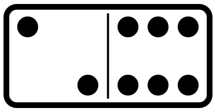


# Problem of the Week

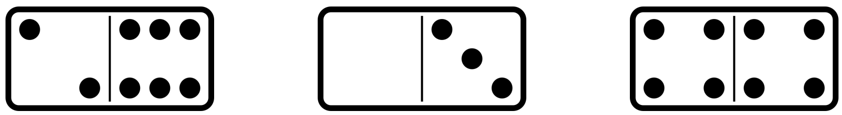
## Problem D and Solution

## Missing Tile

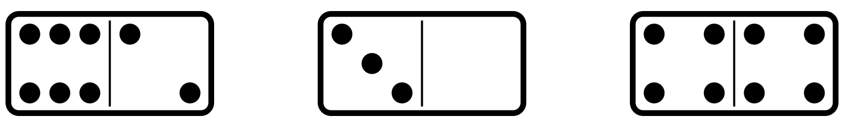
### Problem

A domino tile is a rectangular tile with a line dividing its face into two square ends. Each end is marked with a number of dots (also called pips) or is blank.

The first domino shown below is a $[2, 6]$ domino, since there are 2 pips on its left end and 6 pips on its right end. The second domino shown below is a $[0, 3]$ domino, since there are 0 pips on its left end and 3 pips on its right end. The third domino shown below is a $[4, 4]$ domino, since there are 4 pips on its left end and 4 pips on its right end.



We can also rotate the domino tiles. The first domino shown below is a $[6, 2]$ domino, since there are 6 pips on its left end and 2 pips on its right end. However, since this tile can be obtained by rotating the $[2, 6]$ tile, $[6, 2]$ and $[2, 6]$ represent the same domino. Similarly, the second domino shown below is a $[3, 0]$ domino. Again, note that $[3, 0]$ and $[0, 3]$ represent the same domino.



A 2-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 2, with no two dominoes being the same. A 2-set of dominoes has the following six tiles: $[0, 0]$, $[0, 1]$, $[0, 2]$, $[1, 1]$, $[1, 2]$, $[2, 2]$. Notice that the three dominoes $[1, 0]$, $[2, 0]$, and $[2, 1]$ are not listed because they are the same as the three dominoes $[0, 1]$, $[0, 2]$, and $[1, 2]$.

Similarly, a 9-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 9, with no two dominoes being the same.

Drew and Bennett separate a 9-set of dominoes into two piles. Drew counts all of the pips on the dominoes in the first pile. He counts that there are a total of 213 pips. Bennett counts all of the pips on the dominoes in the second pile. He counts that there are a total of 266 pips. They then realize that one domino is missing from the set. Drew also notes that every domino that has the same number of pips on its left and right ends is accounted for. Which domino is missing from the set?

## Solution

We first determine which dominoes are in a 9-set of dominoes and calculate the total number of pips on all of the dominoes in the set. In a 9-set of dominoes, the number of pips on each end of a domino tile can range from 0 to 9. Since rotating a domino tile does not change the domino, we orient each domino so that the smaller number is always on the left end of the domino. For each possible number on the left end of the domino, we examine the possible numbers that can occur on the right end of the domino, and then calculate the total number of pips on all dominoes with that number of pips on the left end. We compile this information in a table.

| Number on Left End of Domino | Possible Numbers on Right End of Domino | Total Number of Pips on All Dominoes |
|---|---|---|
| 0 | 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 | $1+2+3+4+5+6+7+8+9=45$ |
| 1 | 1, 2, 3, 4, 5, 6, 7, 8, 9 | $9(1)+1+2+3+4+5+6+7+8+9=54$ |
| 2 | 2, 3, 4, 5, 6, 7, 8, 9 | $8(2)+2+3+4+5+6+7+8+9=60$ |
| 3 | 3, 4, 5, 6, 7, 8, 9 | $7(3)+3+4+5+6+7+8+9=63$ |
| 4 | 4, 5, 6, 7, 8, 9 | $6(4)+4+5+6+7+8+9=63$ |
| 5 | 5, 6, 7, 8, 9 | $5(5)+5+6+7+8+9=60$ |
| 6 | 6, 7, 8, 9 | $4(6)+6+7+8+9=54$ |
| 7 | 7, 8, 9 | $3(7)+7+8+9=45$ |
| 8 | 8, 9 | $2(8)+8+9=33$ |
| 9 | 9 | $1(9)+9=18$ |

Therefore, the total number of pips on all of the dominoes in a 9-set of dominoes is

$$45+54+60+63+63+60+54+45+33+18=495$$

Now, the total number of pips in the two piles is $213+266=479$. That leaves a total of 16 pips on the missing tile.

In a 9-set of dominoes, the only tiles with a total of 16 pips are $[8,8]$ and $[7,9]$. Since every domino with the same number of pips on its left and right ends is present, then the $[8,8]$ tile is present. Therefore, the missing tile must be the $[7,9]$ tile.