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February 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

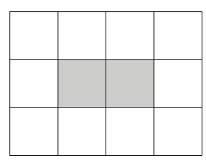
1. *1973 Junior Mathematics Contest, Question 8*

   A rectangular block with dimensions 4 cm $\times$ 3 cm $\times$ 2 cm has its surface painted red, and is then cut into cubes with edge length 1 cm. The number of cubes having exactly one of its faces painted red is

   (A) 0 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 24

   Solution

   One of the 4 cm $\times$ 3 cm faces is illustrated below (12 cubes are visible).

   Each edge cube has more than one red face. Hence only the two central cubes have only one red face.
   Similarly there are two central cubes on the bottom surface.
   The other faces have no central cubes, only edge cubes.
   Thus there are a total of four cubes having exactly one of its faces painted red.

   ANSWER: (B)

2. *1969 Junior Mathematics Contest, Question 28*

   If x, y are integers such that $(x-y)^2 + 2y^2 = 54$, then the only values that x can have are

   (A) 3 or 5

   (B) 7 or 3

   (C) -3, 3, 7, -7, 9, or -9

   (D) -3, 3, or 9

   (E) -3, 3, 7 or 9

   Solution

   Since $(x-y)^2 \geq 0$ and $2y^2 \geq 0$, then $2y^2 \leq 54$ and so $y^2 \leq 27$.
   Since y is an integer, then y^2 is a perfect square, and so can equal $0, 1, 4, 9, 16, 25$.
   Since $(x-y)^2 = 54 - 2y^2$, then the possible values of $(x-y)^2$ are $54, 52, 46, 36, 22, 4$.
   Since x and y are integers, then $(x-y)^2$ is a perfect square; of these possible values, only 36 and 4 are

perfect squares.
Therefore, $y^2 = 9$ and $(x-y)^2 = 36$, or $y^2 = 25$ and $(x-y)^2 = 4$.
In the first case, $y = \pm 3$ and $x - y = \pm 6$, which gives the pairs $(x, y) = (9, 3), (3, -3), (-3, 3), (-9, -3)$.
In the second case, $y = \pm 5$ and $x - y = \pm 2$, which gives the pairs $(x, y) = (7, 5), (-3, -5), (3, 5), (-7, -5)$.
Thus, the possible values of x are $-9, -7, -3, 3, 7, 9$.

ANSWER: (C)

3. *1978 Descartes Contest, Question 7*
The twenty volumes comprising an encyclopedia are clearly numbered from 1 to 20. If ten volumes have blue covers, six have red covers, and the remainder have green covers, determine the number of ways the books can be arranged so that no two books of the same colour are side by side.

Solution
We call the positions of the books $P1, P2, P3, \ldots, P19, P20$.
Consider the 10 blue books.
There must be at least one non-blue book in between each pair of blue books.
Thus, 10 of the positions are allocated to blue books and 9 positions are allocated to spaces between them, leaving 1 of the remaining positions that can either be before the first blue book, after the last blue book, or in between one pair of the books.
This gives the following possible locations for the blue books:

- Extra position before: $P2, P4, P6, \ldots, P18, P20$
- Extra position after: $P1, P3, P5, \ldots, P17, P19$
- Extra position between: $P1, P4, P6, \ldots, P18, P20$, or $P1, P3, P6, P8, \ldots, P18, P20$, or $P1, P3, P5, P8, P10, \ldots, P18, P20$, …, or $P1, P3, P5, \ldots, P15, P18, P20$, or $P1, P3, P5, \ldots, P15, P17, P20$

In each of the first two configurations, there are 10! of arranging the blue books. For each of these arrangements, there are $\binom{10}{6}$ ways of choosing the positions for the red books and 6! ways of arranging these red books. The positions of the 4 remaining green books are fixed, but the books can be arranged in 4! ways.
Therefore, the first two configurations give $2(10!)\frac{10!}{4!6!}(6!)(4!) = 2(10!)(10!)$ arrangements.
In the last set of nine configurations, we have to be careful about the "gap of two", since these books cannot be the same colour either. This gap of two can be filled as red-green or green-red (2 ways). There are 6 ways of choosing the red book in this pair and 4 ways of choosing the green book in this pair. The blue books can be arranged in 10! ways. Of the remaining 8 positions, 5 need to be chosen for the remaining red books after which the the positions for the remaining 3 green books are fixed.
Therefore, these nine configurations give $9(10!)(2)(6)(4)\frac{8!}{5!3!}(5!)(3!) = 9(10!)(48)(8!) = 48(10!)(9!)$ arrangements.
In total, there are $2(10!)(10!) + 48(10!)(9!) = (10!)(9!)(2(10) + 48) = 68(10!)(9!)$ arrangements.

4. *1972 Ontario Senior Mathematics Problems Competition, Question 6*
The sum of the terms in an infinite geometric sequence is 8. If the sum of the second and third terms is 3, determine all possible values of r, the common ratio between any two consecutive terms in the series.

Solution
Let the first term be a and the common ratio r. From the given information, $ar + ar^2 = 3$ and

$\dfrac{a}{1-r} = 8$. Substituting, we obtain

$$\begin{aligned} ar(1+r) &= 3 \\ 8r(1-r)(1+r) &= 3 \\ 8r(1-r^2) &= 3 \\ 8r^3 - 8r + 3 &= 0 \\ r^3 - r + \tfrac{3}{8} &= 0 \end{aligned}$$

By the Factor Theorem, $r - \frac{1}{2}$ is a factor. We obtain $(r-\frac{1}{2})(r^2+\frac{1}{2}r-\frac{3}{4}) = 0$ or $(2r-1)(4r^2+2r-3) = 0$.

By the quadratic formula, $r = \frac{1}{2}$ or $\frac{-2+\sqrt{52}}{8}$ or $\frac{-2-\sqrt{52}}{8}$.

But the sum to infinity of a geometric series can only exist if $|r| < 1$. Thus the third value is inadmissible.

Then, $r = \frac{1}{2}$ or $\frac{-2+\sqrt{52}}{8}$.

5. *1987 Descartes Contest, Question 1b*

   The line $x + y = 1$ is rotated 90° clockwise about the origin. Find the equation of the image line.

   Solution

   The given line has slope -1 and passes through $(0, 1)$.
   If the line is rotated through 90° clockwise, the image line is perpendicular to the given line, so it has slope 1.
   Since the point $(0, 1)$ is on the original line, then after rotating by 90° clockwise, the new line passes through $(1, 0)$.
   Finally, the new line has slope 1 and passes through $(1, 0)$, so has equation $y = x - 1$.