Exponents and Logarithms

Toolkit

Exponents

Let a, b, x, and y be real numbers and let n be an integer with $n \geq 2$. The rules for exponents are

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^0 = 1$, if $a \neq 0$
- $a^{-x} = \dfrac{1}{a^x}$, if $a \neq 0$
- $a^x a^y = a^{x+y}$
- $\dfrac{a^x}{a^y} = a^{x-y}$, if $a \neq 0$
- $(a^x)^y = a^{xy}$
- $a^x \cdot b^x = (ab)^x$
- $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$, if $b \neq 0$

Also, 0^0 is not defined, if it is encountered using any of the above formulae.

Logarithms

Let x and y be positive real numbers. Let a be a positive real number with $a \neq 1$. The equation $y = a^x$ is equivalent to $\log_a y = x$. The rules for logarithms are

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a(x^y) = y \log_a x$
- $\log_a(a^x) = x$
- $a^{\log_a x} = x$
- $\log_a x = \dfrac{1}{\log_x a}$, where $x \neq 1$
- $\log_y x = \dfrac{\log_a x}{\log_a y}$

Sample Problems

1. Given $2\log_5(x-3y) = \log_5(2x) + \log_5(2y)$, calculate the ratio $\dfrac{x}{y}$.

   Solution

   First, we note that in the original equation, if the three logarithmic terms are to be defined, then their arguments must be positive. So $x > 0$, $y > 0$, and $x > 3y$. Now

$$\begin{aligned} 2\log_5(x-3y) &= \log_5(2x) + \log_5(2y) \\ \log_5(x-3y)^2 &= \log_5(4xy). \end{aligned}$$

   Now since the log function takes on each value in its range only once, we have that

$$\begin{aligned} (x-3y)^2 &= 4xy \\ x^2 - 6xy + 9y^2 &= 4xy \\ x^2 - 10xy + 9y^2 &= 0 \\ (x-y)(x-9y) &= 0 \end{aligned}$$

   So $\dfrac{x}{y} = 1$ or $\dfrac{x}{y} = 9$. But from our restrictions we know that $\dfrac{x}{y} > 3$, and so $\dfrac{x}{y} = 9$.

2. Given that m and k are integers, find all values of m and k satisfying the equation

$$9(7^k + 7^{k+2}) = 5^{m+3} + 5^m$$

   Solution

   We factor both sides of this equation to arrive at

$$\begin{aligned} 9(1+7^2)7^k &= 5^m(1+5^3) \\ 9(50)7^k &= 5^m(126) \\ 3^2 \cdot 2 \cdot 5^2 \cdot 7^k &= 5^m \cdot 2 \cdot 3^2 \cdot 7 \end{aligned}$$

   Now since both sides of this equation are products of primes, and integers have unique prime factorizations, it follows that $m = 2$ and $k = 1$ is the only solution.

3. Determine the points of intersection of the curves $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

   Solution

   Again the arguments of the logarithmic functions, $x-2$ and $x+1$, must be positive, which implies that $x > 2$. Now

$$\begin{aligned} \log_{10}(x-2) &= 1 - \log_{10}(x+1) \\ \log_{10}(x-2) + \log_{10}(x+1) &= 1 \\ \log_{10}[(x-2)(x+1)] &= 1 \\ (x-2)(x+1) &= 10 \\ x^2 - x - 2 &= 10 \\ x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \end{aligned}$$

So $x = 4$ or $x = -3$, but from our restriction $x > 2$ and so $x = 4$. The point of intersection is $(4, \log_{10} 2)$ or $(4, 1 - \log_{10} 5)$. Since $\log_{10} 2 + \log_{10} 5 = 1$, these are equivalent answers.

4. Determine all values of x such that $\log_2(9 - 2^x) = 3 - x$.

   Solution

   Once again the argument of the logarithm must be positive, implying that $9 > 2^x$.

$$\begin{aligned} \log_2(9 - 2^x) &= 3 - x \\ 9 - 2^x = 2^{3-x} &= \frac{8}{2^x} \end{aligned}$$

   Substituting $y = 2^x$ we have

$$\begin{aligned} 9 - y &= \frac{8}{y} \\ y^2 - 9y + 8 &= 0 \\ (y-1)(y-8) &= 0 \end{aligned}$$

   Thus, $y = 1$ or $y = 8$. Since $y = 2^x$, we obtain the corresponding values $x = 0$ or $x = 3$. Both of these values satisfy the restriction $9 > 2^x$ and so both are valid solutions.

5. The graph of $y = m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

   Solution

   From the given information we have that $m^2 = 5$ and $n = m^5$. Thus, $m = \pm\sqrt{5}$ with corresponding values $n = (\pm\sqrt{5})^5$. Therefore, $mn = (\sqrt{5})^6 = 125$.

Problem Set

1. Determine the values of x such that $\log_x 2 + \log_x 4 + \log_x 8 = 1$.
2. Determine the values of x such that $12^{2x+1} = 2^{3x+7} \cdot 3^{3x-4}$.
3. What is the sum of the following series?

$$\log_{10} \frac{3}{2} + \log_{10} \frac{4}{3} + \log_{10} \frac{5}{4} + \cdots + \log_{10} \frac{200}{199}$$

4. Given that $x^3 y^5 = 2^{11} \cdot 3^{13}$ and $\frac{x}{y^2} = \frac{1}{27}$, solve for x and y.
5. Given that $\log_8 3 = k$, express $\log_8 18$ in the form $ak + b$ where a and b are rational numbers.
6. Determine the point(s) of intersection of the graphs of $y = \log_2(2x)$ and $y = \log_4 x$.
7. The points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the graph of $y = \log_a x$. A horizontal line is drawn through the midpoint of AB such that it intersects the curve at the point $C(x_3, y_3)$. Show that $(x_3)^2 = x_1 x_2$.
8. The graph of the function $y = ax^r$ passes through the points $(2, 1)$ and $(32, 4)$. Calculate the value of r.
9. Given that $2^{x+3} + 2^x = 3^{y+2} - 3^y$ and x and y are integers, determine the values of x and y.
10. Given that $f(x) = 2^{4x-2}$, calculate, in simplest form, $f(x) \cdot f(1-x)$.
11. Find all values of x such that $\log_5(x-2) + \log_5(x-6) = 2$.
12. Let x be a positive real number with $x \neq 1$. Prove that a, b, and c are three numbers that form a geometric sequence if and only if $\log_x a$, $\log_x b$ and $\log_x c$ form an arithmetic sequence.
13. Determine all real values of x for which

$$3^{x+2} + 2^{x+2} + 2^x = 2^{x+5} + 3^x$$

14. Determine all real numbers x for which

$$(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10000$$

15. Determine all real numbers $x > 0$ for which

$$\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$$