



Problem of the Week

Problem E and Solution

Picking Boxes

Problem

Billy's Box Company sells boxes with the following very particular restrictions on their dimensions.

- The length, width, and height, in cm, must be all integers.
- The ratio of the length to the width to the height must be $4 : 3 : 5$.
- The sum of the length, width, and height must be between 100 cm and 1000 cm, inclusive.

Stefan bought the box with the smallest possible volume, and Lali bought the box with the largest volume less than 4 m^3 .

Determine the dimensions of Stefan and Lali's boxes.

Solution

Since the boxes from Billy's Box Company have integer side lengths in the ratio $4 : 3 : 5$, let $4n$ represent the length of a box in cm, let $3n$ represent the width of a box in cm, and let $5n$ represent the height of a box in cm, where n is an integer. Furthermore, the sum of the length, width and height must be at least 100 cm. It follows that

$$\begin{aligned}4n + 3n + 5n &\geq 100 \\12n &\geq 100 \\n &\geq \frac{100}{12} = 8\frac{1}{3}\end{aligned}$$

Also, the sum of the length, width and height must be at most 1000 cm. It follows that

$$\begin{aligned}4n + 3n + 5n &\leq 1000 \\12n &\leq 1000 \\n &\leq \frac{1000}{12} = 83\frac{1}{3}\end{aligned}$$



There is one other restriction to consider, since the volume of Lali's box is less than 4 m^3 . To convert from m^3 to cm^3 , note that

$$\begin{aligned} 1 \text{ m}^3 &= 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} \\ &= 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm} \\ &= 1\,000\,000 \text{ cm}^3 \end{aligned}$$

Therefore, $4 \text{ m}^3 = 4\,000\,000 \text{ cm}^3$.

It follows that

$$\begin{aligned} (4n)(3n)(5n) &< 4\,000\,000 \\ 60n^3 &< 4\,000\,000 \\ n^3 &< \frac{200\,000}{3} \\ n &< \sqrt[3]{\frac{200\,000}{3}} \approx 40.5 \end{aligned}$$

We also know that n is an integer. Since $n \geq 8\frac{1}{3}$, then the smallest possible integer value of n is 9. Using the dimensions $4n$, $3n$, and $5n$ with $n = 9$, we can determine that the dimensions of Stefan's box are 36 cm by 27 cm by 45 cm.

For Lali's box, since $n \leq 83\frac{1}{3}$ and $n < 40.5$, then the largest possible value of n is 40. Using the dimensions $4n$, $3n$, and $5n$ with $n = 40$, we can determine that the dimensions of Lali's box are 160 cm by 120 cm by 200 cm. This box has a volume of 3.84 m^3 .