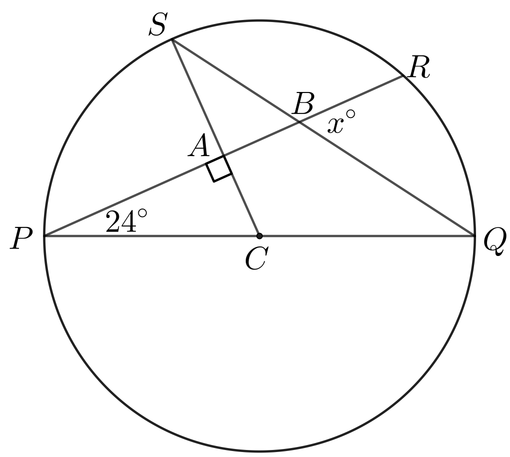
Problem of the Week

Problem D and Solution

Find that Angle

Problem

A circle with centre C has a diameter PQ and radius CS. Chord PR intersects CS and chord SQ at A and B, respectively. If $\angle CAP = 90^\circ$, $\angle RPQ = 24^\circ$, and $\angle QBR = x^\circ$, then determine the value of x.



Solution

Solution 1

Since the angles in a triangle sum to 180°, in $\triangle CAP$, $\angle ACP = 180^\circ - 90^\circ - 24^\circ = 66^\circ$.

Since PQ is a diameter, PCQ is therefore a straight line. Thus, $\angle ACP + \angle QCS = 180^\circ$, and so $\angle QCS = 180^\circ - \angle ACP = 180^\circ - 66^\circ = 114^\circ$.

Since CS and CQ are radii of the circle, we have $CS = CQ$. It follows that $\triangle CSQ$ is isosceles and $\angle CQS = \angle CSQ$. Since the angles in a triangle sum to 180°, we have $\angle CSQ + \angle CQS + \angle QCS = 180^\circ$, and since $\angle CQS = \angle CSQ$, this gives

$$\begin{aligned} 2\angle CSQ + 114^\circ &= 180^\circ \\ 2\angle CSQ &= 66^\circ \\ \angle CSQ &= 33^\circ \end{aligned}$$

Opposite angles are equal, so it follows that $\angle SBA = \angle QBR = x^\circ$ and $\angle SAB = \angle CAP = 90^\circ$.

In $\triangle ABS$, $\angle SBA + \angle SAB + \angle ASB = 180^\circ$. Since $\angle ASB = \angle CSQ = 33^\circ$, we have

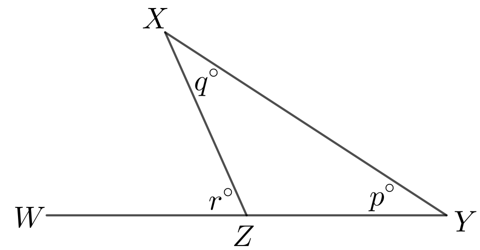
$$\begin{aligned} x^\circ + 90^\circ + 33^\circ &= 180^\circ \\ x + 123 &= 180 \\ x &= 57 \end{aligned}$$

Therefore, $x = 57$.

Solution 2

This solution will use the *exterior angle theorem*. In a triangle, the angle formed at a vertex between one side of the triangle and the extension of the other side of the triangle is called an exterior angle.

The exterior angle theorem state that the measure of an exterior angle of a triangle is equal to the sum of the two opposite interior angles.
For example, in the diagram shown, $\angle XZW$ is exterior to $\triangle XYZ$. The exterior angle theorem states that $r^\circ = p^\circ + q^\circ$.



Since the angles in a triangle sum to 180°, in $\triangle CAP$, $\angle ACP = 180^\circ - 90^\circ - 24^\circ = 66^\circ$.

Since CS and CQ are radii of the circle, we have $CS = CQ$. It follows that $\triangle CSQ$ is isosceles and $\angle CQS = \angle CSQ$. Since $\angle ACP$ is exterior to $\triangle CSQ$, by the exterior angle theorem,

$$\begin{aligned}\angle ACP &= \angle CQS + \angle CSQ \\ 66^\circ &= 2\angle CQS \\ \angle CSQ &= 33^\circ\end{aligned}$$

Since $\angle QBR$ is exterior to $\triangle BQP$, by the exterior angle theorem, $\angle QBR = \angle BPQ + \angle BQP$. Since $\angle QBR = x^\circ$, $\angle BPQ = \angle RPQ$ (same angle), and $\angle BQP = \angle CQS$ (same angle), this gives

$$\begin{aligned}x^\circ &= \angle RPQ + \angle CQS \\ x &= 24 + 33 \\ x &= 57\end{aligned}$$

Therefore, $x = 57$.

Solution 3

This solution uses the *exterior angle theorem* from Solution 2, as well as the *angle subtended by an arc theorem*. This theorem states that the measure of the angle subtended by an arc at the centre is equal to two times the measure of the angle subtended by the same arc at any point on the remaining part of the circle.

Since $\angle SCP$ is the angle at the centre subtended by arc SP, and $\angle SQP$ is an angle subtended by that same arc but on the circle, we know that $\angle SCP = 2\angle SQP$.

Since the angles in a triangle sum to 180°, in $\triangle CAP$, $\angle ACP = 180^\circ - 90^\circ - 24^\circ = 66^\circ$. Thus, since $\angle SCP = \angle ACP$ (same angle), we have $\angle SCP = \angle ACP = 66^\circ$. Therefore, $\angle SCP = 2\angle SQP$ gives $66^\circ = 2\angle SQP$, and thus $\angle SQP = 33^\circ$.

Since $\angle QBR$ is exterior to $\triangle BQP$, by the exterior angle theorem, $\angle QBR = \angle BPQ + \angle BQP$. Since $\angle QBR = x^\circ$, $\angle BPQ = \angle RPQ$ (same angle), and $\angle BQP = \angle SQP$ (same angle), this gives

$$\begin{aligned}x^\circ &= \angle RPQ + \angle SQP \\ x &= 24 + 33 \\ x &= 57\end{aligned}$$

Therefore, $x = 57$.