Functions, Equations and Polynomials
Solutions

1. Subtract the first equation from the second, rearrange the resulting expression and then factor to obtain

$$\begin{aligned} -8x + y + xy - 8 &= 0 \\ xy - 8x + y - 8 &= 0 \\ x(y-8) + y - 8 &= 0 \\ (x+1)(y-8) &= 0 \end{aligned}$$

Therefore, $x = -1$ or $y = 8$. If $x = -1$, then substituting into the first equation and solving we obtain that $y = -9$. If $y = 8$, then substituting into the first equation and solving we obtain $x = 4 \pm 2\sqrt{2}$. Therefore, the solutions are $(-1, -9)$ and $(4 \pm 2\sqrt{2}, 8)$.

2. **Solution 1**

We are asked for the x value of the midpoint of zeros, which is the x value of the vertex of the parabola. The equation is written in vertex form already and so $a = 1$.

Solution 2

Find the x-intercepts:

$$\begin{aligned} (x-1)^2 - 4 &= 0 \\ (x-1)^2 &= 4 \\ x &= 1 \pm 2 \end{aligned}$$

Thus, $x = 3$ or $x = -1$. Thus, $a = \dfrac{-1+3}{2} = 1$.

3. (a) Consider $a = 0$ and $a = 1$ and find the intersection point of the resulting equations, $y = x^2$ and $y = x^2 + 2x + 1$. Subtracting the equations we obtain $0 = 2x + 1$. Therefore, $x = -\dfrac{1}{2}$ and so the intersection point is $\left(-\dfrac{1}{2}, \dfrac{1}{4}\right)$. Now substitute $x = -\dfrac{1}{2}$ into the general equation. Therefore,

$$\begin{aligned} y &= x^2 + 2ax + a \\ &= \frac{1}{4} + 2a \cdot \left(\frac{-1}{2}\right) + a \\ &= \frac{1}{4} \end{aligned}$$

Since $\left(-\dfrac{1}{2}, \dfrac{1}{4}\right)$ satisfies the general equation, it is a point on all of the parabolas.

(b) Now $y = x^2+2ax+a = (x+a)^2+a-a^2$ and so the vertex is at $(-a,\, a-a^2)$. If we represent the coordinates of the vertex by (p,q) we have $p = -a$ and $q = a - a^2$ or $q = -p^2 - p$, the required parabola. Completing the square we obtain

$$q = -\left(p^2 + p + \frac{1}{4}\right) + \frac{1}{4} = -\left(p + \frac{1}{2}\right)^2 + \frac{1}{4}$$

and so we see that the vertex of this parabola is $\left(-\frac{1}{2}, \frac{1}{4}\right)$, the common point found in part (a)

4. Factoring both equations we arrive at:

$$p(1 + r + r^2) = 26 \qquad (1)$$
$$p^2r(1 + r + r^2) = 156 \qquad (2)$$

From equation (1) we can see neither of the factors of its left-hand side are 0. Dividing (2) by (1) gives $pr = 6$. Substituting this relation back into (1) we get

$$\begin{aligned} \frac{6}{r} + 6 + 6r &= 26 \\ 6 - 20r + 6r^2 &= 0 \\ 3r^2 - 10r + 3 &= 0 \\ (3r - 1)(r - 3) &= 0 \end{aligned}$$

Therefore, $r = \frac{1}{3}$ or $r = 3$. Hence $(p, r) = (2, 3)$ or $\left(18, \frac{1}{3}\right)$.

5. We assume, on the contrary, that the coefficients are in geometric sequence. Then $\frac{b}{a} = \frac{c}{b}$ which implies that $b^2 = ac$. But now the discriminant $b^2 - 4ac = -3b^2 < 0$, so that the roots are not real. Thus, we have a contradiction to the condition set out in the statement of the problem and our assumption is false.

6. Let r and s be the integer roots. The equation can be written as

$$\begin{aligned} a(x - r)(x - s) &= a(x^2 - (r + s)x + rs) \\ &= ax^2 - a(r + s)x + ars \\ &= ax^2 + bx + c \end{aligned}$$

with $b = -a(r + s)$ and $c = ars$. Since a, b, c are in arithmetic sequence, we have

$$\begin{aligned} c - b &= b - a \\ a + c - 2b &= 0 \\ a + ars + 2a(r + s) &= 0 \\ 1 + rs + 2(r + s) &= 0 \qquad \text{(we can divide by } a \text{ since } a \neq 0) \\ rs + 2r + 2s + 4 &= 3 \\ (r + 2)(s + 2) &= 3 \end{aligned}$$

Ignoring the order of the factors, we can factor 3 as a product of two integers in two ways: $3 = 1(3)$ or $3 = (-1)(-3)$. Therefore, the two possibilities for the roots of quadratic are: (i) -1 and 1 or (ii) -3 and -5.

7. **Solution 1**

Multiplying out and collecting terms results in $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$. We look for a factoring with integer coefficients, using the fact that the first and last coefficients are 1 and -1, respectively. So

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 + ax + 1)(x^2 + bx - 1)$$

where a and b are undetermined coefficients. However, expanding and comparing coefficients gives $a + b = -6$ and $-a + b = 2$ and $ab = 8$. Since all three equations are satisfied by $a = -4$ and $b = -2$, we have factored the original expression as

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)$$

Factoring these two quadratics gives the roots $x = 2 \pm \sqrt{3}$ and $x = 1 \pm \sqrt{2}$.

Solution 2

We observe that the original equation is of the form $f(f(x)) = x$, where $f(x) = x^2 - 3x + 1$. Now if we can find x such that $f(x) = x$, then $f(f(x)) = x$. So we solve $f(x) = x^2 - 3x + 1 = x$ which gives the first factor $x^2 - 4x + 1$ above. With polynomial division, we can then determine that

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)$$

and continue as in Solution 1.

8. The vertex has $x = 2$ and $y = -16$ and so $A = (2, -16)$. When $y = 0$ we get $0 = x^2 - 4x - 12$ which factors to give us intercepts at -2 and 6. The larger value is 6, and so $B = (6, 0)$. Therefore, we want the line through $(2, -16)$ and $(6, 0)$. Finding the slope of the line and using the second point, the equation of the line is

$$y = \left(\frac{0 + 16}{6 - 2}\right)(x - 6)$$

which simplifies to $y = 4x - 24$.

9. **Solution 1**

Multiplying gives

$$\begin{aligned} x^2 - (b + c)x + bc &= a^2 - (b + c)a + bc \\ x^2 - (b + c)x + a(-a + b + c) &= 0 \end{aligned}$$

The roots are

$$\begin{aligned} x &= \frac{b + c \pm \sqrt{(b + c)^2 - 4a(-a + b + c)}}{2} \\ &= \frac{b + c \pm \sqrt{(b + c)^2 + 4a^2 - 4a(b + c)}}{2} \\ &= \frac{b + c \pm \sqrt{(b + c - 2a)^2}}{2} \end{aligned}$$

Thus, $x = -a + b + c$ or $x = a$.

Solution 2

Observe that $x = a$ is one solution. Rearranging as in the first solution we get

$$x^2 - (b + c)x + a(-a + b + c) = 0$$

Using the sum (or the product) of the roots, we determine that other root is $x = -a + b + c$.

10. Since $x = -2$ is a solution of $x^3 - 7x - 6 = 0$, we know that $x + 2$ is a factor of $x^3 - 7x - 6$. Factoring (or using long division) we obtain

$$\begin{aligned} x^3 - 7x - 6 &= (x + 2)(x^2 - 2x - 3) \\ &= (x + 2)(x + 1)(x - 3) \end{aligned}$$

Thus, the roots are -2, -1 and 3.

11. Let the roots be r and s. Using the sum of the roots and the product of the roots we obtain

$$\begin{aligned} r + s &= \frac{-4(a - 2)}{4} \\ &= 2 - a \end{aligned}$$

and

$$\begin{aligned} rs &= \frac{-8a^2 + 14a + 31}{4} \\ &= -2a^2 + \frac{7}{2}a + \frac{31}{4} \end{aligned}$$

Then

$$\begin{aligned} r^2 + s^2 &= (r + s)^2 - 2rs \\ &= (2 - a)^2 - 2\left(-2a^2 + \frac{7}{2}a + \frac{31}{4}\right) \\ &= 4 - 4a + a^2 + 4a^2 - 7a - \frac{31}{2} \\ &= 5a^2 - 11a - \frac{23}{2}. \end{aligned}$$

It appears that the minimum value should be at the vertex of the parabola $f(a) = 5a^2 - 11a - \frac{23}{2}$, that is, at $a = \frac{11}{10}$ (found by completing the square). But we have ignored the condition that the roots are real. The discriminant of the original equation is

$$\begin{aligned} B^2 - 4AC &= [4(a - 2)]^2 - 4(4)(-8a^2 + 14a + 31) \\ &= 16(a^2 - 4a + 4) + 128a^2 - 224a - 496 \\ &= 144a^2 - 288a - 432 \\ &= 144(a^2 - 2a - 3) \\ &= 144(a - 3)(a + 1). \end{aligned}$$

Thus, we have real roots only when $a \geq 3$ or $a \leq -1$. Therefore, $a = \frac{11}{10}$ cannot be our final answer, since the roots are not real for this value. However $f(a) = 5a^2 - 11a - \frac{23}{2}$ is a parabola opening up and is symmetrical about its axis of symmetry $a = \frac{11}{10}$. So we move to the nearest value of a to the axis of symmetry that gives real roots, which is $a = 3$.

12. Let $g(2) = k$. Since f and g are inverse functions, we know that $f(k) = 2$. We need to solve

$$\begin{aligned} \frac{3k-7}{k+1} &= 2 \\ 3k - 7 &= 2(k+1) \\ k &= 9 \end{aligned}$$

Thus, $g(2) = 9$.

13. Complete the square to obtain

$$\begin{aligned} y &= -2x^2 - 4ax + k \\ &= -2(x^2 + 2ax + a^2) + k + 2a^2 \\ &= -2(x+a)^2 + k + 2a^2 \end{aligned}$$

The vertex is at $(-a, k + 2a^2)$ which we know is $(-2, 7)$. Therefore, solving we obtain $a = 2$ and $k = -1$.

14. Using the sum and the product of the roots we have the four equations:

$$\begin{aligned} a + b &= -c \\ ab &= d \\ c + d &= -a \\ cd &= b \end{aligned}$$

Therefore,

$$\begin{aligned} -(c+d) + cd &= -c \\ cd - d &= 0 \\ d(c-1) &= 0 \end{aligned}$$

But none of a, b, c or d are zero, so $c = 1$. Then we get $d = b$. Substituting $d = b$ into $ab = d$ we get $a = 1$. Then $d = b = -2$. Thus, $a + b + c + d = -2$.

15. The most common way to do this problem uses calculus. However, we make the substitution $z = x - 4$. To get y in terms of z, try

$$\begin{aligned} y &= x^2 - 2x - 3 \\ &= (x-4)^2 + 6x - 19 \\ &= (x-4)^2 + 6(x-4) + 5 \\ &= z^2 + 6z + 5 \end{aligned}$$

Therefore, the value we want to minimize is $\dfrac{y-4}{(x-4)^2} = \dfrac{z^2+6z+1}{z^2} = 1+\dfrac{6}{z}+\dfrac{1}{z^2}$. If we now let $u = \dfrac{1}{z}$, we have the parabola $1+6u+u^2$ which opens up and has its minimum at $u=-3$ with minimum value of -8. Note that since x can assume any real value except 4, we know that z and u will assume all real values except zero. Thus, the minimum value of this expression is -8.

16. **Solution 1**

Since the function g is linear and has positive slope, it is one-to-one and so it is invertible.

This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.

Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.

This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$. Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2-6)$$

Since $g(x) = 2x-4$, we have $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \dfrac{1}{2}y + 2$. Therefore,

$$f(a) = \tfrac{1}{2}(8a^2-6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, it is one-to-one and so it is invertible.

To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x-4$, convert to $y = 2g^{-1}(y)-4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y+4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.

We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.

We can apply the function g^{-1} to both sides successively to obtain

$$\begin{aligned}
f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\
f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}\text{)} \\
f(g^{-1}(x)) &= x^2 + 8x + 15 \\
f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}\text{)} \\
f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\
f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1
\end{aligned}$$

We want to determine the value of $f(\pi)$.

Thus, we can replace $\frac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.

Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.