



Problem of the Week

Problem C and Solution

A Multiple Problem

Problem

How many integers between 100 and 2024 are multiples of both 5 and 7, but are not multiples of 10?

Solution

The integers that are multiples of both 5 and 7 are the integers that are multiples of 35. Now let's determine which multiples of 35 are also multiples of 10. Notice that $1 \times 35 = 35$, which is not a multiple of 10. However, $2 \times 35 = 70$, which is a multiple of 10. In fact, multiplying 35 by any even integer will result in a multiple of 10. This is because 35 is a multiple of 5, and all even integers are multiples of 2. So multiplying 35 by an even integer will result in an integer which is a multiple of both 5 and 2, and thus a multiple of 10. So if we are looking for integers that are multiples of 35 but not multiples of 10, then we must multiply 35 by odd integers only.

The smallest multiple of 35 greater than 100 is $3 \times 35 = 105$. Similarly, the largest multiple of 35 less than 2024 is $57 \times 35 = 1995$. It follows that the number of integers between 100 and 2024 that are multiples of both 5 and 7, but are not multiples of 10, is equal to the number of odd integers between 3 and 57, inclusive. This is equal to the number of odd integers between 1 and 55, inclusive. We know that exactly half of the integers between 1 and 54 are odd, and 55 is an odd integer. So in total, there are $\frac{54}{2} + 1 = 27 + 1 = 28$ odd integers between 1 and 55, inclusive.

Thus, there are 28 integers between 100 and 2024 that are multiples of both 5 and 7, but are not multiples of 10.