Problem of the Week

Problem E and Solution

Pi Hexagons

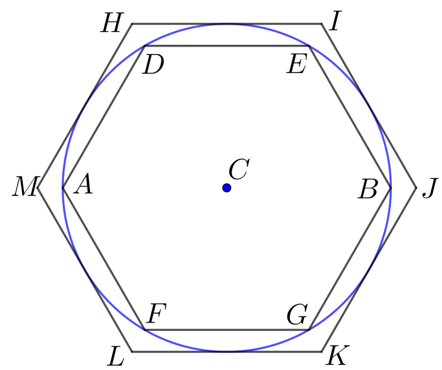
Problem

Pi Day is an annual celebration of the mathematical constant π. Pi Day is observed on March 14, since 3, 1, and 4 are the first three significant digits of π.

Archimedes determined lower bounds for π by finding the perimeters of regular polygons inscribed in a circle with diameter of length 1. (An inscribed polygon of a circle has all of its vertices on the circle.) He also determined upper bounds for π by finding the perimeters of regular polygons circumscribed in a circle with diameter of length 1. (A circumscribed polygon of a circle has all sides tangent to the circle. That is, each side of the polygon touches the circle in one spot.)

In this problem, we will determine a lower bound for π and an upper bound for π by considering an inscribed regular hexagon and a circumscribed regular hexagon in a circle of diameter 1.

Consider a circle with centre C and diameter 1. Since the circle has diameter 1, it has circumference equal to π. Now consider the inscribed regular hexagon $DEBGFA$ and the circumscribed regular hexagon $HIJKLM$.

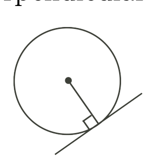


The perimeter of hexagon $DEBGFA$ will be less than the circumference of the circle, π, and will thus give us a lower bound for the value of π. The perimeter of hexagon $HIJKLM$ will be greater than the circumference of the circle, π, and will thus give us an upper bound for the value of π.

Using these hexagons, determine a lower and an upper bound for π.

NOTE: For this problem, you may want to use the following known results:

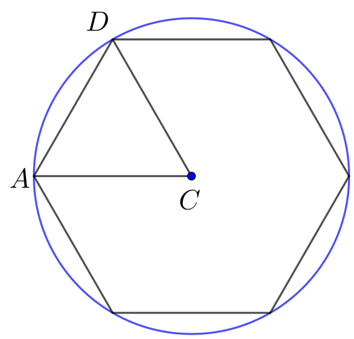
1. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.



2. For a circle with centre C, the centres of both the inscribed and circumscribed regular hexagons will be at C.

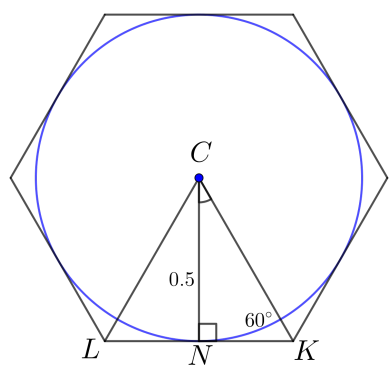
Solution

For the inscribed hexagon, draw line segments AC and DC, which are both radii of the circle.



Since the diameter of the circle is 1, $AC = DC = \frac{1}{2}$. Since the inscribed hexagon is a regular hexagon with centre C, we know that $\triangle ACD$ is equilateral (a justification of this is provided at the end of the solution). Thus, $AD = AC = \frac{1}{2}$, and the perimeter of the inscribed regular hexagon is $6 \times AD = 6\left(\frac{1}{2}\right) = 3$. Since the perimeter of this hexagon is less than the circumference of the circle, this gives us a lower bound for π. That is, this tells us that $\pi > 3$.

For the circumscribed hexagon, draw line segments LC and KC. Since the circumscribed hexagon is a regular hexagon with centre C, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Thus, $\angle LKC = 60^\circ$. Drop a perpendicular from C, meeting LK at N. We know that N must be the point of tangency. Thus, CN is a radius and so $CN = 0.5$. In $\triangle CNK$, $\angle NKC = \angle LKC = 60^\circ$.



Since $\angle CNK = 90^\circ$,

$$\begin{aligned}
\sin(\angle NKC) &= \frac{CN}{KC} \\
\sin(60^\circ) &= \frac{0.5}{KC} \\
\frac{\sqrt{3}}{2} &= \frac{0.5}{KC} \\
\sqrt{3}KC &= 1 \\
KC &= \frac{1}{\sqrt{3}}
\end{aligned}$$

But $\triangle LCK$ is equilateral, so $LK = KC = \frac{1}{\sqrt{3}}$.

Thus, the perimeter of the circumscribed hexagon is $6 \times LK = 6 \times \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \approx 3.46$.

Since the perimeter of this hexagon is greater than the circumference of the circle, this gives us an upper bound for π. That is, this tells us that $\pi < \frac{6}{\sqrt{3}}$.

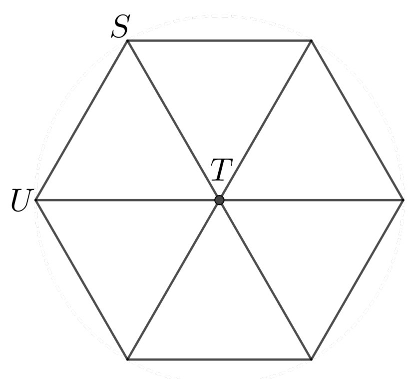
Therefore, the value for π is between 3 and $\frac{6}{\sqrt{3}}$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

Extension: Archimedes used regular 12-gons, 24-gons, 48-gons and 96-gons to get better approximations for the bounds on π. Can you?

Equilateral triangle justification:

In the solutions, we used the fact that both $\triangle ACD$ and $\triangle LCK$ are equilateral. In fact, a regular hexagon can be split into six equilateral triangles by drawing line segments from the centre of the hexagon to each vertex, which we will now justify.

Consider a regular hexagon with centre T. Draw line segments from T to each vertex and label two adjacent vertices S and U.



Since T is the centre of the hexagon, T is of equal distance to each vertex of the hexagon. Since the hexagon is a regular hexagon, each side of the hexagon has equal length. Thus, the six resultant triangles are congruent. Therefore, the six central angles are equal and each is equal to $\frac{1}{6}(360°) = 60°$.

Now consider $\triangle STU$. We know that $\angle STU = 60°$. Also, $ST = UT$, so $\triangle STU$ is isosceles and $\angle TSU = \angle TUS = \frac{180°-60°}{2} = 60°$.

Therefore, all three angles in $\triangle STU$ are equal to $60°$ and so $\triangle STU$ is equilateral. Since the six triangles in the hexagon are congruent, this tells us that all six triangles are all equilateral.