Problem of the Week
Problem D and Solution
Pi Squares

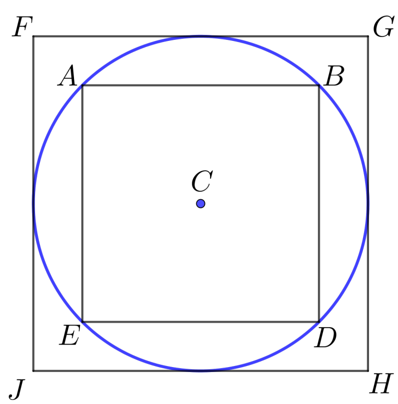
Problem

Pi Day is an annual celebration of the mathematical constant π. Pi Day is observed on March 14, since 3, 1, and 4 are the first three significant digits of π.

Archimedes determined lower bounds for π by finding the perimeters of regular polygons inscribed in a circle with diameter of length 1. (An inscribed polygon of a circle has all of its vertices on the circle.) He also determined upper bounds for π by finding the perimeters of regular polygons circumscribed in a circle with diameter of length 1. (A circumscribed polygon of a circle has all sides tangent to the circle. That is, each side of the polygon touches the circle in one spot.)

In this problem, we will determine a lower bound for π and an upper bound for π by considering an inscribed square and a circumscribed square in a circle of diameter 1.

Consider a circle with centre C and diameter 1. Since the circle has diameter 1, it has circumference equal to π. Now consider the inscribed square $ABDE$ and the circumscribed square $FGHJ$.

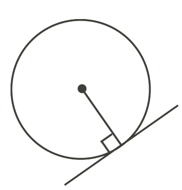


The perimeter of square $ABDE$ will be less than the circumference of the circle, π, and will thus give us a lower bound for the value of π. The perimeter of square $FGHJ$ will be greater than the circumference of the circle, π, and will thus give us an upper bound for the value of π.

Using these squares, determine a lower bound and an upper bound for π.

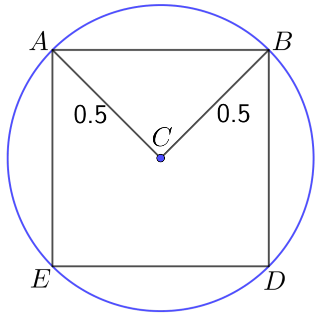
NOTE: For this problem, you may want to use the following known results about circles:

1. For a circle with centre C, the diagonals of an inscribed square meet at 90° at C.
2. For a circle with centre C, the diagonals of a circumscribed square meet at 90° at C.
3. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.



Solution

For the inscribed square $ABDE$, draw line segments AC and BC. Both AC and BC are radii of the circle with diameter 1, so $AC = BC = 0.5$.



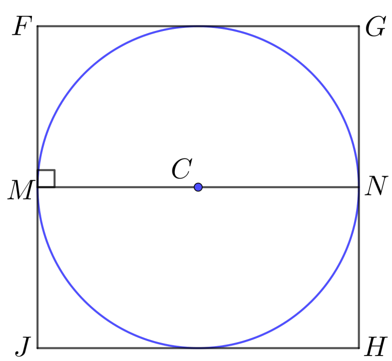
Since the diagonals of square $ABDE$ meet at $90°$ at C, it follows that $\triangle ACB$ is a right-angled triangle with $\angle ACB = 90°$. We can use the Pythagorean Theorem to find the length of AB.

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (0.5)^2 + (0.5)^2 \\ &= 0.25 + 0.25 \\ &= 0.5 \end{aligned}$$

Therefore, $AB = \sqrt{0.5}$, since $AB > 0$.

Since AB is one of the sides of the inscribed square, the perimeter of square $ABDE$ is equal to $4 \times AB = 4\sqrt{0.5}$. This gives us a lower bound for π. That is, we know $\pi > 4\sqrt{0.5} \approx 2.828$.

For the circumscribed square, let M be the point of tangency on side FJ and let N be the point of tangency on GH. Draw radii CM and CN. Since M is a point of tangency, we know that $\angle FMC = 90°$, and thus CM is parallel to FG. Similarly, CN is parallel to FG.



Thus, MN is a straight line segment, and since it passes through C, the centre of the circle, MN must also be a diameter of the circle. Thus, $MN = 1$. Also, $FMNG$ is a rectangle, so $FG = MN = 1$ and the perimeter of square $FGHJ$ is equal to $4 \times FG = 4(1) = 4$. This gives us an upper bound for π. That is, we know $\pi < 4$.

Therefore, a lower bound for π is $4\sqrt{0.5} \approx 2.828$ and an upper bound for π is 4. That is, $4\sqrt{0.5} < \pi < 4$.

Note: Since we know that $\pi \approx 3.14$, these are not the best bounds for π. Archimedes used regular polygons with more sides to get better bounds. In the Problem of the Week E problem, we investigate using regular hexagons to get better bounds.