12, 13, and 14 increases the number of 2s we multiply by but not the number of 5s. So the number of zeros at the end of the product does not change. The next time we multiply by a 5 is when we multiply by 15 since $15 = 5 \times 3$. So 15! will end in exactly three zeros, 000.

Multiplying by 16, 17, 18, and 19 increases the number of 2s we multiply by but not the number of 5s. So the number of zeros at the end of the product does not change. The next time we multiply by a 5 is when we multiply by 20 since $20 = 4 \times 5$. So 20! will end in exactly four zeros, 0000.

Multiplying by 21, 22, 23, and 24 increases the number of 2s we multiply by but not the number of 5s. The next time we multiply by a 5 is when we multiply by 25. In fact, multiplying by 25 is the same as multiplying by 5 twice since $25 = 5 \times 5$. So when we multiply by 25, we will increase the number of zeros on the end of the product by two. So 25! will end in exactly six zeros, 000 000.

Therefore, the smallest positive integer n such that $n!$ ends in exactly six zeros is 25. (It could be noted that 26!, 27!, 28!, and 29! also end in six zeros.)

For the curious,

$$24! = 620\,448\,401\,733\,239\,439\,360\,000$$

and

$$25! = 15\,511\,210\,043\,330\,985\,984\,000\,000$$