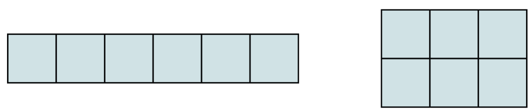
Problem of the Week
Problem A and Solution
Rearranging Rectangles

Problem

Ravi arranges identical square tiles to form rectangles using the following rules:

1. Tiles must line up exactly without any gaps or overlaps.
2. The width of each rectangle must be larger than its height.

Using 6 tiles, Ravi can form two different rectangles. The first has a width of 6 and a height of 1, and the second has a width of 3 and a height of 2, as shown.



(a) Draw all the rectangles Ravi can form with 15 square tiles.

(b) Draw all the rectangles Ravi can form with 24 square tiles.

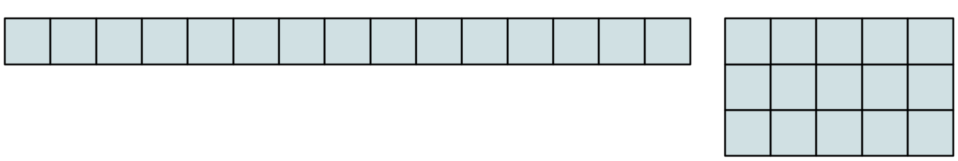
(c) Draw all the rectangles Ravi can form with 17 square tiles.

(d) Can Ravi form more rectangles with 24 square tiles or 35 square tiles?
Justify your answer.

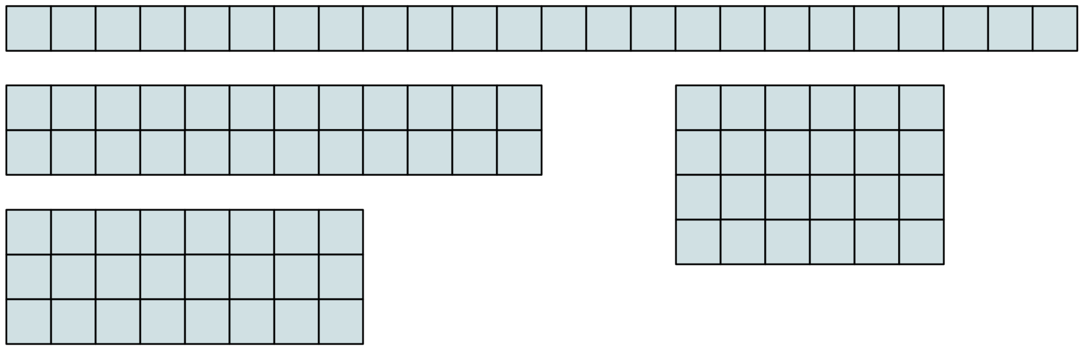
(e) **Challenge:** Ravi has some number of tiles less than 100 and is able to form only one rectangle. What is the largest number of tiles that Ravi could have?

Solution

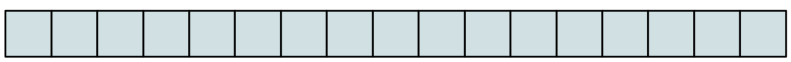
(a) There are two possible rectangles that Ravi can form with 15 tiles.
They have dimensions 15×1 and 5×3.



(b) There are four possible rectangles that Ravi can form with 24 tiles.
They have dimensions 24×1, 12×2, 8×3, and 6×4.



(c) There is only one possible rectangle that Ravi can form with 17 tiles.
It has dimensions 17×1.



(d) There are two possible rectangles that Ravi can form with 35 tiles. They have dimensions 35×1 and 7×5. From part (b) we know that Ravi can form four rectangles with 24 tiles. So Ravi can form more rectangles with 24 square tiles than with 35 square tiles.

Another way to justify the answer is to notice the relationship between the number of rectangles Ravi can form and the number of whole numbers that are *divisors* of the number of tiles we start with in each case. Divisors are the whole numbers that divide exactly into the number of tiles that we start with. The whole number 1 is a divisor of every whole number. A whole number is always a divisor of itself.

There are 4 whole numbers that are divisors of 15: $1,\ 3,\ 5,\ 15$, and Ravi can form 2 rectangles.

There are 8 whole numbers that are divisors of 24: $1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24$, and Ravi can form 4 rectangles.

There are 2 whole numbers that are divisors of 17: $1,\ 17$, and Ravi can form 1 rectangle.

Generally, the more divisors a number has, the more rectangles Ravi can form with that number of tiles. Specifically, if there are an even number of divisors, Ravi can form half that number of rectangles. Note that if there are an odd number of divisors, then one of the rectangles Ravi can form will actually be a square. However, according to Ravi's rules, the width of each rectangle must be larger than its height, so squares are not allowed.

Since there are 4 whole numbers that are divisors of 35: $1,\ 5,\ 7,\ 35$, then we predict that Ravi can form 2 rectangles. This means Ravi can form more rectangles with 24 tiles than with 35 tiles.

(e) One way to figure out the largest number of tiles, less than 100, that can only form one rectangle is to look for the largest number that has only two whole numbers that are divisors of it: 1 and itself. Working backwards, we know that 99 is divisible by 3, and 98 is divisible by 2. However, 97 is not divisible by any whole numbers except 1 and 97. So the largest number of tiles that Ravi could have is 97.

Teacher's Notes

Rearranging identical square tiles into rectangles is a way to find the *factors* of a whole number. A factor of a whole number is another name for a divisor of a whole number.

A *prime number* is a whole number greater than 1 that has exactly two whole number factors: 1 and itself. So 2 is a prime number, but 4 is not a prime number. In our problem of finding the number of rectangles that can be formed by identical square tiles, a prime number of tiles will always result in one rectangle that has a height of 1 and a width equal to the number of tiles.

A *composite number* is a whole number greater than 1 that has more than two whole number factors. A composite number that is a perfect square has an odd number of whole number factors. Most factors come in pairs of two different numbers, where the two numbers multiplied together equal the number you are factoring. With a perfect square, one of the factors of the number is its square root. The square root of a number n is a number that when multiplied by itself is equal to n. So the factor that is equal to the square root does not have a different number to pair up with, which results in an odd number of factors.

In our problem, if we had a number of tiles that is a perfect square of a prime number, this would result in exactly one rectangle, since the width must be larger than its height. In these cases, we could form a square out of the tiles, where the length of the side of the square is equal to the square root of the number. An example of this would be the number 49. It has three whole number factors: 1, 7, and 49, and you can only form one rectangle (with dimensions 1×49) and one square (with dimensions 7×7) out of 49 tiles.